Introductions to Mathematical Reasoning–lecture notes–Fall 2016

1 Introduction: Thinking and communicating about mathematics [1]

This course is aimed at preparing students to use, do, and communicate mathematics at the level of a beginning professional mathematician.

If you have no intention of being a professional mathematician, you will still benefit from developing these skills You will be better able to learn difficult technical subjects, to think in ways that are both abstract and rigorous, to formulate convincing arguments, and to explain difficult ideas so that others can understand them. [2] These are the skills that an aspiring mathematician seeks to master.

These notes are addressed to you, the aspiring mathematician.

What skills does a mathematician need?

Most of you have studied mathematics for more than 12 years. You've learned has been basic mathematical vocabulary, such as *whole number*, *rational number*, *polynomial*, *function*, *quadrilateral*, and *derivative*, and you've learned basic algorithms for solving specific mathematical problems such as *adding fractions*, *long division*, *the quadratic formula*, *finding the maximum value of a function*.

The vocabulary and algorithms are an important part of learning mathematics but they represent only a small part of the arsenal of thinking and communication skills needed by a mathematician, which include:

1. **Thinking in the *abstract mathematical universe*.** To a mathematician, mathematics is not as a collection of problem solving methods. It is a fascinating *abstract universe*, populated by *mathematical objects* of many different types. This mathematical universe exhibits amazing patterns and regularity, and is full of beautiful and surprising connections. The familiar (and not so familiar) problem solving techniques of mathematics emerge as byproducts of the structure of the mathematical universe.

2. **Understanding the classification of mathematical objects.** Objects come in different **types**, such as *real numbers*, *4 by 3 matrices whose entries are real numbers*, and *functions that take as input a list of three real numbers and output a single real number*. The type of an object determines how you can manipulate it. If f is a function that takes as input an ordered pair of real numbers then you (normally) can't give it as input a single real number. Objects may have more than one type, for example, the number 7 is a real number, and it is also an integer. Each different area of mathematics focuses on different

[1] Version 9/4/2016.

[2] For example, the best scores on the LSAT (the standardized test required for law school applicants) are earned by students who majored in Mathematics or Physics.

types of mathematical objects, for example, **linear algebra** is the study of *vector spaces* and *linear transformaitons*, **group theory** is the study of *groups*, and **topology** is the study of *topological spaces*.

3. **Working within the strict rules of the mathematical universe**. The mathematical universe operates according to precise rules, and every mathematician must play within those rules. These rules are not arbitrary, they are based on a thorough understanding of logic, and their purpose is to prevent a mathematician from falsely claiming incorrect statements about the mathematical universe.

4. **Mathematical freedom and creativity.** Mathematical objects can be manipulated in any way that conforms to the rules of that object. Furthermore, a mathematician may create new objects, and new types of objects as long as these obey the general rules of the mathematical universe. This flexibility gives mathematicians a vast scope for intellectual creativity. Remarkably, the answers to questions about one type of mathematical object may involve making connections to seemingly unrelated objects, and even inventing new objects.

5. **Using mathematical definitions properly.** In any field, when you introduce new terminology you have to define it so that others know what the terminology means. In mathematics, this requirement is very strict. A proper mathematical definition must make clear to the (properly trained) reader exactly what is meant with no ambiguity or room for disagreement. Some terms in mathematics (such as matrix, prime number, ring) have standard definitions that almost every mathematician agrees to. Other terms are not standardized, and may be used to mean different things by different writers, or by the same writer in different contexts. The mathematician knows that whenever he uses terminology without a generally agreed meaning he must provide a proper definition.

6. **Using different representations.** The same mathematical idea can be viewed in the mathematical universe in very different ways. A circle can be thought of as a geometric shape in the plane obtained by taking all points at some fixed distance from a specified center point, and can also be thought of as the solution set of the equation $x^2 + y^2 = 1$. The mathematician recognizes that different viewpoints are useful in different situations, and is adept at switching between different viewpoints.

7. **Applying the universal principles of mathematics.** The mathematical universe exhibits amazing patterns and structure, and this structure can be expressed as *universal principles* that hold for all objects of a particular type. For example, we have the Pythagorean theorem: For every right triangle, the square of the length of the hypotenuse must equal the sum of the squares of the two legs. Over thousands of years, mathematicians have discovered many such principles. These universal principles provide the toolkit for solving mathematical problems. The proper application of these principles is the main activity of *applied mathematics*. The skilled mathematician is familiar with known universal principles of many areas of mathematics, and is able to recognize which principles are applicable in different situations, and is able to correctly apply these principles.

8. **Searching for new universal principles.** While mathematicians already know many universal principles of mathematics, *new universal principles of mathematics are established every day*! Establishing a principle involves two key activities: *conjecture* (making educated guesses about possible new universal principles) and *proof* (establishing that a conjectured universal principle is indeed true.) This process of conjecture and proof, is the main activity of *pure mathematicians* and are the central skills of a mathematician.

9. **Extending known universal principles to more general situations, or to analogous situations.** Given a universal principle of mathematics, the mathematician asks himself: is there an even more general principle that applies to more situations? Or, are there different, but analogous situations where the universal principle does not apply, but where one can discover a new universal principle that is analogous to the original principle?

