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Tensor products of singular representations and an extension of the θ-correspondence

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Abstract. In this paper we consider the problem of decomposing tensor products of certain singular unitary representations of a semisimple Lie group G. Using explicit models for these representations (constructed earlier by one of us) we show that the decomposition is controlled by a reductive homogeneous space G'/H'. Our procedure establishes a correspondence between certain unitary representations of G and those of G'. This extends the usual θ-correspondence for dual reductive pairs. As a special case we obtain a correspondence between certain representations of real forms of E_7 and F_4.

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0. Introduction

Let F be a field and ε some fixed additive character of F. If W is a finite dimensional vector space over F endowed with a non-degenerate skew-symmetric form, we can consider an associated Heisenberg group $H(W)$. Denote by ρ_ε an irreducible unitary representation of $H(W)$ on which the center of $H(W)$ operates via the character ε (it is unique by the theorem of Stone and von Neumann). Since the symplectic group $\mathrm{Sp}(W)$ operates on $H(W)$ via its action on the vector space W, it also acts on the representation ρ_ε. The action is trivial on the center of $H(W)$ and therefore, for any $g \in \mathrm{Sp}(W)$, there is an operator $\omega_\varepsilon(g)$ (unique up to scaling) which intertwines ρ_ε with ρ_ε^g. These operators form the *oscillator representation* of $\mathrm{Sp}(W)$. In general, the oscillator representation ω_ε is projective, but it always corresponds to an ordinary representation of a two-fold cover of $\mathrm{Sp}(W)$. This cover is denoted by $\widetilde{\mathrm{Sp}(W)}$ and is called the *metaplectic group*.

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The oscillator representation originated in the works of Segal and Shale and was immortalized by Weil, who used it in his construction of the theta-functions on the metaplectic group [W]. In one of the earliest works on the spectrum of the Weil (oscillator) representation, Gelbart [G1] studied the decomposition of the tensor product $\omega' = \omega^{\otimes k}$, where ω is the oscillator representation of the real symplectic group $\mathrm{Sp}(2m, \mathbb{R})$ and $k \geq 2m$. He demonstrated that for $k = 2m$ all representations of the holomorphic discrete series for $\mathrm{Sp}(2m, \mathbb{R})$ occur in the spectrum of ω'. Kashiwara and Vergne [KV] extended the results of [G1] to tensor products $\omega^{\otimes k}$, $k \geq 1$. In particular, any unitary highest weight representation of $\mathrm{Sp}(2m, \mathbb{R})$ will appear in the decomposition of $\omega^{\otimes k}$ for some appropriate value of k.

Later, the approach of [G1] and [KV] was replaced by the modern approach to the θ-correspondence. One starts with a reductive dual pair of algebraic groups G and G', defined over some local field F, which are mutual centralizers inside a symplectic group Sp. Let ω be an oscillator representation of the metaplectic group $\widetilde{\mathrm{Sp}}$ on a Hilbert space $\mathcal{H}$, and let $\overline{E} \subset \widetilde{\mathrm{Sp}}$ denote the preimage of a reductive subgroup $E \subset \mathrm{Sp}$. Denote by $\mathcal{R}(E)$ the set of (equivalence classes) of continuous irreducible representations of $\overline{E}$ on a locally convex space, which can be realized as quotients of $\mathcal{H}^\infty$ by $\omega(\overline{E})$-invariant subspaces.

Conjecture 0.1 (Howe's duality conjecture)**.** *The set $\mathcal{R}(G \cdot G')$ is the graph of a bijection between all of $\mathcal{R}(G)$ and all of $\mathcal{R}(G')$. Moreover, an element of $\mathcal{R}(G \cdot G')$ occurs as a quotient of ω in a unique way.*

Howe's conjecture has been proved for $F = \mathbb{R}$ or $\mathbb{C}$ [Ho2], and also for all non-Archimedean local fields of odd residue characteristic. The resulting correspondence between the irreducible representations of $\overline{G}$ and those of $\overline{G'}$ is called the Howe duality correspondence, or the θ-correspondence. In general, this correspondence does not preserve unitarity, i.e., a unitary representation of $\overline{G}$ can correspond to a non-unitarizable representation of $\overline{G'}$.

If one member of a dual pair (say G') is much "smaller" than the second group, unitarity is preserved and the duality correspondence is a particularly nice one (this is a so called stable range duality [Ho1], [L1]). We denote by $\widehat{G}(\varepsilon)$ and $\widehat{G'}(\varepsilon)$ the subsets of the unitary duals of $\overline{G}$ and $\overline{G}'$ consisting of those unitary irreducible representations whose restriction to the kernel of the projection $\widetilde{\mathrm{Sp}} \to \mathrm{Sp}$ (i.e., the group $\mathbb{Z}_2$) is a multiple of the non-trivial character of $\mathbb{Z}_2$. Then the Howe correspondence gives an injection $\widehat{G'}(\varepsilon) \hookrightarrow \widehat{G}(\varepsilon)$. In many cases the coverings $\overline{G} \to G$ and $\overline{G}' \to G'$ are trivial, and we obtain an injection of the unitary dual of G' into that of G.

Example. $G' = U(1)$ and $G = U(p, q)$ form a stable range dual pair inside $\mathrm{Sp}(2p+2q, \mathbb{R})$. The representations of $U(p, q)$ appearing in the θ-correspondence for this reductive dual pair are the "ladder" representations, which were given this name because their $\overline{K}$-types lie along a line, i.e., their highest weights are obtained from the highest weight of the lowest $\overline{K}$-type by adding multiples of a single vector.

The duality conjecture also has a global version [G2, 2.5], when F is a global field, $\mathbb{A}$ is an adele ring of F, and one considers a dual pair inside a symplectic group over $\mathbb{A}$. This version is of considerable representation- and number-theoretic interest, since the global duality correspondence provides a way to lift automorphic forms between members of a dual pair [G2], [P]. For example, if $\overline{G'} = \widetilde{SL(2)}$ and $G \simeq PGL(2)$ is realized as an orthogonal group preserving a 3-variables quadratic form $Q(x_1, x_2, x_3) = x_1^2 - x_2x_3$, the θ-correspondence produces the Shimura lifting which associates a modular form of weight $n-1$ to a modular form of half-integral weight $n/2$.

It is a remarkable fact that *every* classical group and *no* exceptional group can be realized as a member of a reductive dual pair. It seems desirable to determine whether one can extend this theory to other reductive groups in a reasonable manner. In this paper we attempt to extend the original ([G1], [KV]) approach to the θ-correspondence. We study the decompositions of tensor products of certain small unitary representations of a real reductive group G and construct a parametrization of the spectra of these tensor products.

Let Ω be a symmetric tube domain of rank n, and $G = \mathrm{Aut}(\Omega)$. The Shilov boundary of Ω is of the form G/P where $P = LN$ is the Siegel–Poincaré parabolic subgroup of G. The nilradical N is abelian and so is isomorphic to its Lie algebra $\mathfrak{n}$, and the Levi subgroup L has finitely many (coadjoint) orbits on $\mathfrak{n}^*$. These orbits are indexed by their "signatures", where a signature p consists of a pair of non-negative integers $p = (p^+, p^-)$ with $|p| \stackrel{\mathrm{def}}{=} p^+ + p^- \leq n$.

