Notes from the Senior Seminar

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These are notes on two hour-long lectures which briefly overviewed Fourier analysis and complex analysis. They were designed to give a crash course in the subject to give some of the background needed for this course on modular forms.

1 Fourier Analysis

1.1 Main Motivation: Sums of Squares

One of the problems which we will have in mind during this course is *"how many ways can a positive integer be written as the sum of four squares?"* Let

$$r_4(n) = |\{(w,x,y,z) \in \mathbf{Z}^4 \mid n = w^2 + x^2 + y^2 + z^2\}|.$$

Thus $r_4(1) = 8$ because we can count apparently identical solutions many times. It is a theorem of Legendre that every positive integer can be written as the sum of four squares, i.e. $r_4(n) > 0$ if $n > 0$.

The reason Fourier analysis enters the subject is because it is convenient to examine this problem using the theta function

$$\theta(y) = \sum_{n \in \mathbf{Z}} e^{-\pi n^2 y},$$

because

$$(\theta(y))^4 = \sum_{m \in \mathbf{Z}} r_4(m) e^{-\pi m y}.$$

We will prove below that the theta function has the transformation property

$$\theta(y) = \frac{1}{\sqrt{y}} \theta\left(\frac{1}{y}\right),$$

which implies that

$$(\theta(y))^4 = \frac{1}{y^2} \theta\left(\frac{1}{y}\right)^4.$$

We will later describe how there is essentially only one function with this transformation property, and we will derive another expansion for it. This will enable us to find a formula for $r_4(n)$. Most notably, $r_4(p) = 8(p+1)$ if p is prime.

1.2 Periodic Functions

The simplest examples of modular forms are periodic functions on the real line, i.e. functions $f : \mathbf{R} \to \mathbf{C}$ such that $f(x+n) = f(x)$ for any $x \in \mathbf{R}$ and $n \in \mathbf{Z}$. The most famous examples of these are the trigonometric functions $\sin(x)$ and $\cos(x)$, which themselves are examples of trigonometric polynomials. These are polynomials in

$$e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x),$$

like

$$\cos(x) = \frac{1}{2}\left(e^{2\pi i x} + e^{-2\pi i x}\right).$$

A limiting type of trigonometric polynomial is an infinite series like

$$\sum_{n \in \mathbf{Z}} a_n e_n(x) \quad , \quad e_n(x) = e^{2\pi i n x}.$$

These are trigonometric polynomials in the case that $a_n = 0$ for all but a finite number of n. They are periodic functions.

If f is an infinitely differentiable function, then f is represented by the Fourier series

$$f(x) = \sum_{n \in \mathbf{Z}} \hat{f}(n) e_n(x),$$

$$\hat{f}(n) = \int_0^1 f(x) e_{-n}(x) dx.$$

Conversely, if one assumes conditions on the coefficients a_n of an infinite series $\sum_{n \in \mathbf{Z}} a_n e_n(x)$, then it converges to a function with certain differentiability properties. A key calculation in Fourier analysis is that

$$\int_0^1 e_n(x) e_m(x) dx = \begin{cases} 1, & n - m = 0 \\ 0, & \text{otherwise} \end{cases}$$

1.3 Fourier Transform

Fourier series are about periodic functions, on the circle $\mathbf{R}/\mathbf{Z}$. There is a similar theory for functions on the real line. In order to get the integrals to converge, it is very convenient to restrict attention to *Schwartz functions*, which are infinitely differentiable functions $f : \mathbf{R} \to \mathbf{C}$ satisfying

$$\lim_{|x| \to \infty} \left| x^n \frac{d^m f}{dx^m}(x) \right| = 0, n, m \geq 0.$$

The most famous example of a Schwartz function is $f(x) = e^{-x^2}$.

The *Fourier transform* is defined as

$$\hat{f}(t) = \int_{\mathbf{R}} f(x) e^{-2\pi i x t} dx.$$

The *Fourier inversion formula*

$$f(x) = \int_{\mathbf{R}} f(t) e^{2\pi i x t} dt$$

holds for Schwartz functions.

There is a function whose Fourier transform is very easy to compute:

Proposition 1.1 *The Fourier transform of* $f(x) = e^{-pix^2}$ *is* $\hat{f}(t) = e^{-2\pi t^2}$.

