CONTEMPORARY MATHEMATICS

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September 23–26, 2003
ICMS, Edinburgh, United Kingdom

Vadim B. Kuznetsov
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This volume contains the proceedings of the "Workshop on Jack, Hall-Littlewood and Macdonald polynomials" held from September 23–26, 2003, at ICMS, Edinburgh, United Kingdom, as well as some material of historical significance including previously unpublished texts.

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Preface

1. Historical perspective

The subject of symmetric functions arose initially in connection with the representation theory of the symmetric group, however it has since found wide applicability. In the last twenty years or so, there have far-reaching new developments in the subject, as well as a general broadening of the areas of applicability, especially within combinatorics, classical analysis and mathematical physics.

The subject has a particularly distinguished history going back to the work of C. G. Jacobi [**Ja**] in the mid-nineteenth century, and to the papers of F. G. Frobenius [**F**], I. Schur [**S**], H. Weyl [**W**], M. A. MacMahon [**M**], and A. Young [**Y**] in the early twentieth century. These papers singled out a certain family of symmetric polynomials, now called Schur functions, which played a significant role in the representation theory of the symmetric group S_n as well as the complex general linear group $GL_n(\mathbb{C})$. This dual role of the Schur functions is often referred to as "Schur-Weyl duality".

The next stage in the development of the subject was the fundamental work of P. Hall [**H**] and D. E. Littlewood [**L**] who independently discovered a one-parameter generalization of the Schur polynomials. Subsequent work by J. A. Green [**G**] and I. G. Macdonald [**M1**] showed that these polynomials, now called the Hall-Littlewood polynomials, play a crucial role in the representation theory of GL_n over finite and p-adic fields.

In the late 1960's, Henry Jack [**J1, J2**] discovered a totally different one-parameter generalization of Schur functions. These polynomials, now called Jack polynomials, include as a special case the zonal polynomials, which are related to the group $GL_n(F)$ with $F = \mathbb{R}$, and had been previously studied by A. T. James [**J**] in connection with multivariate statistics.

In the 1980's, I. G. Macdonald unified these developments by introducing a two-parameter family of symmetric polynomials, now called Macdonald polynomials. The Hall-Littlewood polynomials are a special case of Macdonald polynomials, and arise by specializing one of the parameters to 0. The Jack polynomials too arise as a limiting case when both parameters approach 1 — the Jack parameter is the limiting direction of approach. These polynomials were also independently discovered by K. Kadell, [**Kad**] in connection with his investigation of the Selberg integral.

As explained above, the Macdonald symmetric polynomials are closely related to the group GL_n and hence to root systems of type A. In subsequent work Macdonald constructed analogous polynomials associated to arbitrary root system. These polynomials arise as the 'discrete spectrum' of a class of q-difference operators.

Since the operators are self-adjoint with respect to a certain scalar product, Macdonald polynomials are multivariate orthogonal polynomials. From this point of view, they generalize various classical orthogonal polynomials.

These root system polynomials are connected with earlier work of Macdonald on spherical functions for p-adic groups, which in the present context are obtained by specializing various parameters to 0. On the other hand taking a suitable limit as the parameters approach 1 one obtains the multivariate Jacobi polynomials that had been previously studied by G. Heckman and E. Opdam [**HO**], and which in turn generalize the characters and spherical functions of the corresponding compact Lie groups. Thus Macdonald's results can be seen as a manifestation of Harish-Chandra's "Lefschetz principle". This principle, which was one of the guiding philosophies of Harish-Chandra's work, asserts that representation theoretic results for an algebraic group over a field should have analogues for the same group over other fields. In a certain sense Macdonald polynomials "see" the representation theory of the group $G(F)$ for "every field F".

2. Macdonald Conjectures

Many of the basic properties of Macdonald polynomials were initially formulated as conjectures by Macdonald. These include the constant term formula, the norm formula, the duality/symmetry property. A great deal of research in recent years has been focused on proving these conjectures.

For the Jacobi limit these conjectures were proved by E. Opdam [**Op1**] by the technique of shift operators. Subsequently, I. Cherednik [**C1**], [**C2**] proved the Macdonald conjectures for all reduced root systems. Cherednik's approach involved his theory of double affine Hecke algebras which is one of the major developments in this area. In the non-reduced BC_n-case, Macdonald polynomials are known as Koornwinder polynomials [**K1**], and they can be viewed as the multivariate analogue of the celebrated Askey-Wilson polynomials. In this case the Macdonald conjectures were proved by S. Sahi in [**S3**] following earlier work of J.F. van Diejen [**vD**]. Macdonald's latest book [**M3**] gives an exposition of all these results.