Every non-open orbit (with $|p| < n$) has a canonical L-equivariant measure, and the main result of [S] is the construction of an irreducible representation of the universal cover of G on the associated L^2-space. Usually these representations descend to G, and in all cases they can be viewed as linear representations of a certain double cover of G, denoted by $\overline{G}$. In this paper we consider the problem of decomposing certain tensor products of these representations. More precisely, let $\Pi = \pi_1 \otimes \cdots \otimes \pi_s$ be the tensor product of representations associated to orbits $\mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_s}$ whose signatures $p_i = (p_i^+, p_i^-)$ satisfy

$$(p_1^+ + p_1^-) + \cdots + (p_s^+ + p_s^-) \leq n. \tag{1}$$

Under assumption (1), which is an analogue of the stable range condition for the θ-correspondence, each of the following spaces contains an open and dense L-orbit

$$\mathcal{O} \equiv \mathcal{O}_{p_1} + \ldots + \mathcal{O}_{p_s}, \quad \mathcal{O}' \equiv \mathcal{O}_{p_1} \times \ldots \times \mathcal{O}_{p_s}.$$

Fix a generic point $\xi' = (\xi_1, \ldots, \xi_k)$ in $\mathcal{O}'$ such that $\xi = \xi_1 + \ldots + \xi_k$ is generic in $\mathcal{O}$. We denote the inverse images of L and P in $\overline{G}$ by $\overline{L}$ and $\overline{P}$, respectively. Let S and S' be the stabilizers of ξ' and ξ in $\overline{L}$, and let χ_ξ be the unitary character of N defined by $\chi_\xi(\exp x) \stackrel{\mathrm{def}}{=} \exp(i\langle \xi, x\rangle)$ for $x \in \mathfrak{n}$.

Theorem 0.2. *The restriction of* Π *to* $\overline{P}$ *is isomorphic to* $\nu \otimes \mathrm{Ind}_{S'N}^{\overline{P}}(1 \otimes \chi_\xi)$, *where* ν *is a certain unitary character of* $\overline{P}$.

In general, S, S' are not reductive, however they contain reductive groups G' and H' and a common normal subgroup N such that $S = G' \ltimes Z$ and $S' = H' \ltimes Z$. We consider the direct integral decomposition

$$L^2(G'/H') = \int_{\pi \in \widehat{G'}}^{\oplus} m(\pi)\pi \, d\mu(\pi).$$

For each π occurring in $L^2(G'/H')$, we define $\Theta(\pi) \in \widehat{\overline{P}}$ by

$$\Theta(\pi) = \nu \otimes \mathrm{Ind}_{G'ZN}^{\overline{P}}(\pi \otimes 1 \otimes \chi_\xi).$$

By Mackey theory, $\Theta(\pi)$ is irreducible, and Theorem 0.2 implies

Theorem 0.3. *The restriction of* Π *to* $\overline{P}$ *has the decomposition*

$$\Pi|_{\overline{P}} = \int_{\pi \in \widehat{G'}}^{\oplus} m(\pi)\Theta(\pi) \, d\mu(\pi). \tag{2}$$

The main result of the paper is the following

Theorem 0.4. *For almost every* π *(with respect to the Plancherel measure* $d\mu$*),* $\Theta(\pi)$ *extends to an irreducible representation of* $\overline{G}$*, so that* (2) *is also a* $\overline{G}$*-decomposition. If* $\sum_{i=1}^{s} |p_i| < n$*, this extension is unique.*

Thus the map $\pi \to \Theta(\pi)$ gives a (measurable) bijection between unitary representations of $\overline{G}$ occurring in Π and the unitary representations of G' occurring in $L^2(G'/H')$.

We now discuss some special cases of the above result:

If $s = 2$, then G'/H' is a symmetric space, which is Riemannian if and only if $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$ both have definite signatures (of the form $(p^+, 0)$ or $(0, p^-)$). Positive (resp. negative) definite orbits correspond to the highest (resp. lowest) weight singular representations of $\overline{G}$, and in this case our constructions complement the results on the tensor products of holomorphic and anti-holomorphic discrete series representations in [R].

For Riemannian symmetric spaces, and also for several non-Riemannian ones, we have $m(\pi) \leq 1$. Thus in these cases we deduce that $\pi_1 \otimes \pi_2$ is multiplicity free.

If Ω is the Siegel upper half plane then $G = \mathrm{Sp}(2n, \mathbb{R})$, $\overline{G}$ is the metaplectic group and $G'/H' = O(p^+, p^-)/[O(p_1^+, p_1^-) \times \ldots \times O(p_s^+, p_s^-)]$, where $p^+ = \sum p_i^+$,

$p^- = \sum p_i^-$. In this case our correspondence coincides with the H'-spherical part of the θ-correspondence.

Finally, if Ω is the exceptional tube domain, then $\overline{G} = G$ is the simply connected exceptional group $E_{7(-25)}$. If we take $s = 2$ and $p_1 = (1,0)$, then among the possibilities for G'/H' are the various forms of the Cayley projective plane [A, p. 118], i.e., for $p_2 = (2,0)$, $p_2 = (0,2)$ and $p_2 = (1,1)$, we obtain respectively

$$F_{4(-52)}/\operatorname{Spin}(9),\ F_{4(-20)}/\operatorname{Spin}(9) \quad \text{and} \quad F_{4(-20)}/\operatorname{Spin}(1,8).$$

Note that these symmetric spaces are multiplicity-free (see [V] for the non-Riemannian space $F_{4(-20)}/\operatorname{Spin}(1,8)$) and by Theorem 0.4, so is $\Pi = \pi_1 \otimes \pi_2$.

If we take $p_1 = p_2 = (1,0)$, then Π is a tensor square of a highest weight representation π_1, and a description of the spectrum of this tensor square is a key step in the classification of unitarizable highest weight modules in [EHW].

Just as with the θ-correspondence, we expect that our results will have smooth and global analogues. We shall take up some of these questions in subsequent papers.

1. Notation and Preliminaries

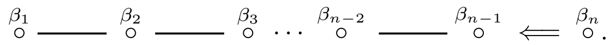
1.1 Groups and subgroups

Let G be one of the following groups:

- $\mathrm{Sp}(2n,\mathbb{R})$ (case I1),
- $U(n,n)$ (case I2),
- $O^*(4n)$ (case I3),
- $O(2,j)$ (case I4),
- $E_{7(-25)}$ (case I5),

and K be the maximal compact subgroup of G. Then $\Omega = G/K$ is a symmetric domain of tube type [He, p. 474]. Taking $G = U(n,n)$ or $O(2,j)$ instead of $SU(n,n)$ or $SO(2,j)$ is not really necessary, but will make some arguments more straightforward.

The restricted root system for each of the groups listed above is of type C_n, where n is the real rank of the group G. Let $\Delta = \{\beta_1, \beta_2, \dots, \beta_n\}$ be the basis of the restricted root system, enumerated in such a way that the corresponding Dynkin diagram is



There exists a one-to-one correspondence between the set of maximal parabolic subgroups of G and the set of maximal subsets of Δ. We will be interested in two parabolic subgroups of G — the Siegel parabolic of G (this corresponds to the set $\Delta \setminus \{\beta_n\}$) and the maximal parabolic subgroup corresponding to the set $\Delta \setminus \{\beta_1\}$. We denote the first by P and the second by P'.

The Levi decomposition of P is $P = L \cdot N$, where the subgroup N is abelian (e.g., for $G = O(2, j)$ we have $n = 2$, $L = \mathbb{R}^* \times O(1, j-1)$ and $N = \mathbb{R}^{1,j-1}$).

