10 Binary relations and binary relationships [11]

10.1 Binary relationships

In everyday usage, we talk about relationships between two people or two objects such as "Bob and Jane are classmates", "Antoine is the father of Katelyn", or "The key is inside the mailbox". Here each of the phrases "are classmates" or "is the father of" or "is inside of" specifies a relationship that holds between the individuals or objects mentioned in the sentence.

In the mathematical universe, a relationship that may hold between two mathematical objects is called a *binary relationship*.

Remark 10.1. The word *binary* is an adjective that means "pertaining to the number two". A binary relationship involves *two objects*. A ternary relationship involves three objects, and a k-ary relationship involves k objects. We will be focusing on binary relationships in this chapter.

Many students may be familiar with the binary system for writing numbers, which is a system that uses *two symbols* (usually 0 and 1) to represent all nonnegative integers. Except for the fact that they each "pertain to the number two", binary relationships have nothing to do with the binary system for writing numbers.

Example 10.1. Some binary relationships that might hold between two real numbers x and y:

x **is less than** y**.** This is abbreviated $x < y$. Similarly we have the binary relationships $x \leq y$, $x \geq y$ and $x > y$.

y **is the square of** x**.** This is abbreviated by $y = x^2$.

x **is within** ε **of** y**.** Here $\varepsilon \geq 0$ and this relationship means $|x - y| \leq \varepsilon$.

y **is the image of** x **under** f**,** where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is some fixed function. This relationship is abbreviated $y = f(x)$. Note that the relationship "y is the square of x" is a special case of this one.

x **and** y **satisfy a specified equation or inequality.** Any equation or inequality involving x and y, such as "$|x - y| \leq 1$", "$x^2 + y^3 = 1$" or "$(x + y)^2 \leq x^3$" defines a relationship between x and y.

Example 10.2. Some binary relationships that might hold between two integers m and n:

m **is the successor of** n**.** This means that $m = n + 1$.

m **is a divisor of** n**.** This is abbreviated $m|n$, and means that n/m is an integer.

m **is coprime to** n**.** This means that m and n have no common divisor greater than 1.

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m **and** n **differ by a multiple of** k**.** Here k is an integer, and we abbreviate this relationship by writing $m \equiv_k n$, and say that *m is congruent to n modulo k* or *m is equal to n mod k*.

Example 10.3. Some binary relationships that might hold between two sets A and B

A **is a subset of** B**.** This is abbreviated $A \subseteq B$

A **is disjoint from** B . This means $A \cap B = \emptyset$.

A **and** B **have the same size.** This is abbrevated $|A| = |B|$.

Example 10.4. Some binary relationships that might hold between two lists: k and ℓ:

k **is a rearrangement of** ℓ**.** This means that every object that appears in either k or ℓ, appears as an entry of both lists the same number of times. For example $(1, 1, 1, 2, 2, 3)$ is a rearrangement of $1, 2, 3, 1, 2, 1)$.

k **and** ℓ **have the same set of entries.** We can form a set from any list consisting of the objects that appear as entries in the list. For example $(1, 1, 4, 2, 1, 2)$ has entry set $\{1, 2, 4\}$. Thus $(1, 1, 4, 2, 1, 2)$ and $(4, 4, 2, 2, 1, 1)$ have the same set of entries.

k **is a prefix of** ℓ**.** This means that there is a list k' such that $\ell = k * k'$, where $*$ is the concatentation operation.

Example 10.5. *Binary relationships between two different types of objects.* In the previous examples, the binary relationships considered are between pairs of objects of the same type. A binary relationship can be between pairs of objects of *different types*:

a **is an entry of** L**.** For a list L of integers and an integer a, "a is an entry of L" means that a appears somewhere on the list L.

b **is a root of** f**.** For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $b \in \mathbb{R}$ we say that *b is a root of f* provided that $f(b) = 0$.

z **is a lower bound of** A**.** If A is a subset of $\mathbb{R}$ and $z \in \mathbb{R}$ we say that z is a lower bound of A provided that for all $x \in A$, $z \leq x$.

Technically, a binary relationship on the pair of sets S,T is an indefinite assertion $A(x, y)$ involving a pair of objects $(x, y) \in S \times T$. We say that the binary relationship is a *binary relationship between S and T*. In the case $S = T$, it is a *binary relationship on S*. For example the relationship "coprime to" introduced above is a binary relationship on $\mathbb{Z}$, and the "is a root of" relationship is relationship between $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}$.

10.2 Binary relations

Next we introduce the term *binary relation*. This terminology sounds quite similar to binary relationship, but they mean different things. Nevertheless, we'll see that binary relationships and binary relations are closely connected.

Definition 10.1. A binary relation R is a mathematical object that is a list of three sets:

- A set called the *source* of R, denoted $\mathbf{Source}(R)$
- A set called the *target* of R, denoted $\mathbf{Target}(R)$
- A set $\mathbf{pairs}(R)$ of ordered pairs that is a subset of $\mathbf{Source}(R) \times \mathbf{Target}(R)$.

If R is a binary relation with source S and target T we say that R is a *binary relation from S to T*. We also might say that R is a binary relation *between S and T*. If the source and target are the same set S, we say that *R is a binary relation on S*.

Remark 10.2. While $\mathbf{Source}(R)$ and $\mathbf{Target}(R)$ are a necessary part of the definition of R, the most important part of R is the set $\mathbf{pairs}(R)$. In fact, it's so important that mathematicians often simply abbreviate the set $\mathbf{pairs}(R)$ by R.

This can be dangerous, since this gives two different meanings to R. The first meaning is the list $(S, T, \mathbf{pairs}(R))$ where S is the source and T is the target of R. The second meaning R might have is just as the set $\mathbf{pairs}(R)$.

This does not cause a problem for the experienced mathematician, who is able to figure out which is meant from context. But for beginners, its better to use the notation R to mean the list $(\mathbf{Source}(R), \mathbf{Target}(R), \mathbf{pairs}(R))$ and use $\mathbf{pairs}(R)$ for the set of ordered pairs.

Remark 10.3. Consider the example of the relation R on $\mathbb{Z}$ consisting of the pairs (i, j) such that $|j - i| \leq 1$ then it is correct to say that $(3, 4)$ is a *pair of the relation* R, but it is not correct to call $(3, 4)$ a *relation*.

Advanced remark 10.4. There are other kinds of relations other than binary relations. For any positive integer k, a k-ary relation consists of a sequence $S_1, S_2, \ldots, S_k$ of sets and a set of lists $(s_1, \ldots, s_k)$ where for each i, $s_i \in S_i$. Binary relations are the same as 2-ary relations. Since binary relations are the most common ones used in mathematics, mathematicians often omit the word binary and use the word "relation" when they mean "binary relation". We will not do that here.