Proof: By definition,

$$\hat{f}(t) = \int_{\mathbf{R}} e^{-\pi x^2 - 2\pi i x t} dx.$$

The result follows by changing variables $x \mapsto x - it$. Of course, this is an illegal change of variables, since it is not a real number, but at the very end we shall see how to justify this step at the end of the second lecture. $\square$

By changing variables in the definition of the Fourier transform we can also easily prove a generalization of the above proposition: that the Fourier transform of $f(x) = e^{-\pi c x^2}$ is $\hat{f}(t) = \frac{1}{\sqrt{c}} e^{-\pi x^2/c}$.

1.4 Poisson Summation

Let $f(x)$ be a Schwartz function. Since it converges so quickly, the function

$$F(x) = \sum_{n\in\mathbf{Z}} f(x+n)$$

converges and defines a periodic function:

$$F(x+m) = \sum_{n\in\mathbf{Z}} f(x+n+m) = \sum_{n\in\mathbf{Z}} f(x+n).$$

Since it is infinitely differentiable, it is equal to its Fourier series:

$$F(x) = \sum_{n\in\mathbf{Z}} \hat{F}(n) e_n(x).$$

Let us compute

$$\hat{F}(m) = \int_0^1 F(x) e_{-m}(x)dx = \int_0^1 \sum_{n\in\mathbf{Z}} f(x+n)e^{-2\pi i m x}dx.$$

We can switch the order of integration since f is a Schwartz function (we can basically do anything to a Schwartz function), and change variables in each integral:

$$\hat{F}(m) = \sum_{n\in\mathbf{Z}} \int_0^1 f(x+n)e^{-2\pi i m x}dx = \sum_{n\in\mathbf{Z}} \int_n^{n+1} f(x)e^{-2\pi i m(x-n)}dx.$$

Since $e^{-2\pi i m(x-n)} = e^{-2\pi i m x}e^{-2\pi i m n} = e^{-2\pi i m x}$, we see that

$$\hat{F}(m) = \sum_{n\in\mathbf{Z}} \int_n^{n+1} f(x)e^{-2\pi i m x}dx = \int_{\mathbf{R}} f(x)e^{-2\pi i m x}dx = \hat{f}(x)$$

by gluing the integrals together. Note that $\hat{F}$ refers to a Fourier coefficient, but $\hat{f}$ refers to a Fourier transform. We have proved

Proposition 1.2 *(The Poisson Summation Formula) If $f(x)$ is a Schwartz function and $\hat{f}$ denotes its Fourier transform, then*

$$\sum_{n\in\mathbf{Z}} f(x+n) = \sum_{n\in\mathbf{Z}} \hat{f}(n) e_n(x).$$

In particular, if $x=0$ this reduces to the Poisson summation formula

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \hat{f}(n).$$

The Poisson summation formula is an amazing formula which relates an average of a function to an average of its Fourier transform.

The identity

$$\theta(y) = \frac{1}{\sqrt{y}}\theta\left(\frac{1}{y}\right)$$

now follows from combining the above propositions. Note that the sum defining the $\theta(y)$ converges very quickly if y is large but very slowly if y is small. It is amazing that one can compute the values for small y from the values at large y.

2 Complex Analysis

Complex analysis the study of certain functions $f : \mathbf{C} \to \mathbf{C}$. Since $\mathbf{C} = \mathbf{R}^2$, one might think that complex-variable calculus is really a special case of multivariable calculus. While notions from that subject are very useful in the calculus of complex functions, there must be a major difference between the two which is large enough to justify another course in the subject. That difference comes from using the *algebraic* property that complex numbers can be multiplied. We will thus place heavy restrictions on the types of functions we will consider. For example, $f(z) = z$ will be acceptible but $f(z) = Re(z)$ will not.

2.1 Complex Differentiation

If we were to consider general functions $f : \mathbf{C} \to \mathbf{C}$, i.e. $f : \mathbf{R}^2 \to \mathbf{R}^2$, it would not make sense to talk about "$\frac{df}{dz}$," for there are many directions in which one could take the derivative. For example, if $f(x+iy) = x = Re(x+iy)$, then

$$\frac{\partial f}{\partial x} = 1 \ , \ \frac{\partial f}{\partial y} = 0.$$

We will say that a function $f : \mathbf{C} \to \mathbf{C}$ is *complex-differentiable* at the point z if the limit

$$\lim_{u \in \mathbf{C}, |u| \to 0} \frac{f(z+u) - f(z)}{u}$$

is a finite number, and we will call that number $f'(z)$, the derivative of f at the point z. Note that this must be the same limiting value regardless of the path u takes converging to zero.