The Langlands decomposition of P' is $P' = M'AN'$, where the radical N' is a two-step nilpotent group with a one-dimensional center ZN', and we can identify it with the real Heisenberg group of dimension $2m+1$. The vector subgroup A is one-dimensional, i.e., $A = \mathbb{R}^*$. For example, $G = E_{7(-25)}$ gives $M'A = SO(2, 10) \times \mathbb{R}^*$ and N' is the Heisenberg group associated with a 32-dimensional real vector space.

The group M' splits into a direct product of a compact factor and a noncompact group, which we denote by G_-. The group G_- belongs to one of the families (I1)–(I4), and we can consider its Siegel parabolic P_-, the nilradical N_- of P_-, etc. In general, all subgroups of G_- will be written with a minus as a subscript.

The information about some of the subgroups we defined above is summarized in the following table.

G	N	M'	G_-	m
$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sym}(n, \mathbb{R})$	$\mathrm{Sp}(2n-2, \mathbb{R})$	$\mathrm{Sp}(2n-2, \mathbb{R})$	$n-1$
$U(n, n)$	$\mathrm{Herm}(n, \mathbb{C})$	$U(1) \times U(n-1, n-1)$	$U(n-1, n-1)$	$2(n-1)$
$O^*(4n)$	$\mathrm{Herm}(n, \mathbb{H})$	$\mathrm{Sp}(1) \times O^*(4n-4)$	$O^*(4n-4)$	$4(n-1)$
$O(2, j)$	$\mathbb{R}^{1,j-1}$	$SL(2, \mathbb{R}) \times O(j-2)$	$SL(2, \mathbb{R})$	$j-2$
$E_{7(-25)}$	$\mathrm{Herm}(3, \mathbb{O})$	$SO(2, 10)$	$SO(2, 10)$	16

1.2. Orbits and representations

The orbits of the natural action of L on $\mathfrak{n}^* = N^*$ are parametrized by pairs of non-negative integers p^+, p^- with $p^+ + p^- \leq n$ [S, 2.1]. The simplest example of this parametrization can be observed for $G = \mathrm{Sp}(2n, \mathbb{R})$, when the group N can be identified with the vector space of $n \times n$ real symmetric matrices, and an arbitrary orbit $\mathcal{O}$ of $L = GL(n, \mathbb{R})$ on $N^* \simeq N$ is defined by the signature of the symmetric matrix $\xi \in \mathcal{O}$. We write p for the pair (p^+, p^-), and $\mathcal{O}_p$ for the corresponding orbit. The rank of the orbit is $p^+ + p^-$, which we denote by $|p|$. If $|p| = n$, the orbit $\mathcal{O}_p$ is open in N^*, otherwise we get small (singular) orbits. By S_p we denote the stabilizer of $\xi_p \in \mathcal{O}_p$ in L.

Suppose now $|p| < n$. Then $O_p = L/S_p$ has an L-equivariant measure $d\mu_p$ which transforms by some positive character δ_p of L. The main result of [S] associates with each nonzero singular orbit $\mathcal{O}_p$ a unitary irreducible representation π_p of $\overline{G}$. Here $\overline{G} = G$ unless $G = \mathrm{Sp}(2n, \mathbb{R})$ or $O(2, j)$ with j odd and $\overline{G}$ is a two-fold cover of G in these two cases (for $G = \mathrm{Sp}(2n, \mathbb{R})$ we can take $\overline{G}$ to be a real metaplectic group).

If H is a subgroup of G, we write $\overline{H}$ for the inverse image of H in $\overline{G}$.

The representation π_p acts on the Hilbert space $L^2(\mathcal{O}_p, d\mu_p)$, and actions of the elements of the maximal parabolic subgroup $\overline{P}$ can be written in a particularly simple manner — the action of the reductive part $\overline{L}$ comes from the action of $\overline{L}$ on $\mathcal{O}_p$ and the unipotent radical N acts by characters

$$\begin{aligned} {[\pi_p(n)h](\xi_p)} &= \chi_{\xi_p}(n)\, h(\xi_p), && n \in N, \quad \xi_p \in \mathcal{O}_p \\ {[\pi_p(l)h](\xi_p)} &= \nu_p(l)\delta_p(l)^{-1/2} h(l^{-1}\xi_p), && l \in \overline{L}, \quad \xi_p \in \mathcal{O}_p. \end{aligned} \tag{3}$$

Here χ_{ξ_p} is the unitary character of the vector space N defined by $\xi_p \in N^*$ and ν_p is a unitary character of $\overline{L}$ (trivial on the identity component of $\overline{L}$)[1].

2. Tensor products $\pi_{p_1} \otimes \ldots \otimes \pi_{p_s}$

We pick s singular orbits $\mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_s}$ such that $|p_1| + \ldots + |p_s| \leq n$ and consider the tensor product of associated representations

$$\Pi = \bigotimes_{i=1}^{s} \pi_{p_i}.$$

The group $\overline{L}$ acts on the set $\mathcal{O}' \overset{\mathrm{def}}{=} \mathcal{O}_{p_1} \times \ldots \times \mathcal{O}_{p_s}$, and up to a set of measure zero, $\mathcal{O}'$ is a single $\overline{L}$-orbit. Note that the set

$$\mathcal{O} \overset{\mathrm{def}}{=} \mathcal{O}_{p_1} + \ldots + \mathcal{O}_{p_s} = \Big\{ \zeta \in N^* \mid \zeta = \sum_{i=1}^{s} \zeta_{p_i},\ \zeta_{p_i} \in \mathcal{O}_{p_i} \Big\}$$

also contains a dense $\overline{L}$-orbit. The representation Π acts in $\bigotimes_{i=1}^{s} L^2(\mathcal{O}_{p_i}, d\mu_{p_i})$, and we can identify this space with $L^2(\mathcal{O}', d\mu')$ where $d\mu'$ is the product measure. If we fix a generic representative $\xi' = (\xi_{p_1}, \ldots, \xi_{p_s}) \in \mathcal{O}'$ and set

$$\xi = \xi_{p_1} + \ldots + \xi_{p_s} \in \mathcal{O}$$

and $\delta = \prod_{i=1}^{s} \delta_{p_i}$, $\nu = \prod_{i=1}^{s} \nu_{p_i}$, we have the following formulas for the actions of $\Pi|_{\overline{P}}$ on $L^2(\mathcal{O}', d\mu')$

$$\begin{aligned} \Pi(l_0) f(l\xi') &= \nu(l_0)\delta(l_0)^{-1/2} h(l_0^{-1} l\xi'), && l_0 \in \overline{L} \\ \Pi(n_0) f(l\xi') &= \chi_{l\xi}(n_0) h(l\xi'), && n_0 \in N. \end{aligned} \tag{4}$$

Let now S' and S be the isotropy subgroups of ξ' and ξ, respectively, with respect to the action of $\overline{L}$ on $\mathcal{O}'$ and $\mathcal{O}$. If $|p_1| + \ldots + |p_s| = n$, the groups S' and S are reductive.

[1] In [S] the characters ν_p and δ_p are denoted by μ and ν respectively.

Example. Take $G = U(n,n)$, $s = 2$ and $p_1 = (k,0)$, $p_2 = (0, n-k)$. Then we can choose

$$\xi_{p_1} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_{p_2} = \begin{pmatrix} 0 & 0 \\ 0 & -I_{n-k} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

It is easy to see that $S = U(k, n-k)$ and $S' = U(k) \times U(n-k)$. The quotient S/S' is a Riemannian symmetric space.