Another set of conjectures was formulated by Macdonald in the type A setting, see [**M2**]. These conjectures are known as the "integrality" and "positivity" conjectures, and are concerned with the expansion of these polynomials in terms of other bases of symmetric functions, e.g. the monomial basis. Macdonald made separate conjectures for Jack polynomials and for symmetric Macdonald polynomials. It has recently been discovered that in the case of Jack polynomials, Jack himself had conjectured some of these properties in an unpublished manuscript [**J3**] shortly before his death. In the case of Jack polynomials, both conjectures were proved by F. Knop and S. Sahi in [**KnS2**]. The "integrality" conjecture for Macdonald polynomials was established in six different papers which appeared roughly at the same time.

The positivity conjecture for Macdonald polynomials proved to be much harder. Garsia and Haiman [**GH**] generalized this to a conjecture for the dimension of a certain doubly-graded S_n-modules, which came to be known as the $n!$ conjecture. In [**H1**] M. Haiman established a spectacular connection between Macdonald polynomials and the geometry of the Hilbert scheme of points in the plane, following a suggestion of C. Procesi. This enabled Haiman to prove the $n!$ conjecture, as well as the related $(n+1)^{n-1}$ conjecture on the dimension of the space of diagonal harmonics [**H2**].

3. Variants of Macdonald polynomials

As explained above, Macdonald polynomials generalize characters of compact group and, like these characters, they are symmetric (*i.e.* invariant with respect to the Weyl group action). It was therefore somewhat surprising when the study of these symmetric polynomials gave rise to a natural family of *non-symmetric* polynomials.

These polynomials were first introduced in the Jacobi setting by E. Opdam [**Op2**], who credits the definition to G. Heckman. In turn, Heckman was motivated by the work of Cherednik who had expressed the Macdonald operators as symmetric polynomials in certain commuting first order operators. These Cherednik operators are trigonometric analogs of operators first considered by C. Dunkl [**Du**]. The nonsymmetric Macdonald polynomials are defined to be the simultaneous eigenfunctions of these Cherednik operators.

The discovery of the non-symmetric polynomials led to substantial simplifications in the theory of Macdonald polynomials. This was crucial in the proof of the integrality and positivity conjectures for Jack polynomials in [**KnS2**]. Generalizing the ideas in that paper to arbitrary root system, Cherednik [**C3**] formalized the theory of intertwiners and used them to give alternate proofs of some of the Macdonald conjectures. Although the non-symmetric polynomials are very useful and natural in the Macdonald theory, they remain somewhat mysterious. For certain special values of the parameters they have been identified with Demazure characters of basic representations of affine Kac-Moody Lie algebras by Y. Sanderson [**San**] for type A, and by B. Ion [**I**] for arbitrary root systems. However for general parameters their representation-theoretic meaning is still obscure.

Another class of polynomials which turned out to be closely connected to Macdonald polynomials are the so-called *interpolation* polynomials. These polynomials were first defined by S. Sahi [**S4**], in connection with joint work with B. Kostant on the Capelli identity. They are symmetric inhomogeneous polynomials, depending on several parameters, and defined by fairly simple vanishing properties. In the special case when the parameters form an arithmetic progression, F. Knop and S. Sahi proved in [**KnS1**] that the top degree terms of the interpolation polynomial is the usual Jack polynomial. A similar result also holds for Macdonald polynomials [**S1, Kn1**].

Many results for Jack and Macdonald polynomials, both symmetric and non-symmetric, continue to hold for the interpolation polynomials. Indeed some of the results are easier to prove in the inhomogeneous setting because of the strong uniqueness result for these polynomials. Results for the homogeneous polynomials can then be deduced by considering the top homogeneous terms. Considerable work on these polynomials was done by A. Okounkov who obtained combinatorial and integral formulas for these polynomials, and also defined their analogs in the BC_n setting. It turns out that special values of interpolation polynomials are the coefficients in the series expansion of the Jack polynomial about the point $x=(1,\dots,1)$ [**OO1**]. Analogous results are true for symmetric and non-symmetric Macdonald polynomials, and in the BC_n setting one obtains a multivariable analog of the hypergeometric series representing the Askey-Wilson polynomials [**O1, O2, S2, Kn2**].