So a binary relationship on S, T is an indefinite assertion involving an object of type S and an object of type T, while a binary relation on S, T consists of S and T and a subset of $S \times T$. These are different things, but there is a close correspondence between binary relationships and binary relations.

Given a binary relationship $A(s, t)$ on S,T, we have an associated binary relation R on S, T whose set of pairs is just the set of all pairs $(s, t) \in S \times T$ that satisfy the relationship. Thus $\mathbf{pairs}(R) = \{(s, t) \in S \times T : A(s, t)\}$.

Example 10.6. Suppose S is the set $\{1,2,3\}$ and T is the set of subsets of $\{4,5,6\}$ and suppose the relatiionship for $s \in S$ and $t \in T$ is $t - s \leq 3$. Then the associated relation has $\mathbf{pairs}(R) = \{(1,4),(2,4),(2,5),(3,4),(3,5),(3,6)\}$.

On the other hand, given a binary relation R on $S \times T$, we have an associated binary relationship on $S \times T$, which is the relationship "$(x, y) \in \mathbf{pairs}(R)$."

Summarizing we have:

> Every binary relationship defined on $S \times T$ is associated to a unique relation on $S \times T$ consisting of all pairs $(s,t) \in S \times T$ that satisfy the relationship, and every binary relation $R \subseteq S \times T$ is associated back to the relationship specified by the requirement "$(s,t) \in \mathbf{pairs}(R)$".

Remark 10.5. Mathematicians usually don't distinguish between a "binary relation" and the corresponding "binary relationship", and simply use the word "binary relation" for both. In most books, when the author refers to the "$<$ relation" for real numbers, he or she might mean the "$<$" *relationship* or the *relation* $\{(x,y) \in \mathbb{R}^2 : x \leq y\}$. It is up to the reader to use the context to figure out which is meant.

For now, we use the different words "relationship" and "relation" because they represent two different (though connected) concepts. Later we will adopt the standard convention of using the word "relation" for both.

Equality of relations. Two relations R and Q are equal if $\mathbf{Source}(R) = \mathbf{Source}(Q)$, $\mathbf{Target}(R) = \mathbf{Target}(Q)$ and $\mathbf{pairs}(R) = \mathbf{pairs}(Q)$

Remark 10.6. In practice, it is almost always clear from the definitions of R and Q whether they have the same source and target, and if they have the same source and target then when we prove $Q = R$ we only have to prove $\mathbf{pairs}(Q) = \mathbf{pairs}(R)$.

Notation for the associated relationship. For binary relation R, the notation xRy is used to abbreviate the indefinite assertion "(x, y) is an ordered pair of the relation R". We write $x\not{R}y$ if $(x,y) \notin R$. Sometimes we introduce a special symbol such as $\sim$ and write $x \sim y$ instead of xRy. Other symbols commonly introduced to denote a particular relationship are $\equiv$ and $\preceq$.

Reverse relations If R is a relation on $S \times T$, the *reverse relation* $\overleftarrow{R}$ is the relation on $T \times S$ with pair set $\{(t,s) : (s,t) \in \mathbf{pairs}(R)\}$.

Image and reverse image Suppose R is a relation from S to T. For $x \in S$, we define the *image of* x *under* R, denoted by $R(x)$, to be the set $\{t \in T : sRt\}$. For $X \subseteq S$ we define the image of X under R, denoted $R(X)$, to be the set $\{t \in T : \exists x \in X \text{ such that } xRt\}$ which is equal to $\bigcup_{x \in X} R(x)$. Notice that when we use the notation $R(z)$, z might be a member of S or a subset of S and it's important to be clear about which is meant.

For $y \in T$, we define the *reverse image of* y *under* R, denoted $\overleftarrow{R}(y)$ to be the set $\{s \in S : sRt\}$. For $Y \subseteq T$ we define the *reverse image of* Y *under* R, denoted $\overleftarrow{R}(Y)$, to be the set $\{s \in S : \exists y \in Y$ such that $sRy\}$ which is equal to $\bigcup_{y \in Y} \overleftarrow{R}(y)$.

Operations on Relations For relation Q, the *complementary relation* Q^c has $\mathbf{Source}(Q^c) = \mathbf{Source}(Q)$ and $\mathbf{Target}(Q^c) = \mathbf{Target}(Q)$ and $\mathbf{pairs}(Q^c) = \mathbf{Source}(Q) \times \mathbf{Target}(Q) \backslash Q$.

For relations Q and R, $Q \cup R$ and $Q \cap R$ are defined by:

- $\mathbf{Source}(Q \cup R) = \mathbf{Source}(Q) \cup \mathbf{Source}(R)$, $\mathbf{Target}(Q \cup R) = \mathbf{Target}(Q) \cup \mathbf{Target}(R)$ and $\mathbf{pairs}(Q \cup R) = \mathbf{pairs}(Q) \cup \mathbf{pairs}(R)$.
- $\mathbf{Source}(Q \cap R) = \mathbf{Source}(Q) \cap \mathbf{Source}(R)$, $\mathbf{Target}(Q \cap R) = \mathbf{Target}(Q) \cap \mathbf{Target}(R)$ and $\mathbf{pairs}(Q \cap R) = \mathbf{pairs}(Q) \cap \mathbf{pairs}(R)$.

The *difference of* Q *and* R, $Q \backslash R$ has the same source and target as Q, and $\mathbf{pairs}(Q \backslash R) = \mathbf{pairs}(Q) \backslash \mathbf{pairs}(R)$.

Viewing functions as relations If $f : S \longrightarrow T$ is a function, then we can define the relationship $y = f(x)$ with source S and target T. The associated relation R_f has source S, target T and $\mathbf{pairs}(R_f) = \{(x, y) \in S \times T : y = f(x)\}$.

Advanced remark 10.7. The point of view we take here is that the function f, and the associated relation R_f are different, but closely related, objects. In many mathematics books, the function f and relation R_f are viewed as being the same object. In fact a function is often *defined* to be a relation with the property that all of the pairs have different first coordinates. We will not use this definition here, but readers should be aware of it when reading other books.

10.3 Binary relations and relationships between two objects from the same set

When the source and target of a relation R are the same set A, we say that the R is a relation on A, and the corresponding relationship is defined on A. Examples 10.1, 10.2, 10.3, and 10.4 are relationships that might hold between two objects coming from the same set.

One of the simplest binary relationships on a set A is the *equality* relationship where $x = y$ means that x and y are the same object. This is associated to the *equaltiy relation* (sometimes called the *diagonal relation*) on A, denoted E_A. $\mathbf{pairs}(E_A) = \{(x, x) : x \in A\}$.

We classify binary relations on a set (and their corresponding relationships) according to certain basic properties.