Proposition 2.1 *(The Cauchy-Riemann Equations) A complex function $f : \mathbf{C} \to \mathbf{C}$ is complex-differentiable at the point $z \in \mathbf{C}$ if and only if*

$$i\frac{\partial f}{\partial x}(x+iy) = \frac{\partial f}{\partial y}(x+iy).$$

Moreover, if we we break f into its real and imaginary parts $f(x+iy) = u(x+iy) + iv(x+iy)$, these conditions hold if and only if

$$\frac{\partial u}{\partial x}(x+iy) = \frac{\partial v}{\partial y}(x+iy)\ ,\ \frac{\partial u}{\partial y}(x+iy) = -\frac{\partial v}{\partial x}(x+iy).$$

Proof: The first version follows from differentiating f in two ways: first let $u \to 0$ through real values, then through imaginary values. The second set is just a reformulation of the first. □

2.1.1 Examples

Any polynomial is differentiable, just as for real variables.

Any convergent power series is also differentiable.

The Cauchy-Riemann equations are violated for $f(z) = \bar{z}$, complex conjugation. The same is true for $f(z) = |z|^2$, except at $z = 0$.

The function $f(z) = 1/z$ is not defined at the origin, but is differentiable everywhere else with $f'(z) = -1/z^2$, using the same calculation as in the real-variable case.

2.2 Holomorphic Functions

One might be tempted to only worry about the points where the Cauchy-Riemann equations hold, but the example of $f(z) = |z|^2$ shows that this is troublesome, since it is only differentiable at one point. We will therefore restrict our attention to *holomorphic functions*. A complex function $f : \mathbf{C} \to \mathbf{C}$ is called holomorphic at the point z if it is complex-differentiable in some open neighborhood of z. Thus, polynomials are examples of functions which are holomorphic everywhere in the complex plane (*entire* functions). But $f(z) = 1/z$ is holomorphic at every complex point except $z = 0$.

2.3 Power Series

It turns out that every holomorphic function can be expanded in a power series, which converges absolutely in any closed disk where the function is holomorphic. Since power series can be differentiated infinitely, a holomorphic function is also infinitely differentiable. More formally, if $f : \mathbf{C} \to \mathbf{C}$ is holomorphic at $w \in \mathbf{C}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{d^n f}{dz^n}(w) \frac{(z-w)^n}{n!}.$$

Suppose for simplicity that $w = 0$ (we can always examine the function $f(z-w)$ for example). Then, writing $z = re^{2\pi i\theta}$ in polar coordinates, we see that if r is fixed, f is a periodic function in θ. Thus, it has a Fourier expansion. But $z^n = r^n e^{2\pi i n\theta}$, so the Fourier expansion is really the power series expansion:

$$f(re^{2\pi i\theta}) = \sum_{n=0}^{\infty} \frac{d^n f}{dz^n}(0) \frac{r^n}{n!} e^{2\pi i n\theta}.$$

Note that since $n \geq 0$, being holomorphic is equivalent to having only positive frequencies contribute to the Fourier expansion.

2.4 Path Integration

Just as in multivariable calculus, it will be important to integrate holomorphic functions around contours. Let us start with the example of $f(z) = z^n, n \in \mathbf{Z}$. Let C be the curve $|z| = 1$, parametrized by $z(t) = e^{2\pi it}, 0 \leq t \leq 1$. Then $dz(t) = 2\pi i z dt$ and

$$\oint_C f(z)dz = \int_0^1 f(e^{2\pi it})(2\pi i)e^{2\pi it}dt = 2\pi i \int_0^1 e^{(2\pi it)(n+1)} = \left\{ \begin{array}{ll} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{array} \right. .$$

Note that $f(z)$ is holomorphic for $n \geq 0$, so the integral of a holomorphic function such as a polynomial, or even a power series which converges on a big disk containing C, is also zero:

$$\oint_{|z|=1} \sum_{n=0}^{\infty} a_n z^n dz = \sum_{n=0}^{\infty} a_n \oint_{|z|=1} z^n dz = 0.$$

