5 Mathematical Scenarios[6]

In common usage, a scenario is the set up for a story: the main characters, important facts about them and their relationships. A *mathematical scenario* consists of:

- Some unspecified mathematical objects, each represented by a variable, called that *active objects* of the scenario.
- Some assumptions about the activer objects (called the *active assumptions* of the scenario). The assumptions are either definite assertions, or indefinite assertions whose free variables are active variables.

A mathematical scenario establishes a situation involving "characters" (the active objects) who are placed in a given "situation" (defined by the active assumptions). As we will see, *mathematical scenarios are the starting point for nearly any investigation or discussion in mathematics.* Despite their central importance to doing and communicating mathematics, the terminology "mathematical scenario" seems to be new to these notes.

Here are some simple examples of mathematical scenarios:

Scenario 1.
Active objects: Real numbers x and y
Active assumptions: $x^2 + y^2 \geq 16$ and $x \leq y$.

Scenario 2.
Active object: A set S of integers
Active assumption: There is no integer bigger than 1 that is a divisor of every member of S.

Scenario 3.
Active objects: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a real number t.
Active assumption: $f(t) > t$.

Scenario 4.
Active objects: $a, b, c, \in \mathbb{Z}$.
Active assumptions: a is a prime number and $a = b \times c$ and $b > 1$ and $c > 1$.

If we substitute particular objects for the active objects then the active assumptions may or may not be true. For example if in the third scenario we substitute the function $x \longrightarrow x^2$ for the function f and 2 for t then the assumption is true, but if we substitute $\frac{1}{2}$ for t then the assumption is false. When we use mathematical scenarios, we view the active assumptions as *requirements* on the active objects..

A choice of values for the active objects that makes the assumption (or assumptions) true is said to *satisfy the assumptions* and to *satisfy the scenario*, and we say that this choice is a *feasible instance* of the scenario.

A choice of values for the active objects that makes the assumption false is said to *violate the assumptions* or *violate the scenario* and is an *infeasbile instance* of the scenario.

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In Scenario 1, for example, $x = 2$ and $y = 5$ is a feasible instance while $x = 2$ and $y = 3$ is an infeasible instance. We can also represent the feasible instance by an ordered pair $(2, 5)$, where we assume that we've fixed the first coordinate to correspond to x and the second to y. In scenario 4, there is no choice of a,b and c that satisfies the requirements.

The set of feasible instances of a scenario Any mathematical scenario has a *set of feasible instances* which consists of all settings of the variables that make the assumptions true. If there is more than one active object, we think of an assignment to the objects as a list of objects, where the length of the list is the number of variables, and the order of the objects in the list should either be clear from context, or made explicit. If we denote the set of feasible instances for the above four scenarios by S_1,. S_2, S_3 and S_4, then we can express these sets using constraint specification:

$$\begin{aligned}
S_1 &= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 16 \text{ and } x \geq y\} \\
S_2 &= \{S \subseteq \mathbb{Z} : \text{ no integer bigger than 1 is a divisor of every member of } S \\
S_3 &= \{(f, t) \in \mathbb{R}^{\mathbb{R}} \times \mathbb{R} : f(t) > t\}. \\
S_4 &= \{(a, b, c) \in \mathbb{Z}^3 : a \text{ is prime}, a = b \times c, b > 1, c > 1\}.
\end{aligned}$$

We have the following terminology for scenarios:

- A scenario is *feasible* if it has at least one instance, which means that the set of feasible instances is nonempty.
- A scenario is *uniquely feasible* if it has exactly one feasible instance.
- A scenario is *infeasible*, *contradictory* or *impossible* if it has no feasiblle instances so that the set of feasible instances is empty.

Example 5.1. Consider the following three scenarios with two active objects, both of which are real numbers x and y.

Scenario A has one assumption $x + y = 3$. This scenario is feasible since, for example, $x = \frac{1}{2}$ and $y = \frac{5}{2}$ is an instance. It is not uniquely feasible since it has other instances also.

Scenario B has the same requirement $x + y = 3$, and also the additional requirements that x and y are integers, $y > x$ and $x > 0$. This scenario is uniquely feasible since $x = 1$ and $y = 2$ is the only instance.

Scenario C has the assumption $x + y = 3$ and $x^2 + y^2 \geq 10$ and $x \geq 0$ and $y \geq 0$. This scenario is infeasible-there is no way to choose real numbers x and y to satisfy all of the conditions.