Lemma 2.1. $\Pi|_{\overline{P}} \simeq \nu \otimes \mathrm{Ind}_{S'N}^{\overline{P}}(1 \otimes \chi_\xi)$ *(L^2-induction). Here ν is a character of $\overline{L}$ extended trivially to $\overline{P}$.*

Proof. We denote the induced representation $\nu \otimes \mathrm{Ind}_{S'N}^{\overline{P}}(1 \otimes \chi_\xi)$ by Π'. Then by the definition of the induced representation, Π' acts on the space $\mathcal{C}$ of square-summable functions satisfying a standard invariance condition

$$\mathcal{C} = \Big\{ f : \overline{L}N \to \mathbb{C} \mid f(ps'n) = \chi_\xi(n)^{-1} f(p) \quad \text{for} \quad p \in \overline{P}, s' \in S', \quad n \in N \Big\}. \tag{5}$$

Let $|p_1| + \ldots + |p_s| < n$ (strict inequality). Then the quasi-invariant measure on the quotient space $\overline{L}N/S'N \simeq \overline{L}/S'$ is transformed by the character $\delta = \prod_{i=1}^s \delta_{p_i}$ of $\overline{L}$, and we get

$$\begin{aligned} \Pi'(l_0) f(ln) &= \nu(l_0)\delta(l_0)^{-1/2} f(l_0^{-1} ln) \\ \Pi'(n_0) f(ln) &= f(n_0^{-1} ln) = f\big(l(l^{-1} n_0^{-1} ln)\big). \end{aligned} \tag{6}$$

We can now define a unitary operator $\Psi : L^2(\mathcal{O}', d\mu') \to \mathcal{C}$ by setting

$$[\Psi h](ln) = \chi_\xi(n)^{-1} h(l\xi').$$

This operator provides an isometry between $L^2(\mathcal{O}', d\mu')$ and $\mathcal{C}$, and it easy to check that Ψ intertwines the actions Π and Π'. Indeed, for the actions of l_0 this is immediate by inspection of formulas (4) and (6), and for the actions of n_0 we get

$$\begin{aligned} \Pi'(n_0)[\Psi h](ln) &= [\Psi h]\big(l(l^{-1} n_0^{-1} ln)\big) = \chi_\xi(l^{-1} n_0^{-1} l)^{-1} \chi_\xi(n)^{-1} h(l\xi') \\ &= \chi_\xi(n)^{-1} \chi_{l\xi}(n_0) h(l\xi') = [\Psi \Pi(n_0) h](ln). \end{aligned}$$

Computations for $|p_1| + \ldots + |p_s| = n$ are almost identical. In this case the space $\overline{L}/S'$ possesses an $\overline{L}$-invariant measure, the action of $l_0 \in \overline{L}$ is given by

$$\Pi'(l_0) f(ln) = f(l_0^{-1} ln),$$

and we set

$$[\Psi h](ln) = \chi_\xi(n)^{-1} \delta(l)^{-1/2} (l\xi').$$

A straightforward computation shows that this operator intertwines the actions of Π and Π'. □

Denote by γ a S'-quasi-regular representation of S in $L^2(S/S')$. That is

$$\big(\gamma(z)f\big)(x) = f(z^{-1}x) \quad \text{for} \quad z \in S,\ x \in Y \stackrel{\text{def}}{=} S/S', \quad f \in L^2(Y). \tag{7}$$

Of course, $\gamma = \mathrm{Ind}_{S'}^{S} 1$, and combining the induction in stages with the fact that the character χ_ξ of N is SN-fixed, we get

$$\Pi|_{\overline{P}} \simeq \nu \otimes \mathrm{Ind}_{S'N}^{\overline{P}}(1 \otimes \chi_\xi) = \nu \otimes \mathrm{Ind}_{SN}^{\overline{P}} \Big((\mathrm{Ind}_{S'}^{S} 1) \otimes \chi_\xi \Big) = \nu \otimes \mathrm{Ind}_{SN}^{\overline{P}}(\gamma \otimes \chi_\xi). \tag{8}$$

The groups S and S' are, generally speaking, not reductive (except when $|p_1| + \ldots + |p_s| = n$). As was discussed in [S, 2.1], the Lie algebras $\mathfrak{s}$ and $\mathfrak{s}'$ of S and S', respectively, can be written as

$$\mathfrak{s} = (\mathfrak{l}_1 + \mathfrak{g}') + \mathfrak{u}$$
$$\mathfrak{s}' = (\mathfrak{l}_1 + \mathfrak{h}') + \mathfrak{u}\,,$$

where $\mathfrak{l}_1, \mathfrak{g}', \mathfrak{h}'$ are some reductive Lie algebras, $\mathfrak{h}' \subset \mathfrak{g}'$ and $\mathfrak{u}$ is a nilpotent radical common for both $\mathfrak{s}$ and $\mathfrak{s}'$. Let G' and H' be the corresponding Lie groups.

In particular, $X = G'/H'$ is a reductive homogeneous space, and we can consider an H'-quasi-regular representation of G' on $L^2(X)$ (denoted by γ'). Then the representation γ of S given by the formula (7) can be obtained by extending γ' trivially from G' to S. Now let

$$\gamma' \simeq \int_{\widehat{G'}}^{\oplus} m(\pi)\pi \, d\mu(\pi)$$

be a decomposition of a quasi-regular representation γ' into a direct integral of unitary irreducible representations of G', where $m : \widehat{G'} \to \mathbb{Z}_+$ is a multiplicity function and $d\mu$ a Plancherel measure for a symmetric space X. Each irreducible representation $\pi \in \widehat{G'}$ can be extended to an irreducible representation $\pi^\vee$ of S. This gives

$$\gamma \simeq \int_{\widehat{G'}}^{\oplus} m(\pi)\pi^\vee \, d\mu(\pi)$$

and substituting this into (8), we obtain the decomposition of Theorem 0.3

$$\Pi|_{\overline{P}} \simeq \int_{\widehat{G'}}^{\oplus} m(\pi)\Theta(\pi) \, d\mu(\pi), \tag{9}$$

where $\Theta(\pi) = \nu \otimes \mathrm{Ind}_{SN}^{\overline{P}}(\pi^\vee \otimes \chi_\xi)$. Note that representations π present in the formula (9) (i.e., those with $m(\pi) > 0$) are H'-spherical representations of G'.

Mackey theory guarantees that all representations $\Theta(\pi)$ are unitary irreducible representations of $\overline{P}$ and $\Theta(\pi) \simeq \Theta(\sigma)$ if and only if $\pi \simeq \sigma$.

The special case $s = 2$ deserves some special attention. In this situation $\Pi = \pi_p \otimes \pi_q$, where $p = (p^+, p^-)$, $q = (q^+, q^-)$. We will write G'_{pq}, H'_{pq} and X_{pq} for G', H' and X, respectively. The quotient space $X_{pq} = G'_{pq}/H'_{pq}$ is then a reductive symmetric space in the sense of [F]. The table below lists these symmetric spaces for different combinations of G, p and q ($|p|+|q| \leq n$) (see [A, 16.7] for the detailed computations in the case of $G = E_7$).