4. Other directions

There are several areas of mathematics where Macdonald polynomials make a natural appearance. To give a complete and historically accurate description of these areas would require a much longer article, and considerably more expertise than we possess. We shall be content here with a brief mention of some of the themes and some of the key names in those areas. Hopefully this will help the interested reader to track down further results and interconnections.

Macdonald polynomials appear in the context of the exactly solvable quantum Calogero-Sutherland model [**Su**] and its generalizations by Olshanetsky-Perelomov [**OP**], Ruijsenaars [**Ru**] and others. This field is closely related to the study of an ideal gas by Haldane [**Ha**] and Shastry [**Sh**]. Considerable work in this area has been carried out by T. Baker and P. Forrester [**BF**].

Another circle of ideas involving Macdonald polynomials centers around the theory of vertex operator algebras, W-algebras, and conformal blocks. We refer the reader to papers by Frenkel and Reshetikhin [**FR**].

The theory of symmetric functions and representations of the symmetric group plays a big role in algebraic combinatorics. We refer the reader to papers by Lascoux, Leclerc and Thibon on the subject [**LLT, LT**].

Jack and Macdonald polynomials are also intimately connected with the study of random phenomena on the symmetric group, such as random partitions and random permutations. We refer the reader to various papers by Vershik-Kerov and Okounkov-Olshanski [**KOO**].

The subject of harmonic analysis on the affine Hecke algebra has been advanced considerably by the work of E. Opdam [**Op2**]. We also refer the reader to papers by I. Cherednik and J. Stokman in this area.

5. About these proceedings

The first part of these proceedings consists of material of historical significance, including some previously unpublished texts. We include here biographical notes on Jack by B. Sleeman and on Hall, Littlewood and Macdonald by A. Morris. We also include reprints of the original papers of Littlewood and Jack, and notes on Hall's (unpublished) results by I. Macdonald. Finally we print, in its entirety, a recently discovered manuscript of Jack together with comments by I. Macdonald.

The second part of these proceedings consists of original contributions to the subject in the form of refereed research papers. For the reader's convenience we briefly describe the mathematical background for some of these papers. As before, the purpose is to give the interested reader an opportunity to follow up on some of the ideas mentioned in the papers. We lack the space and the expertise to provide a complete and historically accurate exposition of the various subjects.

In 1974 T.H. Koornwinder wrote a series of papers [**K2**] dedicated to the orthogonal symmetric polynomials of type A_2 and BC_2. He constructed several shift operators and derived explicit series representations for these polynomials in two variables. These results were generalized by E. Opdam. In [**KN**] A.M. Kirillov and M. Noumi obtained explicit parameter preserving lowering and raising operators for Macdonald polynomials of the type A_n, thereby generalizing the previous results for Jack polynomials due to L. Lapointe and L. Vinet.

Using Heckman-Opdam's [**HO**] theory of multivariate hypergeometric functions, O. Chalykh, K. Styrkas and A. Veselov [**VSC**] proved that the quantum

Calogero-Sutherland model is algebraically integrable for integer values of the parameters. Generalization of this result to Macdonald operators is due to P. Etingof and K. Styrkas [**ES**]. A. Sergeev discovered the relation of the super Jack polynomials introduced in [**KOO**] with the deformed quantum Calogero-Moser systems and Lie superalgebras [**Ser**]. Further generalizations including the difference case and super Macdonald polynomials have been investigated by A. Sergeev and A. Veselov (to appear in this volume) who have shown that these polynomials are the joint eigenfunctions of certain difference operators on algebraic varieties.

P. Etingof and A.A. Kirillov, Jr. have shown in [**EK1**], [**EK2**] how Macdonald polynomials for the root system of type A_n could be interpreted in terms of the representation theory of quantum groups. Namely, Macdonald polynomials arise as traces of certain natural intertwining operators, which generalizes the description of Schur functions as traces of irreducible SL_n-modules. This leads, in particular, to elegant proofs of various Macdonald polynomials identities, such as inner product and symmetry identities. It also, in the affine case, leads to natural elliptic extensions of Macdonald theory.