This is an example of

Theorem 2.2 *(Cauchy) If f is holomorphic in some domain $\Omega \subset \mathbf{C}$, and $C \subset \Omega$ is a curve such that every point inside of C is also contained in Ω, then*

$$\oint_C f(z)dz = 0.$$

Remark 2.3 *The example $f(z) = 1/z$ is consistent with Cauchy's theorem, since it is not holomorphic at $z = 0$, a point inside of the curve $|z| = 1$.*

We will need to use Green's theorem in the proof:

Theorem 2.4 *(Green) If $D \subset \mathbf{R}^2$ is a region with boundary ∂D, oriented positively, and if P, Q are infinitely differentiable functions of x and y, then*

$$\int_{\partial D} (Pdx + Qdy) = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Proof of Cauchy's Theorem: Let D be the interior of C, so that $\partial D = C$. Expand

$$f(z)dz = f(z)(dx + idy) = f(z)dx + if(z)dy.$$

So let $P = f, Q = if$. The Cauchy Riemann equations state precisely that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

at any point of complex differentiability. Since f is holomorphic at every point of D, we have that

$$\oint_C f(z)dz = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0.$$

$\square$

2.5 Poles and other Singularities

Cauchy's theorem tells us that any path integral of z^n is zero so long as $n \geq 0$. But our calculation earlier tells us that any path integral of z^n (assuming of course that the path doesn't include the dangerous point $z = 0$) is also zero unless $n = -1$. The functions $z^{-k}, k > 0$ are examples of functions with singularities called *poles of order* k at the point $z = 0$. This is because if we multiplied them by z^k they would not increase to infinity as $z \to 0$. In fact, $g(z) = z^k f(z)$ would have a removable singularity, and it is a theorem that if we fix $g(z)$ by letting $g(0) = \lim_{z\to 0} g(z)$ it is not only continuous, but holomorphic at $z = 0$. It is clear that if we continued to multiply $f(z)$ by higher powers of z we would still get a holomorphic function. The *order* of the pole is the least value of k for which $z^k f(z)$ does not blow up to infinity as $z \to 0$. Finally, by translating the function $f(z)$ to $f(z-w)$ we can just as easily discuss poles at points other than $z = 0$.

So if $f(z)$ has a pole of order k at $z = 0$, $f(z) = \frac{g(z)}{z^k}$, for a function $g(z)$ which is holomorphic at $z = 0$, and satisfies $g(0) \neq 0$. We saw before that the Fourier expanion in θ of a holomorphic function has only positive frequencies. If there is a pole of order k, it only has frequencies $n \geq -k$. A function is called *meromorphic* if it is holomorphic in a region except that it has poles at isolated points.

There is yet another type of isolated singularity, the *essential singularity*. Here's the classical example. Recall that the function $f(z) = 1/z$ is holomorphic for $z \neq 0$. Also, $g(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is a power series with an infinite radius of convergence, so it uniformly converges to an entire function. If we compose the two, we get the function $h(z) = e^{1/z}$, which is holomorphic for $\{z \in \mathbf{C} \mid z \neq 0\}$. However, its power series is $h(z) = \sum_{n=0}^{\infty} \frac{1}{z^n}\frac{1}{n!}$, having negative powers for all $n < 0$. Thus, even if we multiply it by any z^k, it will still not be holomorphic at $z = 0$. This singularity is thus not a pole, and called essentially because it cannot be simply removed.

These examples suggest we widen our scope from power series to *Laurent series*: funtions of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Our calculations earlier about the path integrals of z^n show that if the Laurent series defining $f(z)$ converges in $0 < z < 2$, for example, then

$$\oint_{|z|=1} f(z)dz = \sum_{n\in\mathbf{Z}} a_n \oint_{|z|=1} z^n dz = 2\pi i a_{-1}.$$

We see that this coefficient a_{-1} completely determines the value of the integral. Because it is all that is left, we call it the *residue* of $f(z)$ at $z = 0$.

