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A New Scalar Product for Nonsymmetric Jack Polynomials

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1 Introduction

Jack polynomials are a remarkable family of polynomials in n variables $x = (x_1, \dots, x_n)$ with coefficients in the field $\mathbb{F} := \mathbb{Q}(\alpha)$ where α is an indeterminate. They arise naturally in several statistical, physical, combinatorial, and representation theoretic considerations.

The symmetric polynomials ([M1], [St], [LV], [KS]) $J_\lambda = J_\lambda^{(\alpha)}$ are indexed by partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The nonsymmetric polynomials $F_\eta = F_\eta^{(\alpha)}$ ([Op], [KS], and §2) are indexed by compositions $\eta = (\eta_1, \dots, \eta_n)$ where $\eta_i \geq 0$ are integers. They constitute orthogonal bases, respectively, for symmetric polynomials and all polynomials, with respect to the scalar product $\langle f, g\rangle_0 \equiv \int_T f(x)\overline{g(x)}d_\alpha x$ where $d_\alpha x$ is the measure $\prod_{i\neq j}(1 - x_i x_j^{-1})^{1/\alpha}dx$ on the n-torus $T \equiv \{x \in \mathbb{C}^n \mid |x_i| = 1\}$.

Now the J_λ are also orthogonal for a second ("combinatorial") scalar product defined by setting $\langle m_\lambda, g_\mu\rangle_s = \delta_{\lambda\mu}$, where $m_\lambda(x)$ is the symmetrized monomial obtained by summing $x^\eta := x_1^{\eta_1} \cdots x_n^{\eta_n}$ over all distinct permutations η of λ; and g_λ is defined as the coefficient of $m_\lambda(y)$ in the expansion

$$\prod_{i,j=1}^{n} \frac{1}{(1 - x_i y_j)^{1/\alpha}} = \sum_\lambda g_\lambda(x) m_\lambda(y).$$

The two scalar products, although related via the Dyson conjecture, ([M1] VI.9), are really quite different in nature. The first has been generalized by Macdonald to arbitrary root systems. The second has intimate connections with the combinatorics of the symmetric group but, to our knowledge, no analogs for other Weyl groups.

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Following Dunkl [Du], we define a nonsymmetric analog of the second product by setting $\langle x^\eta, q_\gamma\rangle = \delta_{\eta\gamma}$, where the polynomials q_η are defined by the expansion

$$\prod_{i=1}^{n} \frac{1}{(1-x_i y_i)} \prod_{i,j=1}^{n} \frac{1}{(1-x_i y_j)^{1/\alpha}} = \sum_\eta q_\eta(x) y^\eta.$$

1.1 Theorem. The polynomials F_η are mutually orthogonal with respect to $\langle,\rangle$. □

We also derive explicit formulas for the norm $\langle F_\eta, F_\eta\rangle$ and the function value $F_\eta(1^n)$.

To motivate our results, we review the corresponding formulas for J_λ: Recall that the *diagram* of a partition λ consists of the points (i, j) in $\mathbb{Z}^2$ with $1 \le i \le n$ and $1 \le j \le \lambda_i$, drawn in "matrix fashion" with i increasing from top to bottom and j from left to right. For $s = (i, j) \in \lambda$, the number of points to the right, left, below, and above s are called its *arm* $a(s) \equiv \lambda_i - j$, *coarm* $a'(s) \equiv j - 1$, *leg* $l(s) \equiv \#\{k > i \mid j \le \lambda_k\}$, and *coleg* $l'(s) \equiv i - 1$.