Asymptotic properties of Macdonald polynomials were investigated by G. Olshanski, in collaboration with S. Kerov and A. Okounkov. In particular, the analog of the Vershik-Kerov asymptotics for the characters of the symmetric and unitary groups for the case of Jack polynomials were obtained in [**KOO**] and [**OO2**], respectively. Remarkably, the same type of asymptotics continues to hold, with minimal and very natural modifications. Recently, J.F. van Diejen suggested a general approach to deriving asymptotics of a class of multivariate orthogonal polynomials as the degree tends to infinity and applied it to Jack polynomials.

Another connection between interpolation and Macdonald polynomials arose recently in the work of T. Miwa and his collaborators. In [**FJMM1**] and [**FJMM2**] they showed, that certain ideals in the algebra of symmetric functions which are of interests in the representation theory of affine Lie algebras have a linear basis of Macdonald polynomials.

V.B. Kuznetsov, V.V. Mangazeev and E.K. Sklyanin have recently completed the long-standing task of factorizing Jack polynomials [**KS, KMS**] by advancing the theories of separation of variables and Bäcklund transformations for quantum integrable systems.

The French group based mainly in Marne-la-Vallee have over the years made considerable contributions to algebraic combinatorics in general and to Macdonald polynomials [**LLT**], [**LT**] in particular.

Elementary proofs of Macdonald conjectures are by now available for the classical root systems, see [**M2**] for the A_n-case and [**R**] for the general BC_n-case. In recent work (math.QA/0309252) E. Rains constructed a family of elliptic biorthogonal functions generalizing the Koornwinder polynomials.

R. Gustafson has discovered a method of evaluating many important hypergeometric integrals [**Gus**] which are intimately connected to Jack and Macdonald polynomials.

In the joint work with M. Lassalle [**LS**], M. Schlosser recently presented an explicit analytic formula for Macdonald polynomials. This was obtained from a recursion for Macdonald polynomials being derived from inverting the Pieri formula. M. Lassalle gave an elementary proof of the expansion formula for Macdonald polynomials in terms of 'modified complete' symmetric functions.

E. Langmann generalized the method used by Sutherland in [**Su**] to derive new explicit formulas for the Jack polynomials. The method is based on the relation of the Jack polynomials to the eigenfunctions of the quantum Calogero-Sutherland system. The results were further generalized to construct a solution of the elliptic Calogero-Moser system.

V.P. Spiridonov generalized Warnaars elliptic extension of a Macdonald multiparameter summation formula to Riemann surfaces of arbitrary genus.

6. The Workshop

The Workshop on "Jack, Hall-Littlewood and Macdonald polynomials" was held at ICMS, Edinburgh, during September 23–26, 2003. The meeting was organised by V.B. Kuznetsov (Leeds), A.O. Morris (Aberystwyth), B.D. Sleeman (Leeds) and A.P. Veselov (Loughborough) and supported by EPSRC and LMS. The Scientific Advisory Committee was A. Okounkov (Princeton) and J.-Y. Thibon (Marne-la-Vallee). 16 one-hour-long lectures were given by: J.F. van Diejen (Talca, Chile), P.I. Etingof (MIT), R. Gustafson (Texas A&M), F. Knop (Rutgers), T.H. Koornwinder (Amsterdam), A. Lascoux (Marne-la-Vallee), I.G. Macdonald (UK), T. Miwa (Kyoto), E. Opdam (Amsterdam), E. Rains (Davis), S. Sahi (Rutgers), M. Schlosser (Vienna), A.N. Sergeev (Balakovo), E.K. Sklyanin (York), V. Spiridonov (Dubna) and J.-Y. Thibon (Marne-la-Vallee). The meeting was attended by 35-40 participants.

Vadim B. Kuznetsov
Siddhartha Sahi

This volume, meant to be a celebration of the work of the pioneers of the theory of symmetric functions, has unfortunately also turned into a memorial for Vadim Kuznetsov, whose untimely death in December 2005 shocked and saddened all who knew him. Vadim invested a great deal of time and effort into the conference and its proceedings, and I would like to think that he would have been pleased with the results. I also want to add a special note of thanks to Alun Morris and Brian Sleeman for their invaluable help, without which this volume would have been greatly delayed.

