The Highest-Lowest Zero and other Applications of Positivity

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Abstract

The first nontrivial zeroes of the Riemann ζ function are $\approx \frac{1}{2} \pm 14.13472i$. We investigate the question of whether or not any other L-function has a higher lowest zero. To do so we try to quantify the notion that the L-function of a "small" automorphic representation (i.e. one with small level and archimedean type) does not have small zeroes, and vice-versa. We prove that many types of automorphic L-functions have a lower first zero than ζ's (see Theorems 1.1 and 1.2). This is done using Weil's explicit formula with carefully-chosen test functions. When this method does not immediately show L-functions of a certain type have low zeroes, we then attempt to turn the tables and show no L-functions of that type exist. Thus the argument is a combination of proving low zeroes exist and that certain cusp forms do not. Consequently we are able to prove vanishing theorems and improve upon existing bounds on the Laplace spectrum on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$. These in turn can be used to show that $SL_{68}(\mathbb{Z})\backslash SL_{68}(\mathbb{R})/SO_{68}(\mathbb{R})$ has a discrete, non-constant, non-cuspidal eigenvalue outside the range of the continuous spectrum on $L^2(SL_{68}(\mathbb{R})/SO_{68}(\mathbb{R}))$, but that this never happens for $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ in lower rank. Another application is to cuspidal cohomology: we show there are no cuspidal harmonic forms on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ for $n < 27$.

1 Introduction

The Riemann ζ function's first critical zeroes are surprisingly large: about $\frac{1}{2} \pm 14.13472i$. Our main interest in this paper is the following question:

> Does any other automorphic L-function have a larger first zero?

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This question was raised by Odlyzko ([O]), who proved that the Dedekind zeta function of any number field has a zero whose imaginary part is less than 14. Odlyzko also proved related conditional results for Artin L-functions.

Every automorphic L-function conjecturally factors into products of standard L-functions of cusp forms on GL_n over the rationals, and we shall be content to discuss these.[1] In fact, by twisting a cuspidal automorphic representation of $GL_n/\mathbb{Q}$ by a power of the determinant, it is possible to shift the zeroes any amount vertically, so we restrict ourselves to studying cuspidal automorphic representations $\pi_= \otimes_{p\leq\infty} \pi_p$ of $GL_n/\mathbb{Q}$ whose central character is normalized to be trivial. In most examples coming from number theory the archimedean type π_∞ is real, i.e. the gamma factors multiplying $L(s,\pi)$ have real shifts. Our first result answers the question for such cusp forms:

Theorem 1.1. *Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ with a real archimedean type and a trivial central character. Then $L(s,\pi)$ has a low zero which either (i) is on the critical axis between $\frac{1}{2} \pm 14.13472i$ or (ii) violates the generalized Riemann hypothesis (GRH) in an effective range.*

When we speak of a zero violating GRH "in an effective range," we mean that should conclusion (i) fail, then one could theoretically find an effective constant $T > 0$ such that the box $(\frac{1}{2}, 1) \times [-T, T]i$ contains a zero. For brevity we will use the following terminology:

Definition: An L-function has a *low zero* if it either vanishes on the critical axis between $\frac{1}{2} \pm 14.13472i$, or violates GRH in an effectively-bounded range (see Section 2.3).

We will use this definition to state unconditional results, but not much is actually gained philosophically or numerically in this problem by assuming GRH.

The L-functions in Theorem 1.1 include those of Dirichlet characters, rational elliptic curves, and conjecturally all rational abelian varieties. Of course they are also expected to include all Artin L-functions, for example L-functions of Galois representations. We have been unable to squeeze our technique to answer Odlyzko's question in full generality, but can prove many cases. For example:

[1]Nevertheless our arguments work under various wider assumptions, as they are mostly sensitive to the analytic properties of the L-function.

Theorem 1.2. *Let π be a cuspidal automorphic representation of GL_2 over $\mathbb{Q}$ with a trivial central character. Then $L(s, \pi)$ has a* low zero *(which is on the critical axis between $\frac{1}{2} \pm 14.13472i$ or else violates GRH in an effective range).*

This includes modular form and Maass form L-functions.