G	p	q	X_{pq}
$\mathrm{Sp}(2n, \mathbb{R})$	p	q	$O(p^+ + q^+, p^- + q^-)/[O(p^+, p^-) \times O(q^+, q^-)]$
$U(n, n)$	p	q	$U(p^+ + q^+, p^- + q^-)/[U(p^+, p^-) \times U(q^+, q^-)]$
$O^*(4n)$	p	q	$\mathrm{Sp}(p^+ + q^+, p^- + q^-)/[\mathrm{Sp}(p^+, p^-) \times \mathrm{Sp}(q^+, q^-)]$
$O(2, j)$	$(1, 0)$	$(1, 0)$	$SO(j-1)/SO(j-2)$
	$(1, 0)$	$(0, 1)$	$SO_0(1, j-2)/SO(j-2)$
$E_{7(-25)}$	$(1, 0)$	$(1, 0)$	$SO(9)/SO(8)$
	$(1, 0)$	$(0, 1)$	$SO_0(1, 8)/SO(8)$
	$(1, 0)$	$(2, 0)$	$F_{4(-52)}/\mathrm{Spin}(9)$
	$(1, 0)$	$(0, 2)$	$F_{4(-20)}/\mathrm{Spin}(9)$
	$(1, 0)$	$(1, 1)$	$F_{4(-20)}/\mathrm{Spin}(1, 8)$

3. Extending $\Theta(\pi)$ to $\overline{G}$

3.1. The N-spectrum

In this section we study low-rank representations of $\overline{G}$. For the classical groups (cases I1, I2, I3 in our list) a complete theory of low-rank representations can be found in Li's paper [L2]. We rely heavily on the ideas and methods of this paper. Our objective here is to extend the low-rank theory of Li so it can be applied to representations of the groups $O(2, j)$ and $E_{7(-25)}$.

Consider the restriction of the representation $\Theta(\pi) = \nu \otimes \mathrm{Ind}_{SN}^{\overline{P}}(\pi^\vee \otimes \chi_\xi)$ to N. This restriction decomposes into a direct integral of unitary characters, and the decomposition is determined by a projection-valued measure on $\widehat{N} = N^*$. This measure is supported on the set $\mathcal{O} \subset N^*$ (an L-orbit of ξ).

Similarly, for any unitary representation τ of $\overline{G}$, we can consider its restriction to the abelian subgroup N and the associated measure μ_τ on N^*. If μ_τ is supported on the single orbit $\mathcal{O}_r \subset N^*$, we say that τ is of *signature* $r = (r^+, r^-)$ and write

$$\operatorname{sign}_N \tau = r.$$

The number $|r| = r^+ + r^-$ is the *rank* of $\mathcal{O}_r$. If μ_τ is supported on one or several orbits of rank k, we write $\operatorname{rank}_N \tau = k$.

It will be convenient to set

$$\operatorname{sign} t = \begin{cases} (1,0), & t > 0 \\ (0,1), & t < 0. \end{cases}$$

Remark. For the representations of classical groups, the notion of rank was introduced in [Ho1] and [L2]. Our definition extends it to $G = E_{7(-25)}$. For $G = O(2,j)$ the definition above differs from the notion of rank in [L2] due to the different choice of the parabolic subgroup P.

We now take a unitary representation σ of $\overline{G}$ and consider $\sigma|_{\overline{M'}N'}$. The group N' is a Heisenberg group defined by an exact sequence

$$1 \to ZN' \to N' \to \mathbb{R}^{2m} \to 1,$$

and the multiplication on N' defines a standard skew-symmetric bilinear form on $\mathbb{R}^{2m}$. The group M' acts on N' by the automorphisms of N', and it also acts trivially on the center ZN'. Because of this we can view M' as a subgroup of $\operatorname{Sp}(2m, \mathbb{R})$.

Now let ρ_t be a unique representation of N' corresponding in the sense of Stone-von Neumann theorem to the character χ_t of $ZN' \simeq \mathbb{R}$, where

$$\chi_t(z) = \exp(2\pi i t z), \ z \in ZN'.$$

We can extend ρ_t to the representation of the semidirect product $\operatorname{Sp}(2m,\mathbb{R})^{\frown} \cdot N'$ using the corresponding oscillator representation ω_t of the metaplectic group $\operatorname{Sp}(2m,\mathbb{R})^{\frown}$. This extension restricts to a representation of a semidirect product $\overline{M'}N'$, and we denote this restriction by $\widetilde{\rho}_t$.

By the results of [HM] the subspace of ZN'-fixed vectors is invariant under the action of $\sigma(\overline{G})$, and without loss of generality we may assume that σ has no ZN' –fixed vectors. Then, according to the Mackey theory, $\sigma|_{\overline{M'}N'}$ decomposes into representations of the form $\kappa_t \otimes \widetilde{\rho}_t$, where all κ_t, $t \in \mathbb{R}^*$ are unitary representations of $\overline{M'}$. We can write

$$\sigma|_{\overline{G}_-N'} = \int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt.$$

We will now describe the N-spectrum of $\widetilde{\rho}_t$. It is known that a real vector space N is endowed with a structure of a simple formally real Jordan algebra with the unit (denoted by e), and L is the structure group of this Jordan algebra. Then ZN' is a one-dimensional subalgebra of N generated by the primitive idempotent c.

This idempotent determines the Peirce decomposition of N [FK, IV.I]:

$$N = N(c,1) + N(c,1/2) + N(c,0).$$

Observe that $N(c,1) = ZN'$, $N(c,1/2) = \mathbb{R}^m$ and $N(c,0) = N_-$.

Example. Take $G = E_{7(-25)}$. Then

$$N = \mathrm{Herm}(3,\mathbb{O}), \quad c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding Peirce decomposition is

$$N = \mathbb{R}c + \mathbb{O}^2 + \mathrm{Herm}(2,\mathbb{O}).$$

Hence $N(c,1/2) = \mathbb{O}^2 = \mathbb{R}^{16}$ and $N(c,0) = \mathrm{Herm}(2,\mathbb{O}) = R^{1,9}$ and this Jordan algebra is in fact the nilradical N_- of the parabolic subgroup $P_- = [\mathbb{R}^* \times SO(1,9)] \cdot \mathbb{R}^{1,9}$ inside $G_- = SO(2,10)$.

The action of $\widetilde{\rho}_t$ on $N(c,1)$ and $N(c,1/2)$ is easy to describe: these spaces lie inside N' and we can model an irreducible representation ρ_t of N' on the Hilbert space $L^2(N(c,1/2))$. To distinguish between an element x of $N(c,1/2)$ and the corresponding vector from $\mathbb{R}^m$, we shall write $\widehat{x}$ for the latter. Then

$$\begin{aligned} \widetilde{\rho}_t(n_1 c) f(x) &= \chi_t(n_1) f(x) \\ &= \chi_t(\mathrm{tr}_N[n_1 c]) f(x), && n_1 \in \mathbb{R} \\ \widetilde{\rho}_t(n_{1/2}) f(x) &= \chi_t(\widehat{n}_{1/2} \cdot \widehat{x}) f(x) \\ &= \chi_t\Big(\frac{1}{2}\,\mathrm{tr}_N[n_{1/2} x]\Big) f(x), && n_{1/2} \in N(c,1/2). \end{aligned} \tag{10}$$

Here tr_N is the standard trace functional on the Jordan algebra N.

Now take $z_0 \in \mathrm{Sym}(m,\mathbb{R}) \subset \mathrm{Sp}(2m,\mathbb{R})$. The action of the oscillator representation $\omega_t(z_0)$ on $L^2(\mathbb{R}^m)$ is given by the formula

$$\omega_t(z_0) f(x) = \chi_t\Big(\frac{1}{2} \widehat{x} z_0 \widehat{x}^t\Big) f(x).$$

Observe that $\widehat{x} z_0 \widehat{x}^t = \mathrm{tr}_{\mathrm{Sym}(m,\mathbb{R})}[(e-c)x^2 z_0]$.