10. **Building examples to help discover and test conjectures, and to aid in the understanding of mathematical ideas.** When a mathematician is presented with a new type of mathematical problem, or a new mathematical definition, a universal principle, or a conjectured universal principle she asks herself: What are specific examples where this problem, or this definition, or this principle, or this conjecture, comes up? How are these examples similar and how are they different? The observed similarities are the basis of new conjectured universal principles. Having guessed a possible new universal principle, the mathematician builds additional examples to test the principle, and makes these examples as varied as possible.

11. **Verifying that conjectures are true: Constructing proofs** Claims about the mathematical universe, especially claims about general principles, must be proved. While general principles are discovered and understood by looking at examples, examples are not enough to prove a general principle: just because a general principle works on some test cases, there is no guarantee that it will work in all cases. The language of mathematics includes a precise notion of what it means to prove a claimed mathematical principle. The ability to construct such proofs and communicate them properly is a fundamental skill, perhaps the main skill to be learned in this course.

12. **Decomposing a problem.** In a particular complicated mathematical situation where we have a goal (such as solve a problem, or prove a universal principle), reaching this goal often requires breakng the problem into a series of simpler steps. The mathematician recognizes the need to decompose the problem, and knows how to do this decomposition so that the resulting problems are simpler than the original.

13. **Recognizing which details are relevant and which are not.** In a given situation, the mathematician is often dealing with many objects, and many properties of these objects. For a given goal, only some of the given informatoin is relevant and the mathematician is able to sort through the given information to focus on the relevant information. In cases where the goal breaks up into a series of steps, the mathematician realizes that the information relevant for accomplishing each step often differs from step to step.

14. **Explaining mathematical ideas clearly and precisely orally and in writing.** Mathematics is expressed in a language that consists of ordinary language (for us, this is English) together with special vocabulary and symbols. This vocabulary has a very precise meaning. When used properly every other mathematician should be able to understand exactly what you mean, and should agree with the correctness of your assertions. A well-written piece of mathematical writing should not just state facts, but should provide enough explanation that the reader can follow along with what is being written.

15. **Active reading of mathematics.** Serious technical material in mathematics requires active involvement. A mathematician typcially reads with pencil and paper in hand to construct examples, draw pictures, make note of confusing points, etc. He stops his reading frequently to test his understanding by constructing examples of the concepts being discussed. He/she is constantly looking for ways to connect what is currently being presented to previously learned concepts.

16. **Curiousity**. When a mathematician observes a mathematical pattern, such as "the sum of the first k odd numbers seems to equal k^2" he wants to know: does this always work? If so, why? Are there other similar patterns to be discovered?

17. **Thinking skeptically and critically.** A mathematician learns to question everything in mathematics and is constantly asking: Is this mathematical claim true? Does the general principle being claimed work on all examples? Does the new mathematical definition being made make sense? Is the proof that is offered correct? Does the written explanation conform to the requirements of good mathematical communication, so that any knowledgeable reader can clearly understand what is being communicated?

This is quite a long list of skills. Each of these skills is itself a complex combination of other skills, and can't be described or learned by some simple step-by-step formula. Professional mathematicians continue to develop and refine these skills throughout their entire career. While you may have been exposed to some of these skills in earlier courses, the central focus of this course is for you to understand and develop these skills, to provide you with a solid foundation for future work in mathematics.

This list of skills does not mention any particular topics in mathematics, because they are needed in all branches of mathematics. This course will (of course) teach some mathematical content, because you can't learn the skills of a mathematician without actually doing mathematics. The mathematical content covered will deal with concepts that appear in many areas of mathematics, and thus forms an important part of the mathematician's foundation of basic mathematical knowledge.

Being a novice mathematician

Any complex skill, playing the piano, rock climbing, playing basketball, being a neurosurgeon, or being a mathematician takes a long time to learn. Beginner's are taught basic rules that will start them on the road to mastering the skill, and they work hard to apply these rules.

It's not surprising then that novices are sometimes puzzled and frustrated to see that the rules that they are required to follow are seemingly ignored by experts.

If you take a rock climbing class, then among the first and most important things taught are safety rules. These rules serve a very important purpose: keeping the beginning rock climber from bodily harm. A beginner may see an experienced climber violate some of the rules. "Hey", says the novice, "why do I need to take this tedious precaution when the experts don't?"

The answer: Because an expert has developed the ability to recognize situations in which a safety rule is not needed. Most beginners lack such judgement, and may be unable to distinguish between a situation where a particular safety precaution is unnecessary, and a situation where it is critically important. So the beginning climber is advised to follow all of the rules all of the time until he has more experience.

In this course, you will be taught rules, about manipulating mathematical objects, about using mathematical definitions, about using variables, and about writing proofs. These rules are intended to protect the novice mathematician from mathematical disaster, namely, *making a mathematical declaration that is mathematically nonsensical*, or *claiming that you have proved something that is false.* The rules you will be taught in this course will protect you from these disasters. Experts recognize that some of these rules are sometimes unnecessary, and they take shortcuts around these rules. As in rock-climbing, judgement and experience are required to know when you can safely ignore a rule. Most novices lack this judgement. As a novice, you should therefore strive to follow the rules.

In some cases, abuse of of a particular rule has become so commonplace that this abuse has become a part of standard mathematical practice. This is a problem for us, because we want to learn standard mathematical practice, but we also don't want to start abusing rules before we're ready. Our preference is to avoid abusing the rules, and so occasionally our notation and recommended practices will disagree with standard practice among mathematicians. When this happens we'll point out the discrepancy in a remark in the text.