Other results can be proven about low zeroes. For example, every L-function which is related to itself by an odd functional equation automatically vanishes at $s = 1/2$. For a fixed degree n, most cuspidal automorphic representations of GL_n over $\mathbb{Q}$ with a trivial central character have low zeroes. In fact, the possible exceptions all lie in a bounded subset of the unitary dual and have bounded level. This subset tends to be devoid of cusp forms, which is why our method is successful. Thus Odlyzko's question is related to vanishing theorems about automorphic forms.

Our technique uses Weil's explicit formula relating the coefficients and zeroes of automorphic L-functions. It is a variation on the Stark-Odlyzko positivity technique, as formulated by Serre, Poitou, Mestre, and others – see [O] for a survey. In particular, one can compute an exact formula for sum of certain test functions over the critical zeroes. If we use a test function which is positive only in a certain range, then finding this sum is positive ensures a zero in that range. On the other hand, if this sum is negative, then we can often construct another test function which is positive in the critical strip, yet whose sum over the zeroes is negative. This contradiction shows that the L-function actually could not have existed to begin with. Our main difficulty is that it is often very difficult to construct this second test function given the failure of the first.

The latter contradiction, of positive terms yielding a negative sum, can be used to prove vanishing theorems about automorphic forms, since they cannot exist when their L-functions do not. Independent of our interest in low zeroes, this leads to applications in group cohomology and spectral theory.

Other applications

One of the consequences of the Ramanujan-Selberg temperedness conjecture is that the discrete cuspidal spectrum of the laplacian Δ on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ is contained in the continuous spectrum of Δ on $L^2(SL_n(\mathbb{R})/SO_n(\mathbb{R}))$. (We always normalize Δ so that this continuous spectrum is the interval $[\frac{n^3-n}{24}, \infty)$.) This consequence should be true more generally for congruence covers of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$, but in this particular case slightly more was proven in [M]:

Theorem 1.3. *([M]): There exists a constant $c > 0$ such that the Laplace eigenvalue of every cusp form ϕ on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ satisfies*

$$\lambda(\phi) > \lambda_1(SL_n(\mathbb{R})/SO_n(\mathbb{R})) + cn.$$

Our new result is superior for small n:

Theorem 1.4. *Let ϕ be a cuspidal eigenfunction of the non-euclidean laplacian Δ on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$. Then ϕ's Laplace eigenvalue satisfies*

$$\lambda(\phi) > \frac{n^3 - 4n}{24} + 25.92\left(1 + \frac{1}{n-1}\right). \tag{1}$$

It can be applied to answer a question of Alexander Lubotzky: when does the eigenvalue of a noncuspidal, square-integrable eigenfunction of the laplacian on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ lie outside $[\frac{n^3-n}{24}, \infty)$?

Theorem 1.5. *If $n \leq 67$, any non-constant eigenfunction of Δ in $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ has Laplace eigenvalue greater than $\frac{n^3-n}{24}$, but the first Laplace eigenvalue of $SL_{68}(\mathbb{Z})\backslash SL_{68}(\mathbb{R})/SO_{68}(\mathbb{R})$ is in fact approximately*

$$12906.6 < \frac{68^3 - 68}{24} = 13098.5.$$

Finally, we can apply our technique to cuspidal cohomology and extend a result in [M], where it was shown that $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ has no *harmonic* cuspidal automorphic forms for $n < 23$:

Theorem 1.6. *The constant-coefficients cuspidal cohomology of $SL_n(\mathbb{Z})$*

$$H^{\cdot}_{cusp}(SL_n(\mathbb{Z}); \mathbb{C}) = 0$$

vanishes for $1 < n < 27$.

The technique used to prove this theorem is related to the one in [M]. Fermigier [F] had a similar, but weaker, result using positivity with a different L-function. Here we combine both methods to go further.

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L-functions was influenced by the discussion in [RS]. Support was provided by National Science Foundation Graduate and Postdoctoral Fellowships and a Yale Hellmann fellowship during stays at Princeton University, Yale University, and the University of California at San Diego. All numerical computations were made with Mathematica v.3 on an Intel Pentium II 300 MHz system running Windows NT 4.0 and Slackware Linux 2.0.30.