Recall that $G_- \subset \mathrm{Sp}(2m, \mathbb{R})$ and

$$N(c,0) = N_- = P_- \cap \mathrm{Sym}(m, \mathbb{R}).$$

For $n_0 \in N(c,0)$ and $x \in N(c,1/2)$ we have

$$\mathrm{tr}_{\mathrm{Sym}(m,\mathbb{R})} \left[(e-c)x^2 n_0\right] = \mathrm{tr}_{N_-} \left[(e-c)x^2 n_0\right]$$

and

$$\widetilde{\rho}_t(n_0) f(x) = \chi_t\Big(\frac{1}{2}\,\mathrm{tr}_N \left[(e-c)x^2 n_0\right]\Big) f(x). \tag{11}$$

Combining formulas (10) and (11), we can write the formula for $\widetilde{\rho}_t(n^0)$, where $n^0 = n_1 c + n_{1/2} + 2n_0$.

$$\widetilde{\rho}_t(n^0) f(x) = \chi_t\Big(\,\mathrm{tr}_N \left(\left[c + \frac{1}{2}x + \frac{1}{4}(e-c)x^2\right] n^0\right)\Big) f(x). \tag{12}$$

We can identify N^* and N by setting $\phi(n')(n'') = \mathrm{tr}_N(n'n'')$ for $n, n' \in N$. It follows from the formula (12) that the N-spectrum of $\widetilde{\rho}_t$ is supported on the elements of the form $n_t(x)$, $x \in N(c, 1/2)$, where

$$n_t(x) = t\Big(c + \frac{1}{2}x + \frac{1}{4}(e-c)x^2\Big).$$

For an arbitrary element $x' \in N(c,1/2)$ there exists a special element of the structure group L, called the Frobenius transformation and denoted by $\tau(x')$. According to Lemma VI.3.1 of [FK], $\tau(x')n_t(x) = n'_1 + n'_{1/2} + n'_0$, where

$$\begin{aligned} n'_1 &= tc \\ n'_{1/2} &= t\Big(2x'c + \frac{1}{2}x\Big) \\ n'_0 &= t\Big(2(e-c)x'^2 c + (e-c)x'x + \frac{1}{4}(e-c)x^2\Big)\,. \end{aligned}$$

In particular, $\tau(-x/2)n_t(x) = tc$.

We can now describe the N-spectrum of $\kappa_t \otimes \widetilde{\rho}_t$. If the N_--spectrum of κ_t is supported on a set $\mathcal{O}(\kappa_t)$, then the support of the N-spectrum of $\kappa_t \otimes \widetilde{\rho}_t$ consists of the elements $n_t(x) + n_-$ where $x \in N(c,1/2), n_- \in \mathcal{O}(\kappa_t)$. Then

$$\tau\left(-\frac{x}{2}\right)\left(n_t(x) + n_-\right) = tc + n_-\ .$$

Suppose now that $\mathrm{sign}_N\,\sigma = r$, where $r = (r^+, r^-)$. Then $\mathrm{sign}_N(\kappa_t \otimes \widetilde{\rho}_t) = r$, i.e.,

$$\mathrm{sign}_N(tc + n_-) = r. \tag{13}$$

It is easy to see that (13) implies $\mathrm{sign}\,t + \mathrm{sign}_{N_-}\,n_- = r$, i.e.,

$$\mathrm{sign}_{N_-}\,\kappa_t = \begin{cases} (r^+ - 1, r^-), & t > 0 \\ (r^+, r^- - 1), & t < 0. \end{cases}$$

We summarize this discussion in the following

Lemma 3.1. *Let σ be a representation of $\overline{G}$, $\operatorname{sign}_N \sigma = r$ and $\sigma|_{\overline{G}_-N'} = \int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt$. Then for any $t \in \mathbb{R}^*$ the N_--spectrum of the representation κ_t is supported on a single L_--orbit, and $\operatorname{sign}_{N_-} \kappa t = r - \operatorname{sign} t$.*

3.2. Von Neumann algebras

Let τ be a representation of some subgroup H of $\overline{G}$. By $\mathcal{A}(\tau, H_0)$ we denote the von Neumann algebra generated by the operators $\tau(h), h \in H_0$, where H_0 is a subgroup of H.

To proceed further we need

Lemma 3.2. *Assume that $\mathcal{A}(\kappa_t, \overline{G}_-) = \mathcal{A}(\kappa_t, \overline{P}_-)$ for all $t \in \mathbb{R}^*$. Then*

$$\mathcal{A}\Big(\int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt \, , \overline{G}_- \Big) \subseteq \mathcal{A}\Big(\int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt \, , \overline{P}_- N' \Big).$$

Proof. The representation ρ_t is an irreducible representation of N', therefore $\mathcal{A}(\rho_t, N_-)$ is the full algebra of bounded operators on $L^2(\mathbb{R}^m)$. Consider the algebra $\mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t, \overline{G}_-)$. This algebra is generated by operators

$$\kappa_t(g_-) \otimes \widetilde{\rho}_t(g_-), g_- \in \overline{G}_-. \tag{14}$$

All these operators lie inside $\mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t, \overline{P}_- N')$. Indeed, the algebra $\mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t, \overline{P}_- N')$ contains the set $\mathcal{B}$ of all operators of the form $\kappa_t(p^-) \otimes a$, where a is an arbitrary bounded operator on $L^2(\mathbb{R}^m)$ and $p^- \in \overline{P}_-$. Combining this fact with the assumption $\mathcal{A}(\kappa_t, \overline{G}_-) = \mathcal{A}(\kappa_t, \overline{P}_-)$, we conclude that the von Neumann algebra generated by $\mathcal{B}$ already contains all operators (14).

Hence $\mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t, \overline{G}_-) \subseteq \mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t, \overline{P}_- N')$ and

$$\mathcal{A}\Big(\int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt \, , \overline{G}_- \Big) \subseteq \int_{\mathbb{R}^*}^{\oplus} \mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t \, , \overline{G}_-) \, dt \subseteq \int_{\mathbb{R}^*}^{\oplus} \mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t \, , \overline{P}_- N') \, dt.$$

But the representations $\widetilde{\rho}_t$ are irreducible and nonisomorphic for different t, and

$$\int_{\mathbb{R}^*}^{\oplus} \mathcal{A}(\kappa_t \otimes \widetilde{\rho}_t \, , \overline{P}_- N') \, dt = \mathcal{A}\Big(\int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt \, , \overline{P}_- N' \Big). \qquad \square$$

Observe that $\overline{P}_- N'$ is a subgroup of $\overline{P}$.

The next theorem is an analogue of [L2, 4.3].

Theorem 3.3. *Let* σ *be a representation of* $\overline{G}$, $\operatorname{sign}_N \sigma = r$ *and* $0 < |r| < n$ *(i.e.,* σ *is a low-rank representation of* $\overline{G}$*). Then* $\mathcal{A}(\sigma, \overline{G}) = \mathcal{A}(\sigma, \overline{P})$*.*

Proof. The groups $\overline{G}_-$ and $\overline{P}$ together generate $\overline{G}$, and it suffices to check that

$$\mathcal{A}(\sigma, \overline{G}_-) \subseteq \mathcal{A}(\sigma, \overline{P}). \tag{15}$$

But $\sigma|_{\overline{G}_-\,N'} = \int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt$, and by Lemma 3.2, the assertion (15) follows immediately if we can show that $\mathcal{A}(\kappa_t, \overline{G}_-) = \mathcal{A}(\kappa_t, \overline{P}_-)$ for all $t \in \mathbb{R}^*$. By Lemma 3.1 all κ_t are representations of rank $|r| - 1$ of the group $\overline{G}_-$, and we can apply the same line of reasoning to them.