Definition 10.2. For a set A and relation R on A, we say that R is

reflexive provided that for all $a \in A$, aRa.

antireflexive provided that for all $a \in A$, $\neg(aRa)$.

symmetric provided that for any $a, b \in A$, aRb implies bRa.

antisymmetric provided that for any $a, b \in A$ if $a \neq b$ and aRb then $b \not R a$. (This is logically equivalent to saying that if aRb and bRa then $a = b$. (Exercise: Check this logical equivalence!) This alternate form is often more useful when proving that a relation in antisymmetric.)

transitive provided that for any $a, b, c, \in A$, if aRb and bRc then aRc.

full provided that for any $a, b \in R$ with $a \neq b$, aRb or bRa. (This is not standard terminology.)

Exercise 10.1. For all of the examples of relations that come from the relationships described in Examples 10.2, 10.3, and 10.4, check which of the above properties hold.

Exercise 10.2.

1. Give an example of a relation that is neither reflexive, nor anti-reflexive. (Thus reflexive and anti-reflexive are not opposites).
2. Give an example of a relation that is neither symmetric, nor anti-symmetric. (Thus symmetric and anti-symmetric are not opposites).

Exercise 10.3. Suppose R is a transitive relation.

1. Prove that for any x and y if xRy then $R(y) \subseteq R(x)$.
2. Suppose that x and y are elements such that $R(y) \subseteq R(x)$. Does this necessarily imply that xRy?

Exercise 10.4.

1. Prove that for any relation R on A, R is reflexive if and only if $\mathbf{pairs}(E_A) \subseteq \mathbf{pairs}(R)$.
2. Formulate and prove a similar statement that says when R is anti-reflexive.

Exercise 10.5.

1. Prove that a relation R on A is symmetric if and only if $R = \overleftarrow{R}$.
2. Formulate and prove a similar statement that says when R is anti-symmetric. (Be careful, this is not quite as straightforward as it may seem.)

There are two types of relations that arise very frequently in mathematics called *partial order relations* and *equivalence relations*. In the next two subsections we discuss these in detail.

10.4 Partial order relations

A partial order relationship on a set is a way of comparing members where we say that one members x is, in some sense, "smaller than or equal to" another. The most familiar example is numbers where "smaller than or equal to" means "$\leq$". Here are some other examples:

Example 10.7. $\mathcal{P}(U)$ (the set of subsets of a universe U) with the relationship $\subseteq$

The set of positive integers with the relationship "is a divisor of". Recall that this relationship is denoted $m|n$.

The set of lists of real numbers with the relationship "is a prefix of".

In each of these cases, the words "is smaller than or equal to" has a different meaning, but they all are reflexive, anti-symmetric and transitive.

Exercise 10.6.

Prove that the each of the relationships in Example 10.7 satisfy the reflexive, anti-symmetric and transitive properties.

With these examples in mind, we make the following definition

Definition 10.3. A relation R on X is a *partial order relation* provided that it is *anti-symmetric* and *transitive*, and *reflexive*. If R is a partial order relation, then we also say that the associated relationship is a partial order.

Exercise 10.7. Consider the following relations on $\mathcal{P}_{\mathbf{fin}}(\mathbb{Z})$

1. The relation consisting of pairs (X, Y) such that $|X| = |Y|$.
2. The relation consisting of pairs (X, Y) such that $|X| \leq |Y|$.
3. The relation consisting of pairs (X, Y) such that either $X = Y$ or $Y \backslash X$ is nonempty and every member of $Y \backslash X$ is greater than every member of X.

For each of these relations decide whether it is a partial order or not, and provide a proof of your answer.

Notation for partial orders When we are discussing a partial order relation R on a set A we often use the notation $x \leq_R y$ to mean xRy, and $x <_R y$ to mean $(xRy) \wedge (x \neq y)$. The symbols $\leq_R$ and $<_R$ emphasize that R is a partial order. Furthermore, if the partial order relation R is clear from context we may omit the subscript "R" and write simply $x \leq y$ and $x < y$. We can do this provided that that we are careful that there is no possibility that the reader will be confuse this with the usual meaning of $\leq$ for numbers.

If R is a partial order on X, then two members x and y of X are said to be *comparable* if $x \leq_R y$ or $y \leq_R x$ and said to be *incomparable* if neither $x \leq_R y$ nor $y \leq_R x$.

Definition 10.4. A partial order on A with the property that any two members of A are comparable is called a *total order* or *linear order* on A.

For example, the $\leq$ relation on $\mathbb{R}$ is a total order.

Exercise 10.8. Show that the partial order relations described in Example 10.7 other then $\mathbb{R}$ with the usual $\leq$ relationship are not total orders.

Our definition of a partial order R on X requires that the relation be reflexive, which means that xRx for all $x \in X$. We could also consider relations that are transitive, anti-symmetric and *anti-reflexive*. Such a relation is called a *strict partial order*.

For example, strict inequality "<" on $\mathbb{R}$ and strict containment $\subset$ on sets are both strict partial orders.

There is a natural correspondence between strict partial orders and partial orders

Proposition 10.1. *For every set A:*

1. *For any partial order relation R on A, we have $\boldsymbol{pairs}(E_A) \subseteq \boldsymbol{pairs}(R)$ and the relation R^- on A with $\boldsymbol{pairs}(R^-) = \boldsymbol{pairs}(R) \backslash \boldsymbol{pairs}(E_A)$ is a strict partial order. (Recall that E_A is the diagonal relation on A with pairs (x, x) for $x \in A$).*

2. *For any strict partial order relation Q on A, we have $\boldsymbol{pairs}(E_A)$ is disjoint from $\boldsymbol{pairs}(Q)$ and the relation Q^+ on A with $\boldsymbol{pairs}(Q^+) = \boldsymbol{pairs}(Q) \cup \boldsymbol{pairs}(E_A)$*

Proof. We prove the first part, and leave the second part as an exercise.

Suppose A is an arbitrary set and R is an arbitrary partial order on A. We first show that $\mathbf{pairs}(E_A) \subseteq \mathbf{pairs}(R)$. Each pair in $\mathbf{pairs}(E_A)$ has the form (a, a) where $a \in A$. Suppose a is an arbitrary member of A, then since R is reflexive $(a, a) \in R$ and so $\mathbf{pairs}(E_A) \subseteq \mathbf{pairs}(R)$.

Now let R^- be the relation on A with $\mathbf{pairs}(R^-) = \mathbf{pairs}(R) \backslash \mathbf{pairs}(E_A)$. We must show that R^- is a strict partial order, so we must show that R^- is anti-reflexive, anti-symmetric and transitive.