2 L-functions and positivity

By conjectures of Langlands the most general automorphic L-function is a product of standard L-functions of cuspidal automorphic representations $\pi = \otimes_{p\leq\infty}\pi_p$ on GL_m over the rational adeles $\mathbb{A}_{\mathbb{Q}}$. These "primitive" L-functions are degree m Euler products

$$L(s,\pi) = \prod_{p \text{ prime}} \prod_{j=1}^{m}(1-\alpha_{p,j}p^{-s})^{-1} \ , \ \alpha_{p,j} \in \mathbb{C}$$

and have completions

$$\Lambda(s,\pi) = \prod_{j=1}^{m} \pi^{(-s+\eta_j)/2}\Gamma\left(\frac{s+\eta_j}{2}\right) L(s,\pi) \ , \ \eta_j \in \mathbb{C}$$

which are entire unless $m = 1$ and $L(s) = \zeta(s)$. We have used the duplication property of the gamma function in writing the gamma factors in this way. The conductor is D, and for π_p unramified, the $\alpha_{p,j}$ are Hecke eigenvalue parameters and the η_j are related to the archimedean parameters of π_∞. With this normalization $\Lambda(s,\pi)$ has the functional equation

$$\Lambda(s,\pi) = \tau_\pi D^{-s}\Lambda(1-s,\tilde{\pi}) \ , \ \tau_\pi \in \mathbb{C} \ , \ |\tau_\pi| = \sqrt{D} \ , \ D > 0,$$

where $\tilde{\pi}$ is the contragredient representation to π. The Jacquet-Shalika ([JS]) bounds imply that

$$\text{Re } \eta_j > -\frac{1}{2}. \tag{2}$$

2.1 Weil's formula

The explicit formula of André Weil equates a sum over the zeroes of an L-function with a sum over its coefficients and gamma factors:

$$\sum_{\Lambda(\frac{1}{2}+i\gamma,\pi)=0} h(\gamma) = 2\text{Re}\left\{\sum_{j=1}^{m} l(\eta_j) - \sum_{n=1}^{\infty}\frac{c_n}{\sqrt{n}}g(\log n)\right\} + g(0)\log D, \tag{3}$$

where g is an even, differentiable real function,

$$\hat{g}(r) = h(r) = \int_{\mathbb{R}} g(x) e^{irx} dx,$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

and

$$\begin{aligned} l(\eta) &= \frac{1}{2\pi}\int_{\mathbb{R}} h(r) \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}\left(\frac{1}{2} + \eta + ir\right) dr \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\left(\frac{-\log \pi}{2}\right) dr + \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'}{2\Gamma}\left(\frac{1}{4} + \frac{\eta}{2} + \frac{ir}{2}\right) dr \\ &= -\frac{\log \pi}{2} g(0) - \frac{1}{2}\int_0^\infty \left(\frac{g(x/2)e^{-(1/4+\eta/2)x}}{1-e^{-x}} - \frac{g(0)}{e^x x}\right) dx. \end{aligned}$$

Here we have made use of the fact that $L(s,\pi)$ is entire; for $\zeta(s)$ and Rankin-Selberg L-functions there is a polar term that will be introduced when needed later on. See [RS] for a proof of (3).

If g is supported in the interval $[-\log 2, \log 2]$ then the formula can be viewed as giving the value of the sum over the zeroes from the gamma factors:

$$\sum h(\gamma) = 2\mathrm{Re} \sum_{j=1}^{m} l(\eta_j) + g(0)\log D. \tag{4}$$

The basis of the positivity technique is the observation that if $h(\gamma) \geq 0$ for each zero, then the sum on the right-hand side of (4) must also be positive. This immediately gives a lower bound on the conductor D, which is the original application of the positivity technique. Fortunately the sum on the right-hand side of (4) is explicitly computable in terms of the η_j's and D; if it is negative then the L-function $L(s,\pi)$ cannot exist and hence neither can the original cusp form π.

Upon assuming GRH, let

$$\cdots \leq \gamma_{-2} \leq \gamma_{-1} \leq 0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$$

be the imaginary parts of the zeroes of $L(s,\pi)$. Let g and $h = \hat{g}$ be chosen so that $h \geq 0$ on $\mathbb{R}$ and let $c > 0$ be a cutoff parameter. Then the function $h_m(r) = h(r)(c^2 - r^2)$ is positive exactly when $|r| < c$ and is the Fourier transform of $g_m = c^2 g + g''$. The support of g_m is of course also contained in $[-\log 2, \log 2]$ provided g is suitably regular. If the sum $2\mathrm{Re}\ \sum_{j=1}^m l_m(\eta_j) + g_m(0)\log D$ in (4) is positive, then $\gamma_1 < c$ or $\gamma_{-1} > -c$, i.e. $L(s,\pi)$ has a small zero. To summarize:

2.2 Criteria

Our strategy will then be, for given archimedean parameters η_j and conductor D, to find a function g of support contained in $[-\log 2, \log 2]$ and for which either

$$2\mathrm{Re}\sum_{j=1}^{m} l(\eta_j) + g(0)\log D < 0$$

(which shows the L-function does not exist) or

$$2\mathrm{Re}\sum_{j=1}^{m} l_m(\eta_j) + g_m(0)\log D > 0$$

(which shows that it must have a *low zero* or violate GRH in an effective range, as discussed below).

2.3 What low zeroes mean without GRH

Even if we do not assume GRH, we may still conclude from

$$2\mathrm{Re}\sum_{j=1}^{m} l_m(\eta_j) + g_m(0)\log D > 0$$

that the sum

$$\sum h_m(\gamma) > 0.$$

Thus, there are zeroes $\rho = \frac{1}{2} + i\gamma$ in the region where $h_m(\gamma) > 0$. We can explicitly compute the functions h_m for our choices of g and examine where they are positive and negative within the critical strip. Since the density of zeroes increases only logarithmically with their height (with an effective constant), and our functions $h_m(z)$ decay polynomially as $z \to \infty$ in the critical strip, the zero must be contained in an effectively bounded region of the critical strip.

As an example, Figure 1 is a contour plot of the function h_{1m} defined at the end of Section 3. The white regions are where Re $h_{1m} > 0$, the black where Re $h_{1m} < 0$.

Figures 2 and 3 contain plots for the other functions we use.

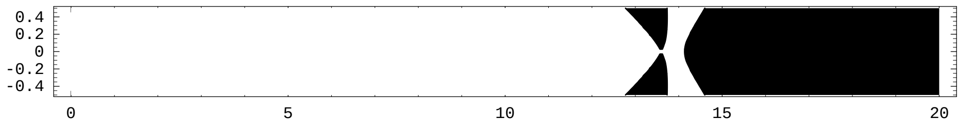

Figure 1: A contour plot of the function Re $h_{1m}(x+iy)$. We have colored the positive set white and the negative one black.

3 A library of functions

The main functions we use in this paper are

$$g_{1,p}(x) = \left(\frac{\left(\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{\pi x}{p})\right)+\sin(\frac{\pi|x|}{p})}{\pi}\right)/\cosh(x/2),$$

$$g_{2,p}(x) = \left(\frac{4\pi\left(1-\frac{|x|}{p}\right)+2\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{2\pi x}{p})+3\sin(\frac{2\pi|x|}{p})}{6\pi}\right)/\cosh(x/2),$$

and

$$g_{3,p}(x) = \frac{\left(\frac{54\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{\pi x}{p})+6\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{3\pi x}{p})+27\sin(\frac{\pi|x|}{p})+11\sin(\frac{3\pi|x|}{p})}{60\pi}\right)}{\cosh(x/2)}.$$

We have normalized $g_{j,p}(0)=1$ and will often write $g_j(x)=g_{j,\log(2)}(x)$. Ignoring the $\cosh(x/2)$'s temporarily, the functions $g_{j,p}$ are rescalings of the convolutions of $(\cos(\frac{\pi x}{2}))^j$ with itself. Without the $\cosh(x/2)$ term they would thus have a positive Fourier transform on the real line, and the $\cosh(x/2)$ term spreads the positivity into the critical strip. Were we to assume GRH we would not need it.

Lemma 3.1. *If an even function $g(x)$'s Fourier transform is positive on the real line, then the Fourier transform of $g(x)/\cosh(x/2)$ is positive in the strip $-\frac{1}{2} < Im\ r < \frac{1}{2}$.*

Proof: The Fourier transform of $\mathrm{sech}(x)$ is

$$\int_{-\infty}^{\infty}\frac{2}{e^x+e^{-x}}e^{irx}dx \quad = \quad \pi\mathrm{sech}(\pi r/2).$$

This has positive real part for $-\frac{1}{2} < \mathrm{Im}\ r < \frac{1}{2}$ and the Fourier transform converts multiplication into convolution, so the smeared $\widehat{g \cdot \mathrm{sech}}$ remains positive in this strip. □