After $|r|$ steps of this process, we reduce the statement of the theorem to the case of representations of rank 0 for some group $\overline{G}_0$, where G_0 is belongs to one of the families I1–I4. Any representation τ of rank 0 decomposes over characters of $\overline{G}_0$ [HM] and it is well known that any character of $\overline{G_0}$ is determined by its restriction to the Siegel parabolic $\overline{P}_0$ (e.g., [L2, 4.2]. Therefore, $\mathcal{A}(\tau, \overline{G}_0) = \mathcal{A}(\tau, \overline{P}_0)$. □

We now return to the problem of decomposing representation $\Pi = \bigotimes_{i=1}^{s} \pi_{p_i}$. The restriction of this representation on $\overline{P}$ is given by (9), and for any $\Theta(\pi) = \nu \otimes \operatorname{Ind}_{SN}^{\overline{P}}(\pi^\vee \otimes \chi_\xi)$ in the decomposition (9)

$$\operatorname{sign}_N \Theta(\pi) = \operatorname{sign}_N \xi = \sum_{i=1}^{s} p_i \, .$$

Therefore Π can be decomposed over the irreducible representations of $\overline{G}$ of signature $\sum_{i=1}^{s} p_i$.

Assume $\sum_{i=1}^{s} |p_i| < n$. Then by Theorem 3.3, any two non-isomorphic irreducible representation from the spectrum of Π restrict to non-isomorphic irreducible representations of $\overline{P}$. Therefore the $\overline{P}$-decomposition (9) gives rise to a $\overline{G}$-decomposition

$$\Pi \simeq \int_{\widehat{G}'}^{\oplus} m(\pi)\theta(\pi) \, d\mu(\pi), \tag{16}$$

where $\theta(\pi)$ is defined for almost every π (with respect to $d\mu$) as a unique irreducible representation of $\overline{G}$ determined by the condition $\theta(\pi)|_{\overline{P}} = \nu \otimes \operatorname{Ind}_{SN}^{\overline{P}}(\pi^\vee \otimes \chi_\xi)$. Obviously $\theta(\pi) \simeq \theta(\sigma)$ if and only if $\pi \simeq \sigma$.

4. Representations of maximal rank

The statement of Theorem 3.3 is certainly false for the representations of maximal possible rank, i.e., when $\operatorname{sign}_N \sigma = r$ and $|r| = n$. Nevertheless, a $\overline{G}$-decomposition (16) can be constructed even when $\sum_{i=1}^{s} |p_i| = n$.

Consider $\sigma = \sigma_1 \otimes \sigma_2$, where σ_1 and σ_2 are representations of $\overline{G}$, $\operatorname{sign}_N \sigma_1 = r_1 = (r^+, r^-)$, $|r_1| = n-1$ and $\operatorname{sign}_N \sigma_2 = (1,0)$. Then $\sigma_1|_{\overline{M}' N'} = \int_{\mathbb{R}^*}^{\oplus} \kappa_t \otimes \widetilde{\rho}_t \, dt$ and $\sigma_2|_{\overline{M}' N'} = \int_{\mathbb{R}_+^*}^{\oplus} \kappa'_u \otimes \widetilde{\rho}_u \, du$, and

$$\sigma|_{\overline{M}' N'} = \iint_{\mathbb{R}^* \times \mathbb{R}_+^*}^{\oplus} (\kappa_t \otimes \kappa'_u) \otimes (\widetilde{\rho}_t \otimes \widetilde{\rho}_u) \, dt \, du. \tag{17}$$

For $t + u \neq 0$ we have $(\widetilde{\rho}_t \otimes \widetilde{\rho}_u)|_{N'} = \rho_t \otimes \rho_u \simeq 1 \otimes \rho_{t+u}$, where 1 is a trivial representation of N' on $L^2(\mathbb{R}^m)$.

$(\widetilde{\rho}_t \otimes \widetilde{\rho}_u)|_{\overline{M}'} = \omega_t \otimes \omega_u \simeq \omega'_{t,u} \otimes \omega''_{t,u}$ where

$$\omega''_{t,u} = \begin{cases} \omega_+, & t+u > 0 \\ \omega_-, & t+u < 0 \end{cases}$$

and

$$\omega'_{t,u} = \begin{cases} \omega_+, & tu/(t+u) > 0 \\ \omega_-, & tu/(t+u) < 0. \end{cases}$$

Here ω_+ and ω_- are the restrictions of two nonisomorphic oscillator representations of $\mathrm{Sp}(2m, \mathbb{R})^{\sim}$ to $\overline{M}'$.

Then $\widetilde{\rho}_t \otimes \widetilde{\rho}_u \simeq \tau_{t,u} \otimes \widetilde{\rho}_{t+u}$, where $\tau_{t,u}(N')$ acts trivially on $L^2(\mathbb{R}^m)$ and $\tau_{t,u}(\overline{M}')$ acts by $\omega'_{t,u}$. The set $t + u = 0$ has measure 0 in $\mathbb{R}^* \times \mathbb{R}^*$ and after a change of variables $t + u = v$ the decomposition (17) becomes

$$\sigma|_{\overline{M}' N'} = \iint_{\mathcal{D}}^{\oplus} (\kappa_t \otimes \kappa'_{v-t} \otimes \tau_{t,v-t}) \otimes \widetilde{\rho}_v \, dt \, dv,$$

where $\mathcal{D} = \{(t,v) \mid t \neq 0, v \neq 0, \, v > t\}$.

If we set $\lambda_v = \int_{(-\infty,v)}^{\oplus} \kappa_t \otimes \kappa'_{v-t} \otimes \tau_{t,v-t} \, dt$, the preceding formula can be rewritten as

$$\sigma|_{\overline{M}' N'} = \int_{\mathbb{R}^*}^{\oplus} \lambda_v \otimes \widetilde{\rho}_v \, dv. \tag{18}$$

By Lemma 3.1 all representations κ_t have signature $r_1 - \operatorname{sign} t$, and all κ'_{v-t} are of rank 0, i.e., decomposable over characters. Therefore

$$\lambda_v|_{\overline{M}'} = \begin{cases} \kappa_v^- \otimes \omega_+, \operatorname{sign}_{N_-} \kappa_v^- = (r^+, r^- - 1) & \text{if } v < 0 \\ (\kappa_v^- \otimes \omega_-) \oplus (\kappa_v^+ \otimes \omega_+), & \\ \quad \operatorname{sign}_{N_-} \kappa_v^- = (r^+, r^- - 1), & \\ \quad \operatorname{sign}_{N_-} \kappa_v^+ = (r^+ - 1, r^-) & \text{if } v > 0. \end{cases} \tag{19}$$

Remark. If the signature r is semi-definite (i.e. $r = (|r|, 0)$ or $(0, |r|)$), some of the signatures in the formula above will involve negative numbers, which is of course impossible. To simplify notation, we agree that in this case corresponding summands are simply absent from the decomposition (18).

