3 Universal principles in mathematics: An introduction [4]

Having introduced some basic objects of study in mathematics, let's turn to the question of what the goal of that study is. On a very basic level, we want to discover the "facts" of mathematics. But there are many facts, we're particularly interested in the *important* ones. Let's start with an example of two mathematical facts:

Fact A**.** $(7-3)(7+3) = 7^2 - 3^2$

Fact B**.** For any two numbers a and b, $a^2 - b^2 = (a+b)(a-b)$.

The first is a fact from arithmetic about the two numbers 3 and 7.

The second fact is related to the first, but is a much more interesting and powerful fact than the first. It asserts a *universal principle* that *works for any two numbers.*

This universal principle implies the first fact, and an endless number of others, such as:

Fact C**.** $(7101036 - 3354251)(7101036 + 3354251) = 7101036^2 - 3354251^2$

Propositions A and C are said to be *instances* of the universal proposition B. They are obtained by substituting specific values for a and b. Since, as we know from high school algebra, Proposition B is a true proposition, it tells us that Fact C must be true also. Notice that Fact C is not something that would be easy to verify directly (even with a calculator).

Universal principles are the most important type of facts. Universal principles allow us to summarize *vast amounts of knowledge* in a *single sentence.*

Every intellectual endeavor, such as astronomy, microbiology, economics, or history, is the search for *universal principles* which reflect a truth that applies in many situations: "galaxies rotate","living organisms use oxygen to convert biological fuel into usable energy","As the supply of a good goes up, the price goes down", or "flu shots protect people against the flu". Universal principles in the sciences and social sciences (such as the ones stated above) are often only approximations to the truth. The principle "living organisms use oxygen to convert biological fuel into usable energy" does not apply to all living organisms, since some do this conversion without oxygen. The principle "the supply of a good goes up, the price goes down" is not always true since there are many other factors that determine the price of a good, and this principle is understood to mean that increases in supply of a good *tend to be* accompanied with a reduction in price. These principles capture some general knowledge that apply broadly but may be incorrect in some situations..

The universal principles of mathematics concern relationships and patterns that are present among mathematical objects. Unlike universal principles in other fields, universal principles of mathematics are required to be *completely clear and unambiguous*, clearly stating the situations to which it applies, clearly excluding situations to which it does not apply, and clearly stating the conclusion that must hold in the situations to which they do apply.

[4]Version 9/22/2016.

How do we know that a proposed universal principle is indeed true? In the physical, biological and social sciences, universal principles are conjectured based on a combination of observation (experiment) and reasoning based on broader knowledge of the field. Once formulated, a general principle in the sciences is proved or disproved by experiment. A general principle in these fields is always open to reconsideration if new evidence comes up that contradicts it.

In mathematics, possible universal principles are also formulated based on a combination of reasoning and observation (of examples). However, the confirmation of universal principles in mathematics is not done by experiment. Instead it is done by *deductive proof.* Deductive proof is a central part of mathematics, and will become a central focus of this course. For now, we will look at some examples of the statements (and not the proofs) of some universal principles of mathematics.

Earlier we mentioned the following famous examples of a universal principle:

Universal Principle 3.1. (Pythagorean Theorem) For any right triangle, the square of the length of its hypotenuse is equal to the sums of the squares of its two legs.

For example, if T is a right triangle whose legs have length 3 and 7, then the hypotenuse h must satisfy $h^2 = 7^2 + 3^2 = 58$ and so the length of the hypotenuse must be $\sqrt{58}$.

Remark 3.1. In stating this general principle, I've assumed that the reader is already familiar with the following terminology:

- *right triangle*,
- *hypotenuse* of a right triangle, and
- *leg* of a right triangle.

Obviously, a reader who doesn't know what a right triangle is, or doesn't know what a hypotenuse is, won't be able to understand this principle.

When reading any general principle, or any mathematical sentence, the first thing to ask yourself is "Do I know the precise meaning of each piece of mathematical terminology used in this sentence?" If the answer is no, STOP! Find out what every term means. It's not enough to have a vague idea what the terminology means. If you don't know the precise meaning of every term means, it is unlikely that you'll be able to work correctly with the principle.