Proof that R^- is anti-reflexive. Suppose $a \in A$ is arbitrary. We must show $(a, a) \notin \mathbf{pairs}(P^-)$. We have $(a, a) \in E_A$ and by the definition of $\mathbf{pairs}(P^-)$, $(a, a) \notin \mathbf{pairs}(P^-)$.

Proof that R^- is anti-symmetric. Suppose x and y are arbitrary members of A we have to prove that xR^-y and $x \neq y$ implies $y \not{R}^- x$. We'll actually prove something stronger: ifxR^-y then $y \not{R}^- x$. (**Question**: *Why is it enough to prove this?*) Assume for contradiction that xR^-y and yR^-x. Since $\mathbf{pairs}(R^-) \subseteq \mathbf{pairs}(R)$ we have xRy and yRx and since R is anti-symmetric, $x = y$. But then xR^-x which is impossible by the definition of R^-.

Proof that R^- is transitive. Suppose x,y,z belong to A. Assume xR^-y and yR^-z. We must show that xR^-z. Since xR^-y and yR^-z and also $\mathbf{pairs}(R^-) \subseteq \mathbf{pairs}(R)$, we have xRy and yRz and therefore xRz. We need to show that xR^-z. Suppose for contradiction that $(x, z) \notin \mathbf{pairs}(R^-)$. Since $(x, z) \in \mathbf{pairs}(R)$ it must be that $(x, z) \in \mathbf{pairs}(E_A)$ which implies $x = z$. But then yR^-z implies yR^-x and since also xR^-y and R^- is anti-symmteric we have $x = y = z$. But then xR^-x which contradicts that R^- is anti-reflexive. Therefore xR^-z as required to prove that R^- is transitive. □

Exercise 10.9. Prove the second part of Proposition 10.1.

Introducing a partial order When we prove universal principles about partially order relations, we'll need to introduce an arbitrary partial order relation. We have to be a little careful about this because a partial order relation involves both a set A and the relation R, and also we may want to introduce a special symbol such as $\preceq$ to represent the less than or equal relation in the relation.

If A is a specific set (such as the set of integers), or A is an unspecified set that was previously introduced into the scenario, we might write one of the following:

- "Suppose R is a partial order on A."
- "Suppose R is a partial order on A with relation symbol $\prec$. " Here besides introduing R, we are also saying that the notation $x \prec y$ will represent that $(x, y) \in R$. (The particular choice of "$\prec$" is arbitrary; we are free to choose another symbol as long as we tell the reader.)

Often, the set A and the set R are introduced at the same time. We might write:

- "Suppose A is an arbitrary set and suppose R is a partial order relation on A with relation symbol $\prec$."
- "Suppose R is a partial order on A with relation symbol $\prec$. "

In the first introduction, we first introduce A and then introduce R to be a partial order on A. In the second shorter introduction, the set A is introduced by implication. This is similar to when we introduce a function by saying: "Suppose $f : A \longrightarrow B$ is an arbitrary function" which may serve to introduce the two sets A and B and the function f. The interpretation of this sentence depends on whether A was already active. If it was already active, then we are simply introducing R. If A was not already active we are introducing both R and A.

There is yet another form for introducing a partially ordered set. We might just say

Suppose A is an arbitrary partially ordered set with relation symbol $\preceq$.

The term *partially ordered set* means a set that has a partial order relation associated to it. Here we introduced the set A, and introduced the partial order relation on A, but rather than give a name to the relation, we announce that the notation $x \preceq y$ will denote that (x, y) belongs to the partial order relation.

We might shorten this further and say simply:

Suppose A is an arbitrary partially ordered set.

When we use this form, we don't specify the name of the relation or the relationship symbol. In this case, when we want to say that (x, y) is a member of the relation, we write $x \leq_A y$. We can omit the subscript A and write $x \leq y$ provided that there is no risk in confusing this relationship with another partial order.

Some special subsets of a partially ordered sets. Throughout this section A denotes a fixed partially ordered set. We use the symbols $\leq$, $<$, $\geq$ and $>$ to denote the relation symbol for A.

Previously we discussed intervals of real numbers, such as $[1, 3)$ and $(-4, \infty)$. These concepts can be extended to define subsets of any partially ordered set.

Definition 10.5. Suppose $x, y \in A$.

- $[x, y]_A$ denotes the set of elements z such that $x \leq_A z$ and $z \leq_A y$.
- $(x, y]_A$ denotes set of elements z such that $x <_A y$ and $y \leq_A z$.
- $[x, y)_A$ denotes the set of elements z such that $x \leq_A y$ and $y <_A z$.
- $(x, y)_A$ denotes the set of elements z such that $x <_A y$ and $y <_A z$.
- $[x, \uparrow)_A$ denotes the set of z such that $x \leq_A z$.
- $(x, \uparrow)_A$ denotes the set of z such that $x <_A z$.
- $(\downarrow, x]_A$ denotes the set of z such that $z \leq_A x$.
- $(\downarrow, x)_A$ denotes the set of z such that $z < Ax$

If the partially ordered set is clear from context, we omit the subscripts and write simply $[x, y]$ for $[x, y]_A$. A subset of A of any of these forms is called an *interval* of A. We also consider A itself to be an interval of A. The first four types of intervals are referred to as *bounded intervals*. The fifth and sixth intervals are called *up-intervals* and the seventh and eighth are called *down intervals*.

The $\uparrow$ and $\downarrow$ notation is not standard.

Proposition 10.2. *For $x, y \in A$:*

1. $[x, x] = \{x\}$ *and* $[x, x) = (x, x] = (x, x) = \emptyset$.
2. *If x is not less than or equal to y then* $[x, y] = [x, y) = (x, y] = (x, y) = \emptyset$.

Exercise 10.10. Prove Proposition 10.2.

Here are some examples of intervals in some basic posets.

Example 10.8. The real numbers with the usual $\leq$ order. For real numbers x, y, $[x, y]$, $[x, y)$, etc. have their usual meanings. The set $[x, \uparrow)$ is the real interval $[x, \infty)$ and $(\downarrow, x)$ is the interval (∞, x).

Integers with the usual $\leq$ order. In this case $[x, y]$ for $x \leq y$ is the set $\{x, x+1, \ldots, y\}$ consisting of consecutive integers starting from x and ending at y. For the partially ordered set all of the intervals (x, y), $[x+1, y)$, $(x, y-1]$ and $[x+1, y-1]$ are the same; which is not true in the case of real number intervals.

Positive integers ordered by divisiblity. In this case, for positive integers x and y, the interval $[x, y]$ consists of numbers that are both a multiple of x and a divisor of y. For example $[6, 60]$ is the set $\{6, 12, 30, 60\}$.

Subsets of a universe U with the containment order. In this case, for subsets A and B of U, the interval $[A, B]$ consists of those sets that contain A and are contained in B. For example $[\{1, 3\}, \{1, 2, 3, 4\}]$ is the set $\{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.