Mathematical scenarios play a central role in thinking and communicating about mathematics: They are essential for *formulating mathematical problems*, em providing the context for a mathematical definition, and for discussing both existential and universal assertions. Most importantly, they *provide the conceptual and logical framework for mathematical proofs.*

Scenarios and Existential assertions An existential assertion has the form: "There exists an object x of type T satisfying $P(x)$". This principle is associated with the scenario whose active object is of type T and is represented by x, and whose assumption is $P(x)$. The existential principle simply makes the claim that this associated scenario is feasible.

Scenarios and unique existential assertions A unique existential assertion has the form: "There exists a unique object x of type T satisfying $P(x)$". This principle is associated with the scenario whose active object is of type T and is represented by x, and whose assumption is $P(x)$. The unique existential principle simply makes the claim that this associated scenario has exactly one feasiable solutions.

Scenarios and universal assertions A universal assertion has the form: "For any object x of type T that satisfies $A(x)$, we must have $C(x)$.' We can associate the principle to the mathematical scenario with active object x and assumption $A(x)$, which we refer to as the *hypothesis* of the universal assertion. The principle says that any feasible instance of the hypothesis must satisfy $C(x)$.

Let's analyze some previously stated universal assertions from this point of view. For Universal Principle 3.5 we have:

Input. Positive integers a and b

Assumption. a is a positive integer, b is a positive integer and b is prime.

Conclusion. b is a divisor of $a^b - a$.

For Universal Principle 3.7, we have:

Input. The sets A, B and C

Assumption. $A \neq B$.

Conclusion. $A \cup C \neq B \cup C$ or $A \cap C \neq B \cap C$.

The following terminology is helpful in formulating what it means for a universal assertion to be true.

1. A *test case* of a universal assertion is a *feasible instance* of the associated mathematical scenario.

2. A *successful test case* of a universal assertion is a test cqse that makes conclusion true.

3. A *counterexample* or *unsuccessful test case* for a universal assertion is an instance for which the conclusion is false.

For Universal Principle 3.5, we have:

- The choice $a = 8$ and $b = 3$ is a test case since it satisfies the assumption. It is also a successful test case because it also satisfies the conclusion, since 3 is a divisor of $8^3 - 8 = 504$.
- Setting $a = 5$ and $b = 4$ is not a test case because it does not satisfy the assumption that b is prime. Since it is not a test case, it is neither a successful test case or a counterexample.

In general notice that:

- Every test case is either a successful test case or a counterexample but not both.
- An assignment of values to the active objects that makes the assumpton *false* is not a test case, and so can not be either a successful test case or a counterexmample.

Using this terminology we can say:

> A universal principle is a universal assertion for which every test case is successful, or equivalently, the assertion has no counterexamples.

Here's another example. Consider the following two assertions:

Assertion D. Every prime number is odd.

Assertion E. There is no largest prime number.

For assertion D:

- The choice $k = 11$ is a test case (since 11 is prime) and is a successful test case (since 11 is odd).
- The choice $k = 15$ is not a test case since 15 is not prime.
- The choice $k = 2$ is a test case (since 2 is prime), and is a counterexample since 2 is not odd.

Since Assertion D has a counterexample, it is *not* a universal principle.

Assertion E does not look like a universal assertion but it turns out that it is a universal assertion in disguise. To formulate this as a universal assertion, observe that Assertion B has the following meaning: If you give me any prime number, I can give you a larger one. In other words, in the scenario where n is a prime number we want to conclude that there is a prime number m that is larger than n. So Assertion E is equivalent to:

Assertion E′. For every prime number n there is a larger prime number.

Here are some successful test cases:

- Choose $n = 3$. Then n is prime and 7 is a larger prime number. (Notice we have many other choices besides 7.)

- Choose $n = 17$. Then 37 is a larger prime number.

How about $n = 12553$. For one ting it's not clear whether 12553 is a test case, which in this case requires that it be prime. If it is a test case, then to be successful we'd need a larger prime number.

As usual with a universal assertion, even if we check a few successful test cases, we can't be sure that the assertion is true. Later we'll see that this universal assertion is indeed true (and so is a universal proposition). In fact it is one of the most famous (and oldest) universal propositions known.

Vacuously true universal assertions. For a universal assertion of the form "for all x that satisfy $A(x)$, we have $C(x)$" we saw that this assertion is true provided that every test case is successful. What if there are no test cases at all? Can this happen?

It certainly can happen. For example, consider the universal assertion: For any real number x, if $x^2 < -1$ then $x \geq 1000$. Notice that there are no real numbers that satisfy the assumption, and therefore there are no test cases.

Now, in this case is the universal assertion true or false. Such a universal assertion is considered to be true since there are no counterexamples, and a universal assertion with no counterexamples is, by definition, true.