Universal Principle 3.2. For any real number a and any real number b, the average of the squares of a and b is at least the square of the average of a and b, that is: $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

For example, if $a = 12$ and $b = 17$ then the average of the squares is $(144 + 289)/2 = 216.5$ while the square of their average is $((12 + 17)/2)^2 = 210.25$. This is a principle in the area of *real number inequalities*: here the expressions $\frac{a^2+b^2}{2}$ and $\left(\frac{a+b}{2}\right)^2$ are determined by the choice of a and b and the principle says that for any choice of a and b, the first expression is always less than or equal to the second.

When given any universal principle in mathematics, one question a mathematician asks herself is: Can this principle be extended to cover more general situations? For example, the previous principle applies to any pair of numbers. It turns out that we can generalize this principle so that it applies to any finite list of numbers.

Universal Principle 3.3. For any (finite) list of real numbers, the average of the squares of the numbers is at least the square of the average of the numbers.

For example, for the list of numbers $4, 6, 3, 4$, the average is 4.25, and the square of the average is $18\frac{1}{16}$, while the average of the squares is $\frac{1}{4}(16 + 36 + 9 + 16) = 19\frac{1}{4}$.

The next general principle expresses a basic fact of algebra that you probably learned in high school. Recall that a *monomial* in the variable x is an expression of the form cx^n where n is a nonnegative integer called the *degree* of the monomial, and c is a real number called the *coefficient* of the monomial. A monomial is said to be *nonzero* if its coefficient is not zero. A polynomial in variable x is a sum of a finite number of monomials, such as $5x^4 + \pi x^5 + (-5/4)x^2$. A *root* of the polynomial p is a number which when substituted for x makes the polynomial evaluate to 0. For example the roots of $x^3 + (-9)x^2 + 15x + (-7)$ are 1 and 7, since if you substitute 1 for x the polynomial evaluates to 0, and if you substitute 7 for x the polynomial evaluates to 0, but if you substitute any other number, the polynomial evaluates to a nonzero number. A polynomial is said to be nonzero if there is at least one number such that substituting that number for x results in a nonzero value for the polynomial.

Universal Principle 3.4. For any polynomial p, if p is nonzero then the number of roots of p is at most the largest degree of any of the monomials in the sum.

In the example given just before the statement, the largest degree of any of the monomials in the sum is 3, and the polynomial has exactly 2 roots.

Here's a general principle in the field of *number theory*, which is the study of the integers and their properties:

Universal Principle 3.5. For any positive integer a and positive integer b, if b is a prime number then $a^b - a$ is a multiple of b.

For example, if $a = 3$ and $b = 7$ then $a^b - a = 3^7 - 3$ is equal to 2184, which is indeed a multiple of 7, since it is equal to 7 times 314.

This is a somewhat surprising principle, and a mathematician reading this principle for the first time, would ask herself: Is this really always true? If so why?

Here's another strange and surprising principle from number theory:

Universal Principle 3.6. For any positive integer n such that n is prime, and $n-1$ is divisible by 4, there are positive integes a and b such that $n = a^2 + b^2$.

For example, the number 29 is a prime number such that $29 - 1$ is divisible by 4 and the conclusion is true by choosing $a = 5$ and $b = 2$ since $29 = 5^2 + 2^2$. The number 73 is a prime number such that $73 - 1$ is divisible by 4, and the conclusion is true by choosing $a = 8$ and $b = 3$, since $73 = 8^2 + 3^2$.

The next principle applies to elementary set theory:

Universal Principle 3.7. For any two sets A,B and C, if $A \neq B$ then $A \cup C \neq B \cup C$ or $A \cap C \neq B \cap C$.

Notice that in the conclusion there are two possible conditions, and the principle guarantees that at least one of them is true, but doesn't say which. For example, if A is the set $\{1, 2, 3, 4\}$ and B is the set $\{2, 3, 4, 5\}$ and C is the set $\{1, 3, 5\}$ then $A \cup C = B \cup C = \{1, 2, 3, 4, 5\}$ and $A \cap C = \{1, 3\}$ while $B \cap C = \{3, 5\}$ and so $A \cap C \neq B \cap C$, as asserted by the principle.

We've now seen various examples of some universal principles. One of the main things we'll be studying in this course is: given a possible universal principle, how do mathematicians know that the principle is true? This question will lead us to the idea of *mathematical proof*. Before we can start discussing mathematical proof, we'll need to address a more basic question: how do mathematicians communicate about mathematics?