Minumum versus Minimal; Maximum versus Maximal When we deal with subsets of a partially ordered set, we sometimes want to talk about the "least" or "greatest" member of the set. However, this does not always make sense. For general partial orders, we need to make some careful definitions.

Definition 10.6. Suppose X is a subset of the partially ordered set A.

minimal element A member z of X is a *minimal element* of X provided that there are no members x of X such that $x < z$.

minimum element A member z of X is a *minimum element* of X provided that for all $x \in X$, $z \leq x$.

maximal element A member z of X is a *maximal element* of X provided that there are no members x of X such that $x > z$.

maximum element A member z of X is a *maximum element* of X provided that for all $x \in X$, $z \geq x$.

A minimum element of a set X plays the role of the "least" element; it is less than or equal to every other member of X. Similarly a maximum element of X plays the role of the "greatest" element.

A minimal element is somewhat different. When we say that x is minimal in X we are not saying that x is less than or equal every other member of X; we are only saying that X does not contain any members less than x. Similarly when we say that x is maximal in X we are saying that X does not contain any member greater than x.

The terms *minimal element* and *minimum element*, and the terms *maximal element* and *maximum element*, are easy to confuse. The concepts are closely related, but they are not the same. Let's look at some examples.

Example 10.9.

1. Consider the partially ordered set $\mathcal{P}(\{1,2,3,4\})$ with the $\subseteq$ order. Let $X = \{\{1,2\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}$. In this set $\{1,2\}$ and $\{2,3,4\}$ are both minimal. There is no minimum element. The element $\{1,2,3,4\}$ is both a maximal element and a maximum element.

2. Consider the set $\mathbb{R}$ with the usual order. For the set $[1,3)$, 1 is both a minimal element and a minimum element. There are no maximal elements or maximum elements.

3. Consider the set of integers under the divisibility order. In the subset P of all prime numbers, every member is both minimal and maximal.

Here are some important basic facts about minimal, minimum, maximal and maximum elements.

Proposition 10.3. *For any partially ordered set A and for any subset X of A we have:*

1. *A has at most one maximum element and at most one minimum element.*

2. *If A has a maximum element then that element is maximal in A, and there are no other maximal elements.*

3. *If A has a minimum element then that element is minimal in A, and there are no other minimal elements.*

Exercise 10.11. Prove Proposition 10.3.

Exercise 10.12. Show that the following statement is false: For any partially ordered set A, subset S of A and $x \in S$, if x is the unique minimal element of S, then x is a minimum for S.

Remark 10.8. By the proposition, a subset X of a partially ordered set has at most one maximum and at most one minimum. If X has a maximum it is common to refer to it as "*the* maximum member of X" rather than "*a* maximum member of X". However, when talking about a maximal member of X we refer to it as "*a* maximal member of X" rather than "*the* maximal member of X" since there may be more than one of them.

To summarize, a subset X of A may have no minimal elements, exactly one minimal element, a finite number of minimal elements, or infinitely many minimal elements, and the situation is similar for maximal elements. However, X may have at most one minimum element and at most one maximum element. A minimum element must be minimal, and a maximum element must be maximal, but a minimal element is not necessarily a minimum and a maximal element is not necessarily a maximum.

In the special case that the partially ordered is a total ordered set, then the notions of minimal and minimum are the same, and the notions of maximum and maximal are the same:

Proposition 10.4. *For any totally ordered set A and for any subset X of A:*

1. *A member x of X is a minimal element of X if and only if it is the minimum element of X.*

2. *A member x of X is a maximal element of X if and only if it is the maximum element of X.*

Exercise 10.13. Prove Proposition 10.4

Exercise 10.14. Consider the set of positive integers under the divisibility order $m|n$ if m is a divisor of n. Consider the subset consisting of all powers of 2, together with all odd prime numbers. What are the minimal members? What are the maximal members?

Upper and lower bounds The concept of upper and lower bound is a familiar one from calculus. If S is a subset of real numbers, we say that the number b is an upper bound on S provided that b is greater than or equal to every member of S, and b is a lower bound on S provided that b is less than or equal to every member of S.

We can extend this definition to any partially ordered set.

Definition 10.7. Suppose A is a partially ordered set and $X \subseteq A$.

lower bound for X A *lower bound for* X is a member of A that is less than or equal to every member of X.

- We say that X is *bounded below* by z if z is a lower bound for X.
- We write $LB(X)$ for the set of lower bounds of X.
- We say that X is *bounded below* if $LB(X) \neq \emptyset$.

upper bound for X An *upper bound for* X is a member of A that is greater than or equal to every member of X. (Note: an upper bound for X does not have to be a member of X.)

- We say that X is *bounded above* by z if z is an upper bound for X.
- We write $UB(X)$ for the set of upper bounds of X.
- We say that X is *bounded above* if $UB(X) \neq \emptyset$.

Example 10.10. 1. Consider the set $\mathbb{R}$ with the usual order. For the subset $Z = \{-100, 0, 100, 200, 300\}$, every number less than or equal to -100 is a lower bound and every number greater than or equal to 300 is an upper bound. Thus $LB(Z) = (-\infty, -100]$ and $UB(Z) = [300, \infty)$. Also -100 is a minimum and also is minimal and 300 is a maximum and is maximal. For the open interval $(0, 1]$, every number less than or equal to 0 is a lower bound and any number greater than or equal to 1 is an upper bound. The set has no minimum or minimal element; the number 1 is both a maximal element and a maximum of the set.

2. Consider the set $\mathcal{P}(\mathbb{Z})$ of all subsets of the set of integers with the containment order. For the subset $\{\{1, 3\}, \{1, 3, 5\}, \{1, 3, 6, 7\}\}$, any subset of $\{1, 3\}$ is a lower bound and any superset of $\{1, 3, 5, 6, 7\}$ is an upper bound. The set $\{1, 3\}$ is a minimum for the subset and also minimal. The sets $\{1, 3, 5\}$ and $\{1, 3, 6, 7\}$ are maximal elements of the subset but the subset has no maximum.

Proposition 10.5. *Suppose A is a partially ordered set, suppose X is a subset of A and suppose $x \in X$.*

1. *The element x is a minimum for X if and only if $x \in X$ and x is a lower bound for S.*

2. *The element x is a maximum for S if and only if $x \in X$ and x is an upper bound for S.*

Exercise 10.15. Prove Proposition 10.5.