Define $b_\lambda = \prod_s b(s)$, $c_\lambda = \prod_s c(s)$, and $c'_\lambda = \prod_s c'(s)$, where $b(s) = \alpha a'(s) + n - l'(s)$, $c(s) = \alpha a(s) + l(s) + 1$, and $c'(s) = \alpha(a(s) + 1) + l(s)$. Then Stanley has shown [St]:

$$j_\lambda \equiv \langle J_\lambda, J_\lambda\rangle_s = c_\lambda c'_\lambda \quad \text{and } J_\lambda(1^n) = b_\lambda.$$

For a composition η, we define its diagram, arms, and coarms just as for a partition. However, the leg and coleg at $s = (i, j) \in \eta$ are defined to be sums of their "lower" and "upper" parts. Thus we write $l(s) = ul(s) + ll(s)$ and $l'(s) = ul'(s) + ll'(s)$ where

$$ll(s) = \#\{k > i \mid j \le \eta_k \le \eta_i\}; \quad ul(s) = \#\{k < i \mid j \le \eta_k + 1 \le \eta_i\};$$
$$ll'(s) = \#\{k > i \mid \eta_k > \eta_i\}; \quad ul'(s) = \#\{k < i \mid \eta_k \ge \eta_i\}.$$

(If η *is* a partition then $l(s)$ and $l'(s)$ agree with the previous definition.)

We now define constants $d_\eta = \prod_{s\in\eta} d(s)$, $d'_\eta = \prod_{s\in\eta} d'(s)$, and $e_\eta = \prod_{s\in\eta} e(s)$, where $d(s) = \alpha(a(s) + 1) + l(s) + 1$; $d'(s) = \alpha(a(s) + 1) + l(s)$; and $e(s) = \alpha(a'(s) + 1) + n - l'(s)$.

With this notation, our second main result is:

1.2 Theorem. $f_\eta \equiv \langle F_\eta, F_\eta\rangle = d_\eta d'_\eta$. □

In the course of proving this, we obtain short, new proofs of the following fundamental results which, since they are not directly related to the combinatorial product $\langle,\rangle$, can also be derived from more general results of Macdonald [M2] and Cherednik [C2]:

1.3 Theorem. $F_\eta(1^n) = e_\eta$. □

1.4 Theorem. $j_\lambda^{-1} J_\lambda = \sum_\eta f_\eta^{-1} F_\eta$, summed over all distinct permutations η of λ. □

Here is a brief sketch of the argument. First of all, the E_η are simultaneous eigenfunctions for a family of differential-reflection operators—the Cherednik operators (see [C1], [KS], [Op]), and we deduce Theorem 1.1 from properties of these operators. Next, we use the action of the symmetric group (as described in [KS]) to derive recursion lemmas for the constants e_η, d_η, and d'_η. This proves Theorem 1.3 and reduces Theorem 1.2 to the case where η is a partition. We complete the proof by deducing Theorems 1.2 and 1.4 from Stanley's results in the symmetric case.

Added in proof. C. Dunkl has pointed out that Lemma 5.2 is the "${}_1F_0$ hypergeometric series" from [Y] and [BO].

2 Preliminaries

For the reader's convenience we provide here an abbreviated introduction to some of the known results on Jack polynomials, emphasizing those needed in subsequent sections.

The symmetric Jack polynomial J_λ is characterizable by two properties: first, it is an eigenfunction of a certain (Debiard–Sekiguchi) differential operator; second, it is of the form $c_\lambda m_\lambda +$ "lower terms in the dominance order." Here "lower" means an $\mathbb{F}$-combination of m_μ such that $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all i, with equality for $i = n$.

The nonsymmetric Jack polynomial F_η is similarly characterizable: first, it is an eigenfunction of the *Cherednik* operators $\xi_1, \ldots, \xi_n$, and second it is of the form $d_\eta x^\eta +$ "lower terms in the dominance order."

The dominance order on compositions (and monomials) is defined by combining the order on partitions with the Bruhat order on the symmetric group S_n (all written $\geq$): for each composition η there is a unique *minimal* permutation w_η such $\eta^+ \equiv w_\eta^{-1}\eta$ is a partition (also unique). We say that $\eta \geq \zeta$ if either $\eta^+ > \zeta^+$, or if $\eta^+ = \zeta^+$ but $w_\eta \leq w_\zeta$.