We defined modified functions

$$g_m = \frac{c^2 g(x) + g''(x)}{c^2 g(0) + g''(0)},$$

which also have $g_m(0) = 1$. (Of course we multiplicatively normalize $g(0) = g_m(0) = 1$ to compare the explicit formulas from various test functions.) These are used for showing the presence of low zeroes, and since we do not assume GRH for this, we will actually use $g = g_{1,p}(x)\cosh(x/2)$, $g_{2,p}(x)\cosh(x/2)$, or $g_{3,p}(x)\cosh(x/2)$. Thus

$$g_{1m,p,c}(x) = \frac{\frac{\pi^2\left(-1+\frac{|x|}{p}\right)\cos(\frac{\pi x}{p})}{p^2} + \frac{\pi\sin(\frac{\pi|x|}{p})}{p^2} - \frac{c^2\left(-\left(\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{\pi x}{p})\right)-\sin(\frac{\pi|x|}{p})\right)}{\pi}}{c^2 - \frac{\pi^2}{p^2}}$$

$$g_{2m,p,c}(x) = \frac{\frac{c^2\left(4\pi\left(1-\frac{|x|}{p}\right)+2\pi\left(1-\frac{x}{p}\right)\cos(\frac{2\pi x}{p})+3\sin(\frac{2\pi|x|}{p})\right)}{6\pi} - \frac{\frac{8\pi^3\left(1-\frac{x}{p}\right)\cos(\frac{2\pi x}{p})}{p^2}+\frac{4\pi^2\sin(\frac{2\pi|x|}{p})}{p^2}}{6\pi}}{c^2 - \frac{4\pi^2}{3p^2}}$$

and

$$g_{3m,p,c}(x) =$$

$$\frac{c^2\left(54\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{\pi x}{p}) + 6\pi\left(1-\frac{|x|}{p}\right)\cos(\frac{3\pi x}{p}) + 27\sin(\frac{\pi|x|}{p}) + 11\sin(\frac{3\pi|x|}{p})\right)}{60\pi\left(c^2-\frac{9\pi^2}{5p^2}\right)} +$$

$$\frac{54\pi^3\left(-1+\frac{|x|}{p}\right)\cos(\frac{\pi x}{p}) + 54\pi^3\left(-1+\frac{|x|}{p}\right)\cos(\frac{3\pi x}{p}) + 81\pi^2\sin(\frac{\pi|x|}{p}) - 63\pi^2\sin(\frac{3\pi|x|}{p})}{60\pi p^2\left(c^2-\frac{9\pi^2}{5p^2}\right)}.$$

Since we are interested in finding zeroes in the range from $\frac{1}{2} \pm 14.13472i$, we will now take $c = 14.13472$ and write

$$g_{1m}(x) = g_{1m,\log(2),14.13472}(x),$$

$$g_{2m}(x) = g_{2m,\log(2),14.13472}(x),$$

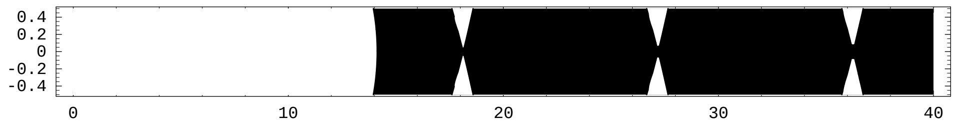

Figure 2: A contour plot of the function Re $h_{2m}(x+iy)$. We have colored the positive set white and the negative one black.

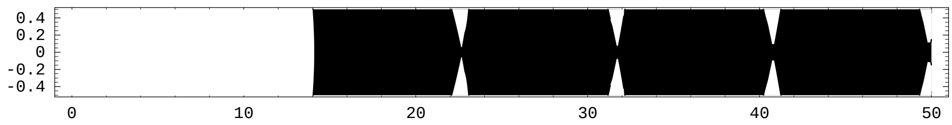

Figure 3: A contour plot of the function Re $h_{3m}(x+iy)$. We have colored the positive set white and the negative set black.