Theorem 10.6. *Let S be a set and consider the powerset $\mathcal{P}(S)$ as a partially ordered set under the inclusion order. Then for any $\mathcal{B} \subseteq \mathcal{P}(S)$:*

$$\begin{aligned} UB(\mathcal{B}) &= [\bigcup_{B \in \mathcal{B}} B, \uparrow). \\ LB(\mathcal{B}) &= (\downarrow, \bigcap_{B \in \mathcal{B}} B]. \end{aligned}$$

Proof. For the first equality, we first show $UB(\mathcal{B}) \subseteq [\bigcup_{B\in\mathcal{B}} B, \uparrow)$. Suppose Z is an upper bound on $\mathcal{B}$. We must show $Z \in [\bigcup_{B\in\mathcal{B}} B, \uparrow)$, which means we must show $\bigcup_{B\in\mathcal{B}} B \subseteq Z$. Suppose z is an arbitrary member of $\bigcup_{B\in\mathcal{B}} B$ We must show $z \in Z$. Since $z \in \bigcup_{B\in\mathcal{B}} B$, there is a member C of $\mathcal{B}$ such that $z \in C$. Since Z is an upper bound for $\mathcal{B}$ and $C \in \mathcal{B}$ we have $C \subseteq Z$ and so $z \in C$ implies $z \in Z$. Thus we've shown $UB(\mathcal{B}) \subseteq [\bigcup_{B\in\mathcal{B}} B, \uparrow)$.

Now suppose Z is an arbitrary member of $[\bigcup_{B\in\mathcal{B}} B, \uparrow)$, which implies $\bigcup_{B\in\mathcal{B}} \subseteq Z$. We must show Z is an upper bound of $\mathcal{B}$. Suppose C is an arbitrary member of $\mathcal{B}$; we must show $C \subseteq Z$. Since $C \in \mathcal{B}$, $C \subseteq \bigcup_{B\in\mathcal{B}} B$ and this is a subset of Z as required.

The proof of the second equation is left as an exercise. □

Exercise 10.16. Prove the second equation in Theorem 10.6.

Selecting a minimal or maximal element In many mathematical proofs we encounter the following situation. We have a partially ordered set A and a subset X of A. As part of the proof, it would be useful to select a minimal element of A or a maximal element of A, or even better, a minimum element of A or a maximum element of A. The problem is that we can't always to this. If X is the set of real numbers and $A = (0, 1)$ then the instruction "Let m be the least member of A" or "Let m be a minimal member of A" is invalid because A has no minimal member.

So it is important for us to have conditions that allow us to select a minimal, minimum, maximal or maximum member of X.

The first such condition pertains to subsets of the integers.

Axiom 10.7. (Well ordering principle for the integers) For any subset X of $\mathbb{Z}$:

1. If X has a lower bound, then X has a minimum element.

2. If X has an upper bound then X has a maximum element.

This is one of the basic properties of the integers, and we take it as an *axiom* which means that mathematicians except this principle without proof.

Example 10.11. The well-ordering axiom is not true of the set of real numbers. The subset $(0, 1)$ is both bounded above and below, but has no minimum or maximum member.

Every set of nonnegative integers has a lower bound, namely 0, so Axiom 10.7 implies:

Corollary 10.8. *(Well-ordering principle for nonnegative integers) Every nonempty subset of nonnegative integers has a minimum element.*

The above principle applies only to subsets of the integers. Here's a principle that applies to any partially ordered set.

Axiom 10.9. (Well-foundedness principle for finite sets).

1. For any partially ordered set A and finite subset X of A, if X is nonempty then X has at least one minimal element and at least one maximal element.

2. For any totally ordered set A and finite subset X of A, if X is nonempty then X has a minimum element and a maximum element.

Remark 10.9. This principle only promises us that a finite subset of a partially ordered set has a *minimal* and a *maximal* member, but it does not guarantee that there is a *minimum* and a *maximum* member.

One very important consequence of Axiom 10.9 is the folowing:

Proposition 10.10. *Suppose that A is a partially ordered set and X is a finite subset and b is a member of X. Then there is a minimal member of X that is less than or equal to b and a maximal member of X that is greater than or equal to b.*

Proof. Suppose A is an arbitrary partially ordered set and X is an arbitrary subset of A and b is an arbitrary member of X. We need to prove (1) There is a minimal member of X that is less than or equal to b, and (2) There is a maximal member of X that is greater than or equal to b. We prove (1) and leave the proof of (2) as an exercise.

Proof of (1) Let Y be the set consisting of those members of X that are less than or b. Then Y is finite (since it is a subset of the finite set X) and Y is nonempty, since $b \in Y$. Therefore by the first part of Axiom 10.9, Y has a minimal member, which we'll call w. We claim that w is a minimal member of X that is less than or equal to b. By definition of Y, we know $w \leq b$. We must show that w is a minimal member of X. (Note that we chose w to be a minimal member of Y, but this does not automatically imply that w is a minimal member of X) We prove that w is a minimal member of X by contradiction. Suppose for contradiction that w is not a minimal member of X. Then there is a member z of X such that $z < w$. Since $z \leq w$ and $w \leq b$, by transitivity we have $z \leq b$. Since $z \in X$ and $z \leq b$ then $z \in Y$. But then z is a member of Y that is less than w, and this contradicts that w is a minimal member of Y. So we conclude that w is a minimal member of X, as required to complete the proof of (1). □

Exercise 10.17. Complete the proof of Proposition 10.10 by proving part (2).

10.5 Equivalence relations

In many situations we classify objects of a particular set into groups of "similar" objects. We might classify real numbers into three types "positive", "negative" and "zero", In geometry we classify triangles according to their sequence of angles (such as 30,60,90 triangles) and two triangles with the same sequence of angles are said to similar. We can classify finite sets according to the number of elements they have; we say that two sets are *equal-sized* if they have the same number of elements.

In such a relation, the set of objects A being classified is partitioned into *classes* and the relation consists of all pairs of objects that belong to the same class.

Definition 10.8. Equivalence relations. Suppose X is a set and let $\mathcal{P}$ be a partition of X. The *equivalence relation $R_{\mathcal{P}}$ induced by the partition $\mathcal{P}$* is the set of pairs $(x, y) \in X \times X$ such that x and y are in the same part of $\mathcal{P}$. The parts of the partition are called the *equivalence classes* of the equivalence relation.

Equivalence relations have a very simple structure: once you know the partition it's easy to see which elements are related to each other.

Example 10.12.

- Take the set of all lists. Every list has an associated set of the entries that appear. For example, the set associated to the list $(1, 3, 5, 3, 6, 3, 1)$ is $\{1, 3, 5, 6\}$. If we classify lists according to the associated set of entries we get an equivalence relation on lists.
- We can also classify lists according to their lengths so that all lists with the same length are in the same equivalence class.
- Consider what happens when you round a real number to the closest integer. This rounding rule is ambiguous for numbers like 3.5 since it is equally close to 3 and 4; in this case let's say we round the integer down. Let's say that two real numbers are *approximately equal* if they round to the same integer, so 8.8 and 9.3 are approximately equal. The relationship *approximately equal* defines an equivalence relation.
- Let's classify integers according to the largest power of 2 they are divisible by. Thus 24 and 40 are divisible by 8 but not 16 and are classified the same.