The ξ_i are defined as follows: let s_{ij} be the operator which interchanges x_i and x_j. Then $N_{ij} = \frac{1}{x_i - x_j}(1 - s_{ij})$ preserves $\mathbb{F}[x]$, and $\xi_i = \xi_i^x := \alpha x_i \partial_i + \sum_{j<i} N_{ij} x_j + \sum_{j>i} x_j N_{ij}$. If we write $\zeta = (0, -1, \ldots, -n+1)$, and put $\overline{\eta} = \alpha\eta + w_\eta \zeta$, then F_η can also be characterized up to a multiple by the eigen-equations $\xi_i F_\eta = \overline{\eta}_i F_\eta$. As noted in [KS], the eigenvalues can also be written as $\overline{\eta}_i = \alpha\eta_i - (k'_i + k''_i)$ where

$$k'_i = \#\{k < i \mid \eta_k \geq \eta_i\}; \quad k''_i = \#\{k > i \mid \eta_k > \eta_i\}.$$

The polynomials J_λ and F_η are "integral" forms of the Jack polynomials. However we shall also find it convenient to work with the "monic" forms $P_\lambda = J_\lambda / c_\lambda$ and $E_\eta = F_\eta / d_\eta$.

The following proposition, (Cor. 4.2, Lemma 2.4, and Prop. 4.3 in [KS]), describes the action on E_η of the transpositions $s_i \equiv s_{i\,i+1}$, and of the operator Φ which acts on compositions by $\Phi(\eta) := (\eta_2, \dots, \eta_n, \eta_1 + 1)$, and on functions by $\Phi f(x_1, \dots, x_n) = x_n f(x_n, x_1, \dots, x_{n-1})$.

2.1. Proposition. (1) $E_{\Phi\eta} = \Phi E_\eta$.

(2) If $\eta_i = \eta_{i+1}$, then $s_i E_\eta = E_{s_i\eta}$.

(3) If $\eta_i > \eta_{i+1}$, then $dE_\eta = (ds_i + 1)E_{s_i\eta}$ with $d = \overline{\eta}_i - \overline{\eta}_{i+1}$. □

We shall also need the following weak form of Theorem 1.4 (see [KS], Lemma 2.3):

2.2. Proposition. P_λ is a linear combination of E_η such that $\eta^+ = \lambda$. □

3 The Cauchy formula

By a simple argument ([St], Prop. 2.1), the assertion $\langle J_\lambda, J_\mu\rangle_s = j_\lambda \delta_{\lambda\mu}$ is equivalent to the "Cauchy" formula $\prod_{i,j=1}^n \frac{1}{(1-x_i y_j)^{1/\alpha}} = \sum_\lambda j_\lambda^{-1} J_\lambda(x) J_\lambda(y)$. By the same argument, to establish $\langle F_\eta, F_\gamma\rangle = f_\eta \delta_{\eta\gamma}$ we need to prove the nonsymmetric analog $\Omega = \sum_\eta f_\eta^{-1} F_\eta(x) F_\eta(y)$ for

$$\Omega \equiv \prod_{i=1}^n \frac{1}{(1 - x_i y_i)} \prod_{i,j=1}^n \frac{1}{(1 - x_i y_j)^{1/\alpha}}.$$

We start by showing the following.

3.1. Lemma. For each $i = 1, \dots, n$ we have $\xi_i^x \Omega = \xi_i^y \Omega$. □

Proof. Let τ be the operator on $\mathbb{F}(x, y)$ which interchanges x_i and y_i for each i. Since $\tau(\Omega) = \Omega$, and $\tau(\xi_i^x \Omega) = \xi_i^y \Omega$, it suffices to prove that $\frac{\xi_i^x \Omega}{\Omega}$ is τ-invariant.