Lemma 4.1. *Let $\sigma = \sigma_1 \otimes \sigma_2$, where σ_1 and σ_2 are representations of $\overline{G} = \overline{O(2,j)}$, $\operatorname{sign}_N \sigma_1 = r$, $|r| = 1$ and $\operatorname{sign}_N \sigma_2 = (1,0)$. Then $\mathcal{A}(\sigma, \overline{G}) = \mathcal{A}(\sigma, \overline{P})$.*

Proof. Consider $P^1 = P_- \times O(j-2)$ – a parabolic subgroup of $M' = SL(2) \times O(j-2)$. It suffices to prove that for all v

$$\mathcal{A}(\lambda_v, \overline{M}') = \mathcal{A}(\lambda_v, \overline{P^1}).$$

Indeed, this fact combined with formula (18) and Lemma 3.2 gives $\mathcal{A}(\sigma, \overline{M}') \subseteq \mathcal{A}(\sigma, \overline{P})$, and the statement of the lemma follows.

Analysis of (19) shows that $\lambda_v|_{\overline{M'}} = \chi_v \otimes \omega$, where χ_v decomposes over characters and ω is an oscillator representation restricted to $\overline{M'}$. Without loss of generality we may take $\omega = \omega_+$. Two factors of M' form a dual reductive pair inside $\mathrm{Sp}(2(j-2), \mathbb{R})$ and the spectrum of ω_+ is very well known: $\omega_+ = \bigoplus_i \eta_1^{(i)} \otimes \eta_2^{(i)}$, where $\eta_1^{(i)}$ and $\eta_2^{(i)}$ are irreducible highest weight representations of $\overline{SL(2)}$ and $O(j-2)$ respectively, and each $\eta_1^{(i)}$ and $\eta_2^{(i)}$ occurs only once in the decomposition. Observe that each $\eta_1^{(i)}|_{\overline{P}_-}$ is irreducible. Therefore $\mathcal{A}(\eta_1^{(i)} \otimes \eta_2^{(i)}, \overline{M}') = \mathcal{A}(\eta_1^{(i)} \otimes \eta_2^{(i)}, \overline{P^1})$, and $\mathcal{A}(\omega, \overline{M}') = \mathcal{A}(\omega, \overline{P^1})$. Similarly, $\mathcal{A}(\chi_v \otimes \omega, \overline{M}') = \mathcal{A}(\chi_v \otimes \omega, \overline{P^1})$. Hence any irreducible component of $\chi_v \otimes \omega$ is irreducible when restricted to $\overline{P^1}$ and uniquely determined by this restriction, and $\mathcal{A}(\chi_v \otimes \omega, \overline{M}') = \mathcal{A}(\chi_v \otimes \omega, \overline{P^1})$. □

Remark. It is easy to see (by inspection of the above argument) that the statement of the lemma remains true if we replace $\sigma = \sigma_1 \otimes \sigma_2$ with $\sigma = \bigoplus_{i=1}^k \sigma_1^{(i)} \otimes \sigma_2^{(i)}$, where $\operatorname{sign}_N \sigma_1^{(i)} = r$ and $\operatorname{sign}_N \sigma_2^{(i)} = (1,0)$, $1 \leq i \leq k$. We can also replace $\overline{G} = \overline{O(2,j)}$ with $\overline{SO(2,j)}$.

Lemma 4.2. *Let $\sigma = \sigma_1 \otimes \sigma_2$, where σ_1 and σ_2 are representations of $G = E_{7(-25)}$, $\operatorname{sign}_N \sigma_1 = r$, $|r| = 2$ and $\operatorname{sign}_N \sigma_2 = (1,0)$. Then $\mathcal{A}(\sigma, G) = \mathcal{A}(\sigma, P)$.*

Proof. In this case $M' = G_- = SO(2,10)$. Once again, it suffices to check that for all $v \in \mathbb{R}^*$

$$\mathcal{A}(\lambda_v, G_-) = \mathcal{A}(\lambda_v, P_-). \tag{20}$$

From (19) we see that λ_v is either a tensor product of two representations of rank 1 (in this case the assertion of (20) follows immediately from Lemma 4.1) or

$$\lambda_v = (\kappa_v^- \otimes \omega_-) \oplus (\kappa_v^+ \otimes \omega_+),$$

where $\operatorname{sign}_{N_-} \kappa_v^- = (1,0)$, $\operatorname{sign}_{N_-} \kappa_v^+ = (0,1)$. This can occur only when $r = (1,1), v > 0$. But $\operatorname{sign}_{N_-} \omega_- = (0,1)$, $\operatorname{sign}_{N_-} \omega_+ = (1,0)$ and we find ourselves in a situation described in the remark to Lemma 4.1.

Therefore, (20) holds for all v. □

Corollary 4.3. *Let* $\sigma = \pi_{p_1} \otimes \pi_{p_2} \otimes \pi_{p_3}$ *be a representation of* $G = E_{7(-25)}$*,* $|p_1| = 1$*,* $|p_2| = 1$*,* $p_3 = (1,0)$*. Then* $\mathcal{A}(\sigma, G) = \mathcal{A}(\sigma, P)$*.*

Proof. Set $\sigma_1 = \pi_{p_1} \otimes \pi_{p_2}$, $\sigma_2 = \pi_{p_3}$. Then $\operatorname{sign}_N \sigma_1 = p_1 + p_2$, $\operatorname{sign}_N \sigma_2 = (1,0)$ and the lemma above can be applied. □

We now return to our study of the tensor product $\Pi = \bigotimes_{i=1}^{s} \pi_{p_i}$, $\sum_{i=1}^{s} |p_i| \leq n$.

Theorem 4.4. $\mathcal{A}(\Pi, \overline{G}) = \mathcal{A}(\Pi, \overline{P})$.

Proof. If $\sum_{i=1}^{s} |p_i| < n$, the statement of this theorem follows from Theorem 3.3. Hence we can restrict our attention to the case $\sum_{i=1}^{s} |p_i| = n$.

If $\overline{G} = \overline{O(2,j)}$, the only possible case is $s = 2$, and we may always assume $p_2 = (1,0)$ and apply Lemma 4.1. Similarly, for $\overline{G} = E_{7(-25)}$ we can take $p_s = (1,0)$, and the theorem follows from Lemma 4.2 for $s = 2$ and Corollary 4.3 for $s = 3$.

Finally, in the classical cases (I1–I3) the statement follows immediately from [L2, 4.7–4.8]. Indeed, for these groups each of the representations π_{p_i} appears in the Howe duality correspondence for an appropriate stable range dual pair (G'_i, G) and all irreducible representations from the spectrum of Π appear in the duality correspondence for the pair (G', G), which is still in the stable range. Therefore any irreducible representation from the spectrum of Π is irreducible when restricted to $\overline{P}$ and uniquely determined by this restriction, and $\mathcal{A}(\Pi, \overline{G}) = \mathcal{A}(\Pi, \overline{P})$. □

Therefore the $\overline{P}$-decomposition (9) gives rise to a $\overline{G}$-decomposition of Π with respect to the same measure $d\mu$ and multiplicity function $m(\pi)$

$$\Pi \simeq \int_{\widehat{G}'}^{\oplus} m(\pi) \theta(\pi) \, d\mu(\pi). \tag{21}$$

Comparing (9) and (21) we see that $\theta(\pi)$ is a unitary irreducible representation of $\overline{G}$ which can be defined (for almost every π with respect to $d\mu$) as a unique irreducible representation from the spectrum of Π satisfying the condition $\theta(\pi)|_{\overline{P}} = \Theta(\pi)$, where $\Theta(\pi) = \nu \otimes \operatorname{Ind}_{SN}^{\overline{P}}(\pi^\vee \otimes \chi_\xi)$.

Theorem 0.4 is thus proved.

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