2.6 Example and Application to Factorization

If $f(z)$ is holomorphic in $|z| < 2$ (2 is arbitrary here, the point is simply that it is greater than 1 so that this domain contains the contour $|z| = 1$ and its interior), then

$$f(z) = \sum_{n=0}^{\infty} \frac{d^n f}{dz^n}(0)\frac{z^n}{n!}.$$

So $f(z)/z^k, k > 0$, has a pole of order k at $z = 0$ and its residue there is $\frac{d^{k-1}f}{dz^{k-1}}(0)\frac{1}{(k-1)!}$. This proves that

$$\frac{d^k f}{dz^k}(0) = \frac{k!}{2\pi i}\oint_{|z|=1} \frac{f(z)}{z^{k+1}}dz.$$

Amazingly, we can calculate derivatives by integrating.

Theorem 2.5 *(Liouville) Every bounded entire function is actually a constant function.*

Proof: We are given that there is some $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbf{C}$. By the formula above ("Cauchy's formula")

$$f'(w) = \oint_{|z-w|=R} \frac{f(z)}{(z-w)^2}dz$$

for any $R > 0$. Parametrize the contour by $z = w + Re^{2\pi it}, 0 \leq t \leq 1$. Then

$$f'(w) = 2\pi i R \int_0^1 f(w + Re^{2\pi it})R^{-2}e^{-4\pi it}dt,$$

$$|f'(w)| \leq \frac{2\pi}{R}\int_0^1 |f(w + Re^{2\pi it})|e^{-4\pi it}dt$$
$$\leq \frac{2\pi}{R}\int_0^1 M dt \leq M/R.$$

If $R \to \infty$ then this goes to zero, showing that $f'(z) = 0$ for any z, which means that $f(z)$ is constant.

Corollary 2.6 *(Fundamental Theorem of Algebra) If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with complex coefficients, then there exists complex roots $\alpha_1, \ldots, \alpha_n$ of this polynomial such that*

$$p(z) = a_n(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n).$$

Proof: By induction, it suffices to show that there is some complex point where $p(z)$ vanishes. If this is not true, then $f(z) = \frac{1}{p(z)}$ defines an entire function (by composition of $1/z$ and $p(z)$. Furthermore, since $|p(z)|$ increases as $|z|$ does, $f(z)$ is bounded, and hence a constant, meaning $p(z)$ is constant. Thus either $n = 0$ and $p(z) = a_n = a_0$ is constant to begin with, or else it really had a root. $\square$

2.7 Residues and the General form of Cauchy's Theorem

We only defined residues before at $z = 0$, but as always, the same notion can be extended everywhere in the complex plane. Formally, suppose that $f(z)$ is defined in a region $\Omega \subset \mathbf{C}$ and is holomorphic except at isolated singularities (this means that the points where $f(z)$ is not holomorphic can be separated from each other by small open balls). If $w \in \Omega$, then $f(z)$ has the Laurent series

$$f(z) = \sum_{n \in \mathbf{Z}} a_n (z - w)^n.$$

The a_n depend of course on w, and in fact, Cauchy's theorem tells us that

$$a_n = \oint_C f(z)(z-w)^{-1-n} dz$$

around a small circle C that encloses at most one of the isolated singularities, w (if it is indeed a singularity). We define the residue of $f(z)$ at $z = w$, denoted $res(f(z); w)$, as $a_{-1} = \oint_C f(z)dz$ above. Of course, if $f(z)$ is holomorphic at $z = w$ then $a_n = 0$ for $n < 0$ and the residue is zero.

Theorem 2.7 *(Extended form of Cauchy's Theorem) Let $\Omega \subset C$ be a region in the complex plane, and let $f : \Omega \to \mathbf{C}$ be a function which is holomorphic in Ω except at isolated singularities. If C is any closed curve contained in Ω, and f is holomorphic at all but a finite number of points called $w_1, \ldots, w_k$ inside the curve C, then*

$$\frac{1}{2\pi i} \oint_C f(z)dz = \sum_{j=1}^{k} res(f(z); w_j).$$

Proof: The idea is to shrink the original contour to the singular points $w_1, \ldots, w_k$. The integrals around each is just the residue, and summing these up recovers the theorem. It's like a linear combination of what we discussed before. $\square$

There are many applications of this theorem to contour shifts. For example, when you shift a contour through a region where the function is holomorphic, we already discussed how the integral doesn't change (join the two together to get a closed loop, where the integral is zero). If you hit a pole, the residue theorem tells you how to adjust. That can be used to justify the integral we did earlier in computing the Fourier transform of $e^{-\pi x^2}$. It thus enables quite complicated integrals to be computed by summing over just a handful of residues.