Since $N_{ij} x_j = x_i N_{ij} - 1$ we get $\xi_i^x = \alpha x_i \partial_i + \sum_{j<i} x_i N_{ij} + \sum_{j>i} x_j N_{ij} - (i-1)$.

$$\text{Now} \quad \frac{\alpha x_i \partial_i \Omega}{\Omega} = -\frac{\alpha x_i y_i}{1 - x_i y_i} - \sum_j \frac{x_i y_j}{1 - x_i y_j} = -\frac{\alpha x_i y_i}{1 - x_i y_i} + n - \sum_j \frac{1}{1 - x_i y_j},$$

$$\text{and} \quad \frac{N_{ij}\Omega}{\Omega} = \frac{1}{x_i - x_j}\left[1 - \frac{(1 - x_i y_i)(1 - x_j y_j)}{(1 - x_i y_j)(1 - x_j y_i)}\right] = \frac{y_i - y_j}{(1 - x_i y_j)(1 - x_j y_i)}.$$

So, writing $f \equiv g$ to denote equivalence modulo τ-invariant terms, we get

$$\begin{aligned}
\frac{\xi_i^x \Omega}{\Omega} &\equiv -\sum_{j \neq i} \frac{1}{1 - x_i y_j} + \sum_{j<i} \frac{x_i (y_i - y_j)}{(1 - x_i y_j)(1 - x_j y_i)} + \sum_{j>i} \frac{x_j (y_i - y_j)}{(1 - x_i y_j)(1 - x_j y_i)} \\
&\equiv -\sum_{j \neq i} \frac{1}{1 - x_i y_j} - \sum_{j<i} \frac{1}{1 - x_j y_i} + \sum_{j>i} \frac{1}{1 - x_i y_j} \\
&\equiv -\sum_{j<i} \left(\frac{1}{1 - x_i y_j} + \frac{1}{1 - x_j y_i}\right) \equiv 0.
\end{aligned}$$

The lemma follows. ■

Proof (of Theorem 1.1). As noted above, it suffices to prove the existence of an expansion of the form $\Omega = \sum_\eta f_\eta^{-1} F_\eta(x) F_\eta(y)$ for some constants f_η.

Expanding Ω in terms of the monomial basis x^η, and re-expressing in terms of $F_\eta(x)$, we deduce that there are polynomials C_η in $\mathbb{F}(y)$ such that $\Omega = \sum_\eta F_\eta(x) C_\eta(y)$.

Applying the lemma to this expression, and comparing coefficients of $F_\eta(x)$ we obtain the equations $\xi_i^y C_\eta(y) = \overline{\eta}_i C_\eta(y)$. It follows that C_η is a multiple of F_η. ■

4 The recursion lemmas

In this section we determine how e_η, d_η and d'_η. transform under s_i and Φ.

4.1 Lemma. For all compositions η, we have $d_{\Phi\eta}/d_\eta = \overline{\eta}_1 + \alpha + n = e_{\Phi\eta}/e_\eta$. □

Proof. We think of the diagram of $\Phi\eta$ as obtained from that of η by removing the top row, moving all the other rows up one unit, adding a point to the top row, and appending this row to the bottom of the diagram.

To prove the first equality, we consider the new point to have been *prefixed* to the top row of η. It is easy to check from the definitions that the arms and legs of all other points are unchanged after this procedure, while for the new point $s = (n, 1) \in \Phi\eta$ we have $a(s) = \eta_1$ and $l(s) = \#\{k > 1 \mid \eta_k \le \eta_1\}$. This gives $d_{\Phi\eta}/d_\eta = \alpha(a(s)+1) + l(s) + 1 = \overline{\eta}_1 + \alpha + n$.