Exercise 10.18. Precisely describe the partition into equivalence classes for the approximately equal relation.

Each of these equivalence relations is obtained by taking a set and partitioning it into classes, called equivalence classes.

Earlier in this section, we identified some important special properties of relations. We now show that three of these properties are satisfied by every equivalence relation.

Theorem 10.11. *Every equivalence relation is reflexive, symmetric and transitive.*

Proof. Suppose X is an arbitrary set and $\mathcal{P}$ is an arbitrary partition and let R be the associated equivalence relation. We claim that R is reflexive, symmetric and transitive.

Proof that R is reflexive. Suppose $x \in X$ is arbitrary. Then x belongs to exactly one set $\mathcal{P}$ and so xRx.

Proof that R is symmetric. Suppose x, y are arbitrary members of X. Assume xRy. Then there is a set of $\mathcal{P}$ that contains both x and y so also yRx.

Proof that R is transitive. Suppose x, y, z are arbitrary members of X. Assume xRy and yRz. Then there are sets $A, B \in \mathcal{P}$ such that $x, y \in A$ and $y, z \in B$. Since $y \in A \cap B$ and $\mathcal{P}$ is a partition, we have $A = B$. Then $x, z \in A$ and so xRz. □

Remarkably it turns out that the converse of Theorem 10.11 is true. Namely if we have any relation that is known to be reflexive, symmetric and transitive, then it must be an equivalence relation.

Theorem 10.12. *Suppose R is a relation on X that is transitive, symmetric, and reflexive. Then the set of sets $\mathcal{P} = \{R(x) : x \in X\}$ is a partition of X and $R = R_{\mathcal{P}}$.*

Remark 10.10. We defined an equivalence relation to be a relation that comes from a partition by saying two objects are related if they belong to the same part of the partition. Another way to define equivalence relation (which appears in many other books) is to say that an equivalence relation is a relation that is transitive, symmetric and reflexive. We refer to the first definition as the *partition-based* definition of equivalence relations and the second definition as the *property-based* definition. Theorems 10.11 and 10.12 together imply the remarkable fact that these two very different definitions lead to the same thing.

Proof. Suppose R is an equivalence relation on X and let $\mathcal{P} = \{R(x) : x \in X\}$. We claim that $\mathcal{P}$ is a partition of X and that $R = R_{\mathcal{P}}$.

First we prove that $\mathcal{P}$ is a partition of X. We must show that (i) for all $x \in X$, there is a member of $\mathcal{P}$ that has x as a member, and (ii) for all $A, B \in \mathcal{P}$ if $A \cap B \neq \emptyset$ then $A = B$.

Proof of (i). Suppose $x \in X$ is arbitrary. Then $R(x) \in \mathcal{P}$. Since R is reflexive $x \in R(x)$.

Proof of (ii). Suppose $A, B \in \mathcal{P}$. Assume that $A \cap B \neq \emptyset$. We must show $A = B$. For this we must show $A \subseteq B$ and $B \subseteq A$. By definition of $\mathcal{P}$ there are objects we'll call $a, b \in X$ such that $A = R(a)$ and $B = R(b)$.

Proof that $A \subseteq B$. Let $x \in A$ be arbitrary. Then by definition of $A = R(a)$, we have aRx. We must show $x \in B$, which means we must show bRx. Since $A \cap B \neq \emptyset$ there is an object we'll call y such that $y \in A$ and $y \in B$ and thus aRy and bRy. By symmetry of R we have yRa and by transitivity bRy and yRa implies bRa. Since bRa and aRx we have bRx by transitivity, as required to show $A \subseteq B$.

The proof that $B \subseteq A$ is analogous by interchanging A and B and a and b in the above proof.

So we've shown that $\mathcal{P}$ is a partition. It remains to show that $R = R_{\mathcal{P}}$. For this we must show that for all $x, y \in X$ xRy if and only if $xR_{\mathcal{P}}y$. Suppose $x, y \in A$. We must prove two things: (a) if xRy then $xR_{\mathcal{P}}y$ and (b) if $xR_{\mathcal{P}}y$ then xRy.

Part (a). Assume xRy. Then $y \in R(x)$ and also $x \in R(x)$ and so $x, y \in R(x)$ and so $xR_{\mathcal{P}}y$.

Part (b). Assume $xR_{\mathcal{P}}y$. Then there is a set $S \in \mathcal{P}$ that contains both x and y. Now $x \in R(x) \in \mathcal{P}$ and so $R(x) \cap S \neq \emptyset$. By the definition of a partition, $S = R(x)$ and since $y \in S$ we have $y \in R(x)$ and so xRy as required. □

Given an equivalence relation R the parts of the associated partition $\mathcal{P}_R$ are called the *equivalence classes* of R.

Remark 10.11. The theorem tells us that if R is an equivalence relation on X then the set $\{R(a) : a \in X\}$ is a partition of X. This does not mean that all of the sets $R(a)$ are different. What it does mean is that for any two members a, b of X we either have $R(a) = R(b)$ or $R(a) \cap R(b) = \emptyset$.

The importance of Theorem 10.12 is that it happens often in mathematics, that we have a relation on a set X where we can prove that it is en equivalence relation by showing that it is reflexive, symmetric and transitive, even though we don't know the equivalence classes. By studying the relation and using Theorem 10.12 we can figure out the equivalence classes. We now look at an important example.

Equivalence modulo an integer k We now discuss in detail some important equivalence relations on the set of integers.

Suppose k is a positive integer. For integers x and y we say that *x is congruent to y modulo k*, or *x is congruent to y mod k*, written $x \equiv_k y$, if $x - y$ is divisible by k. This relation is called the *congruence modulo k* relation or simply the *mod k* relation for short.

Exercise 10.19. Prove that, for every positive integer k, congruence modulo k is a reflexive, symmetric and transitive relation on $\mathbb{Z}$.

Example 10.13. When $k = 2$, two numbers are congruent modulo 2, if they differ by an even number. It is easy to see that there are exactly two equivalence classes, the even numbers and the odd numbers.

In the following discussion, think of k as an arbitrary fixed positive integer. We'll denote by C_k the equivalence class of the congruence modulo k relation. By Theorem 10.12, the equivalence classes of C_k are the sets $C_k(j) = \{n \in \mathbb{Z} : n - j \text{ is divisible by k}\}$. Since j can be any integer, this gives us an infinite list of possible equivalence classes. But many of these equivalence classes are the same. We'd like to know: (1) Is the number of different equivalence classes finite (no matter what k is)? (2) If so, how many are there? The following theorem tells us that for every positive integer k, there are exactly k equivalence classes, one for each of the integers $\{0, \ldots, k-1\}$.