Siddhartha Sahi
July 2006

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- First letter from Henry Jack to G. de B. Robinson [16.4.76]; Note added by Prof. W.N. Everitt, Department of Mathematics, University of Dundee on 5 Sept. 1980; Second letter from Henry Jack to G. de B. Robinson [4 May 76]; Reply from G. de B. Robinson to the two letters by Henry Jack [June 28, 1976]; Letter from Professor W.N. Everitt to G. de B. Robinson [14 October 1980].
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2C. Asymptotics of denominators in binomial formula.

We fix an arbitrary partition μ and let n go to infinity. As in Theorem 1.4, we assume that the parameters a, b may depend on n. We write them as a_n, b_n and assume that the limits (1.7) exist.

PROPOSITION 2.4. *The denominator* (2.8) *in* (2.7) *has the following asymptotics*

$$C(n,\mu;\theta;a_n,b_n) \sim \frac{H(\mu;\theta)}{H'(\mu;\theta)}\, 4^{|\mu|}\,\theta^{|\mu|}\,(\theta+\bar{a})^{|\mu|}\cdot n^{2|\mu|}\,, \tag{2.9}$$

where

$$H(\mu;\theta)=\prod_{(i,j)\in\mu}((\mu_i-j)-\theta(\mu'_j-i)+1), \qquad H'(\mu;\theta)=\prod_{(i,j)\in\mu}((\mu_i-j)-\theta(\mu'_j-i)+\theta)\,,$$

and $\bar{a}=\lim a_n/n$ *as in* (1.7).

PROOF. Recall that $C(n,\mu;\theta;a_n,b_n)$ is the product of two terms, $I_\mu(\mu;\theta;\sigma_n+\theta n)$ and $\mathfrak{J}_\mu(1,\dots,1;\theta,a_n,b_n)$, where $\sigma_n=(a_n+b_n+1)/2$. We claim that, as $n\to\infty$, the following two asymptotic relations hold

$$I_\mu(\mu;\theta;\sigma_n+\theta n) \sim H(\mu;\theta)\,2^{|\mu|}\,(\theta+\bar{\sigma})^{|\mu|}n^{|\mu|}\,, \tag{2.10}$$

$$\mathfrak{J}_\mu(\underbrace{1,\dots,1}_{n};\theta,a_n,b_n) \sim \frac{1}{H'(\mu;\theta)}\,2^{|\mu|}\left(\frac{\theta+\bar{a}}{\theta+\bar{\sigma}}\right)^{|\mu|}\theta^{|\mu|}\,n^{|\mu|}\,, \tag{2.11}$$

where

$$\bar{\sigma}=\lim_{n\to\infty}\frac{\sigma_n}{n}=\frac{\bar{a}+\bar{b}}{2}\,.$$

Clearly, (2.10) and (2.11) imply (2.9).

The first relation immediately follows from (2.6), let us check the second relation.

The following is the general formula, due to Opdam, for the value of a multivariate Jacobi polynomial, indexed by a weight μ, at the unit element, see [**HS**], Part I, Theorem 3.6.6,

$$\prod_{\alpha>0}\frac{\Gamma\left((\mu+\rho,\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\alpha/2}\right)}{\Gamma\left((\mu+\rho,\alpha^\vee)+\frac{1}{2}k_{\alpha/2}\right)}\,\frac{\Gamma\left((\rho,\alpha^\vee)+\frac{1}{2}k_{\alpha/2}\right)}{\Gamma\left((\rho,\alpha^\vee)+k_\alpha+\frac{1}{2}k_{\alpha/2}\right)}\,,$$

where $\alpha^\vee$ stands for the root dual to α, and $k_{\alpha/2}=0$ if the root $\alpha/2$ does not exist.

In our case, the polynomial in question is just $\mathfrak{J}_\mu(\,\cdot\,;\theta,a_n,b_n)$, and the unit element is identified with the point $(1,\dots,1)$. Next, we have

$$\rho=((n-1)\theta+\sigma_n,\dots,\theta+\sigma_n,\sigma_n)$$

and there are 4 types of the positive roots α

$$\varepsilon_i-\varepsilon_j\,,\quad \varepsilon_i+\varepsilon_j\quad (1\le i<j\le n),\qquad \varepsilon_i\,,\quad 2\varepsilon_i\quad (1\le i\le n)$$