For the second equality, we consider the new point to have been *appended* to the top row of η. This time the coarms and colegs of other points stay the same, and for the new point $s = (n, \eta_1 + 1) \in \Phi\eta$ we have $a'(s) = \eta_1$ and $l'(s) = \#\{k > 1 \mid \eta_k > \eta_1\}$. Thus $e_{\Phi\eta}/e_\eta = \alpha(a'(s) + 1) + n - l'(s) = \overline{\eta}_1 + \alpha + n$. ■

4.2 Lemma. We have $e_{s_i\eta} = e_\eta$ for all η. Also if $\eta_i > \eta_{i+1}$ with $d = \overline{\eta}_i - \overline{\eta}_{i+1}$, then $d_{s_i\eta}/d_\eta = (d+1)/d$ and $d'_{s_i\eta}/d'_\eta = d/(d-1)$. □

Proof. Comparing definitions of $l'(s)$ and $\overline{\eta}$ we see that $e_\eta \equiv \prod_s [\alpha(a'(s) + 1) + n - l'(s)]$ can be rewritten as $\prod_{i=1}^n \prod_{j=1}^{\eta_i} (n + \overline{\eta}_i - j\alpha)$. Since $\overline{\eta} = w_\eta(\alpha\eta^+ + \zeta)$ is a permutation of $\overline{s_i\eta} = w_{s_i\eta}(\alpha\eta^+ + \zeta)$, from the last expression it follows that $e_\eta = e_{s_i\eta}$.

For the assertions about d_η and d'_η, note that $d = \alpha(\eta_i - \eta_{i+1}) + (k'_{i+1} - k'_i) + (k''_{i+1} - k''_i)$. Moreover, for the point $s = (i, \eta_{i+1} + 1) \in \eta$ we have $\eta_i - \eta_{i+1} = a(s) + 1$ and

$$\begin{aligned} k'_{i+1} - k'_i &= \#\{k < i+1 \mid \eta_k \ge \eta_{i+1}\} - \#\{k < i \mid \eta_k \ge \eta_i\} \\ &= 1 + \#\{k < i \mid \eta_i > \eta_k \ge \eta_{i+1}\} = 1 + ul(s). \end{aligned}$$

Similarly $k''_{i+1} - k''_i = ll(s)$, and we deduce that $d = \alpha(a(s) + 1) + l(s) + 1$.

Now the diagram of $s_i\eta$ is obtained from that of η by switching rows i and $i+1$. This increases the leg of the point $s = (i, \eta_{i+1}+1) \in \eta$ by 1, while other legs and all arms remain the same. Thus

$$d_{s_i\eta}/d_\eta = \frac{\alpha(a(s)+1) + (l(s)+1) + 1}{\alpha(a(s)+1) + (l(s)+1)} = \frac{d+1}{d}.$$

The assertion about d'_η follows similarly. ■

5 The explicit formulas

We first establish Theorem 1.3.

Proof (of Theorem 1.3). Specializing $x = 1^n$ in Proposition 2.1 we get the recursions

(1) $E_{\Phi\eta}(1^n) = E_\eta(1^n)$ for all η, and

(2) if $\eta_i > \eta_{i+1}$ then $E_{s_i\eta}(1^n) = \frac{d}{d+1} E_\eta(1^n)$ with $d = \overline{\eta}_i - \overline{\eta}_{i+1}$.

By the previous lemmas, e_η/d_η satisfies the same recursions. Checking the trivial case $\eta = (0, \ldots, 0)$, we deduce $E_\eta(1^n) = e_\eta/d_\eta$ for all η. The result follows. ■

5.1. Lemma. If Theorem 1.2 holds for partitions then it holds for all compositions.

Proof. First note that the nonsymmetric Cauchy formula of §3 may be rewritten as $\Omega = \sum_\eta r_\eta E_\eta(x) E_\eta(y)$ where $r_\eta = f_\eta^{-1} d_\eta^2$. Next, observe that $s_i^x \Omega = s_i^y \Omega$, where the superscripts indicate action in the respective variables. Combining these, we get

$$\sum r_\eta (s_i E_\eta)(x) E_\eta(y) = \sum r_{s_i\eta} E_{s_i\eta}(x) (s_i E_{s_i\eta})(y). \qquad (*)$$

If $\eta_i > \eta_{i+1}$ and $d = \overline{\eta}_i - \overline{\eta}_{i+1}$, then $s_i E_{s_i\eta} = -\frac{1}{d} E_{s_i\eta} + E_\eta$ and $s_i E_\eta = \frac{1}{d} E_\eta + \frac{d^2-1}{d^2} E_{s_i\eta}$. The first follows directly from Proposition 2.1 and the second after multiplication by $ds_i - 1$. Computing coefficients of $E_{s_i\eta}(x) E_\eta(y)$ on both sides of $(*)$ we get $r_{s_i\eta} = \frac{d^2-1}{d^2} r_\eta$.