Theorem 10.13. *For any positive integer k,*

1. *The sets $C_k(j)$ for $j \in \{0, \ldots, k-1\}$ are all different.*
2. *Every integer belongs to one of the classes $C_k(0), \ldots, C_k(k-1)$*
3. *The partition of $\mathbb{Z}$ into equivalence classes modulo k is $\{C_k(0), \ldots, C_k(k-1)\}$.*

Proof	**Commentary**
Suppose k is an arbitrary positive integer. For simplicity we drop the subscript and write simply C instead of C_k.	

We prove the first part. Suppose i and j are arbitrary members of $\{0, \ldots, k-1\}$ and assume $i \neq j$. We must show $C(i) \neq C(j)$. By definition of $C(i)$ we have $i \in C(i)$, so to show that $C(i) \neq C(j)$, it is enough to show that $i \notin C(j)$. Suppose for contradiction that $i \in C(j)$. Then $(i-j)/k$ is an integer, and so is $|i-j|/k$. Since i and j are between 0 and $k-1$, $0 \leq |i-j| \leq k-1$ and so $0 \leq |i-j|/k < 1$. The only integer in that interval is 0, which means $|i-j| = 0$ and so $i = j$. But this gives a contradiction since $i \neq j$. So $i \notin C(j)$.

Proof of the first part is a straightforward proof by contradiction. We want to show that no two of the sets are the same. The sets are indexed by the integers in $\{0, \ldots, k-1\}$ so we pick an arbitrary i and j from that set, and assume $i \neq j$, and show that $i \in C(i)$ (by definition) and $i \notin C(j)$. When we consider $i-j$, we use absolute value to avoid having to divide into cases depending on whether $i > j$ or $i \leq j$.

Next we prove the second part of the theorem. Suppose that n is an arbitrary integer. We must show that there is a $j \in \{0, \ldots, k-1\}$ such that $n-j$ is divisible by k.

Our job is to give instructions to find an integer j so that $j < k$ and $j \geq 0$ and $n-j$ is divisible by k. Our strategy will be this: We'll ignore the first requirement and consider the set of all nonnegative integers m such that $n-m$ is divisible by k. To have any hope we need that this set is nonempty. Once we know the set is nonempty we can use the well-ordering principle, Corollary 10.8, to say that this set has a smallest member. We will then show that this smallest member must be less than k and therefore satisfies all the requirements.

Let W be the set of all nonnegative integers m such that $n-m$ is divisible by k. We must pick j from W. First we show that W is nonempty. We split into cases according to whether $n \geq 0$ or not.

Case 1. Assume $n \geq 0$. Then $n \in W$.

We split into cases here because when $n \geq 0$ the argument that W is nonempty is very simple, but that argument does not work when $n < 0$. For $n < 0$ we need a different proof.

Case 2. Assume $n < 0$. We have to show that there is a nonnegative integer t so that $n-t$ is divisible by k. We note that nk is divisible by k and less than or equal to n so we let $t = n - nk$. This is nonnegative since $nk \leq n$ and $n - t = nk$ is divisible by k. So $t \in W$.

Since W is a subset of the nonnegative integers that is nonempty, by Corollary 10.8, W has a smallest member, which we will call j. Clearly $j \geq 0$. We also claim that $j < k$. Suppose for contradiction that $j \geq k$. Let $j' = j - k$. Then $j' \in W$ since $j' \geq 0$ and $b - j' = (b - j) + k$ is divisible by k since $m - j$ is. But this contradicts the choice of j as the smallest member of W. So $j < k$. So we have found a $j \in \{0, \ldots, k-1\}$ in such that $n \in C(j)$ as required to complete the second part.

For the third part, since the sets $C(j)$ are equivalence classes for the relation C, we know from Theorem 10.12 that any two of them are either the same or disjoint. From the first part that the sets $C(0), \ldots, C(k-1)$ are all different, so they must be all disjoint. By the second part of the theorem, their union is all of $\mathbb{Z}$, and so they partition $\mathbb{Z}$. □

Having picked the integer j, we now have to demonstrate that j has the required properties. We do this by contradiction. Using proof by contradiction is very common in proofs that use the well-ordering principle to select a smallest or largest member of a set of numbers.

One important consequence of the above theorem is:

Theorem 10.14. *(Quotient-remainder theorem) Suppose k is an arbitrary positive integer. For every integer n there are is a unique integer q and a unique number $r \in \{0, \ldots, k-1\}$ such that $n = qk + r$.*

Remark 10.12. This theorem states a familiar fact taught in middle school, when you divide an integer n by a positive integer k you get a quotient q and a remainder r that is between

Proof. Suppose k is a positive integer and suppose that n is an integer. Let $\{C_0, \ldots, C_{k-1}\}$ denote the equivalence classes modulo k as given by Theorem 10.13. There is an integer that we will call r such that $r \in \{0, 1, \ldots, k-1\}$ and $n \in C_r$. Let $w = n - r$. Since $b \in C_j$ we have that w is a multiple of k. Letting $q = w/k$ we have $n = qk + r$ as required. □

Some additional examples of equivalence relations. Here are some additional examples of relations. In the exercise that follows you'll show that each is an equivalence relation.

Example 10.14. Some additional examples of relations that are reflexive, symmetric and transitive (and are therefore equivalence relations).

- Real Numbers under rational equivalence xRy if there are integers a, b such that $ax = by$.

- Integers under odd multiple equivalence: xRy if there are odd integers a, b such that $ax = by$.
- Integers under *square equivalence* xRy: if there are integers a, b such that $xa^2 = yb^2$.
- For S a subset of $\mathbb{R}$ we have the following relation: xRy if the interval $I[x, y] \subseteq S$ (where $I[x, y] = [\min(x, y), \max(x, y)]$.)
- Ordered pairs of real numbers with $(a, b)R(c, d)$ if and only if $ad = bc$.

Exercise 10.20. For each relation in Example 10.14, prove that the relation is reflexive, symmetric and transitive.

Exercise 10.21. For parts 2,3 and 5 of Example 10.14 determine a simple description of the equivalence classes and prove that your description is correct.

Exercise 10.22. Let R be a relation on X. Let us say that two elements x and y are R-twins if for every $z \in X$ we have xRz if and only if yRz and zRx if and only if zRy. Let $T = T(R)$ be the relation on X where xTy if x and y are R-twins.

1. Prove that T is an equivalence relation on X.
2. Give an example of a reflexive relation R on a 6 element set for which $T(R)$ has two equivalence classes of size 3.
3. Suppose R is the divisibility relation on $\mathbb{Z}$. What is $T(R)$?