From Lemma 4.2 we have $d_{s_i\eta} = \frac{d+1}{d} d_\eta$ and $d'_{s_i\eta} = \frac{d}{d-1} d'_\eta$. Combining these we deduce that $f_\eta^{-1} d_\eta d'_\eta = r_\eta d'_\eta d_\eta^{-1}$ is s_i-invariant, and the result follows. ■

To continue we need the following formula for symmetric Jack polynomials.

5.2. Lemma. We have $\displaystyle\prod_{i=1}^{n} \frac{1}{(1-x_i)^r} = \sum_\lambda k_\lambda \frac{J_\lambda(x)}{j_\lambda}$ with $\displaystyle k_\lambda = \prod_{s\in\lambda} [\alpha(r + a'(s)) - l'(s)]$. □

Proof. Expanding the left side in terms of m_λ and then in terms of J_λ, we deduce that the k_λ are *polynomials* in $\mathbb{F}[r]$. Next, by the symmetric Cauchy formula in $m \geq n$ variables,

$$\prod_{i,j=1}^{m} \frac{1}{(1-x_i y_j)^{1/\alpha}} = \sum_\mu j_\mu^{-1} J_\mu(x_1, \ldots, x_m) J_\mu(y_1, \ldots, y_m).$$

Let $\lambda = (\mu_1, \dots, \mu_n)$ be the first n components of μ. By the "stability" of Jack polynomials ([St] Prop. 2.5), $J_\mu(x_1, \dots, x_n, 0, \dots, 0)$ equals $J_\lambda(x_1, \dots, x_n)$ if $\mu_{n+1} = \dots = \mu_m = 0$ and equals *zero* otherwise. Thus setting $x_{n+1} = \dots = x_m = 0$ and $y_1 = \dots = y_m = 1$ in the m-variable formula, and using the expressions for j_μ and $J_\mu(1^m)$, we obtain

$$\prod_{i=1}^{n} \frac{1}{(1-x_i)^{m/\alpha}} = \sum_\lambda \prod_{s\in\lambda} [\alpha a'(s) + m - l'(s)]\, j_\lambda^{-1} J_\lambda(x).$$

This implies the formula for $r = m/\alpha$ and, by polynomiality of k_λ, in general. ■

Finally we prove Theorems 1.2 and 1.4.

Proof (of Theorems 1.2 and 1.4) . Setting $y_1 = \dots = y_n = 1$ in the nonsymmetric Cauchy formula, and using Lemma 5.2 and Theorem 1.3 to evaluate the two sides, we get

$$\sum_\eta f_\eta^{-1} e_\eta F_\eta = \sum_\lambda \prod_{s\in\lambda} [\alpha(\frac{n}{\alpha} + 1 + a'(s)) - l'(s)] \frac{J_\lambda}{j_\lambda} = \sum_\lambda e_\lambda j_\lambda^{-1} J_\lambda.$$

Using Proposition 2.2 and Lemma 4.2 ($e_\eta = e_{\eta^+}$), this gives $j_\lambda^{-1} J_\lambda = \sum_{\eta^+=\lambda} f_\eta^{-1} F_\eta$ which is Theorem 1.4. Comparing coefficients of x^λ on both sides we get $j_\lambda^{-1} c_\lambda = f_\lambda^{-1} d_\lambda$, which can be rewritten as $f_\lambda = c'_\lambda d_\lambda$. Since $c'_\lambda = d'_\lambda$, we deduce Theorem 1.2 for partitions and, by Lemma 5.1, in general. ■

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