and

$$g_{3m}(x) = g_{3m,\log(2),14.13472}(x).$$

The Fourier transforms of these functions are

$$h_{1m}(r) = \frac{-8p^3\pi^2\left(c^2-x^2\right)\cos(\frac{px}{2})^2}{\left(-c^2p^2+\pi^2\right)(\pi-px)^2(\pi+px)^2},$$

$$h_{2m}(r) = \frac{128p\pi^4\left(c^2-x^2\right)\sin(\frac{px}{2})^2}{\left(3c^2p^2-4\pi^2\right)\left(-4\pi^2x+p^2x^3\right)^2},$$

and

$$h_{3m}(r) = \frac{2304p^3\pi^6\left(c^2-x^2\right)\cos(\frac{px}{2})^2}{\left(5c^2p^2-9\pi^2\right)\left(9\pi^4-10p^2\pi^2x^2+p^4x^4\right)^2}.$$

We show the contour plots of the functions h_{2m} and h_{3m} in Figures 2 and 3, the plot of h_{1m} having been presented above in Figure 1.

4 The highest lowest zero for π_∞ real

We restate

Theorem 4.1 (=1.1). *Let $\pi = \otimes_{p \leq \infty} \pi_p$ be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$ with a trivial central character and whose archimedean type π_∞ is real. Then $L(s, \pi)$ has a* low zero.

First we will note that for a fixed degree m, L-functions with large η_j's or large conductor D must have low zeroes. This is because Stirling's formula implies that

$$l(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}} \left(\frac{1}{2} + \eta + ir \right) dr$$

has a positive real part for η large. Thus, the lowest zero is only an issue for "small" archimedean parameters η_j and small conductor – partly because $l(\eta)$ is bounded from below in Re $\eta > -\frac{1}{2}$ (which we may assume by (2)).

We will present two different proofs of Theorem 1.1.

Picture Proof of Theorem 1.1: Figures 4, 5, and 6 indicate that $l_1(\eta) < l_{3m}(\eta)$ for $\eta \geq -\frac{1}{2}$, so the theorem follows from Criteria 2.2.

□

Less-Pictorial Proof of Theorem 1.1: This proof also relies on numerical computation, but demonstrates how a proof can be made even if the function l is not strictly less than the modified l_m. It uses l_{2m} instead of l_{3m}.

We noted before in Criteria 2.2 that if

$$2 \sum_{j=1}^{m} l_{2m}(\eta_j) + \log D \geq 0$$

then there is a indeed a *low zero*, while if

$$2 \sum_{j=1}^{m} l_1(\eta_j) + \log D \leq 0,$$

the L-function actually cannot exist to begin with. Thus we are reduced to dismissing the situation where

$$\sum_{j=1}^{m} l_{2m}(\eta_j) < 0$$

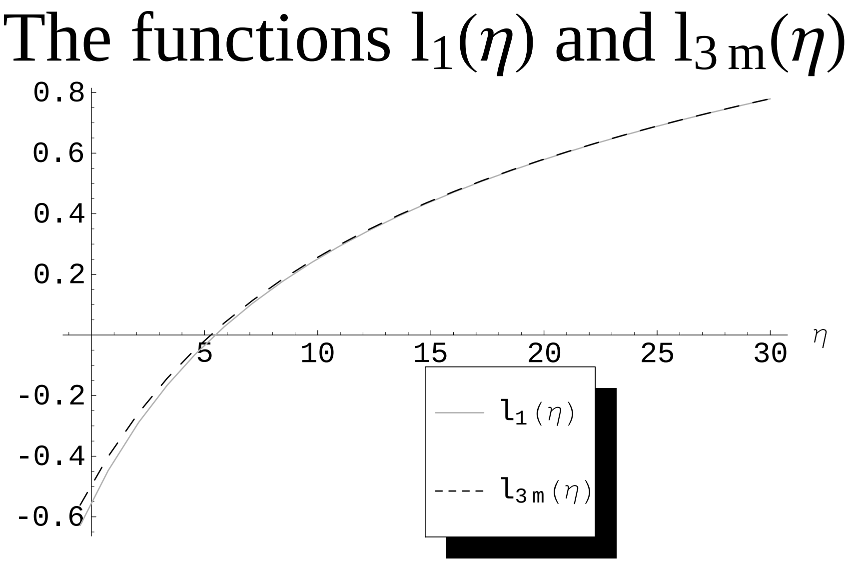

Figure 4: The functions $l(\eta)$ and $l_{3m}(\eta)$.

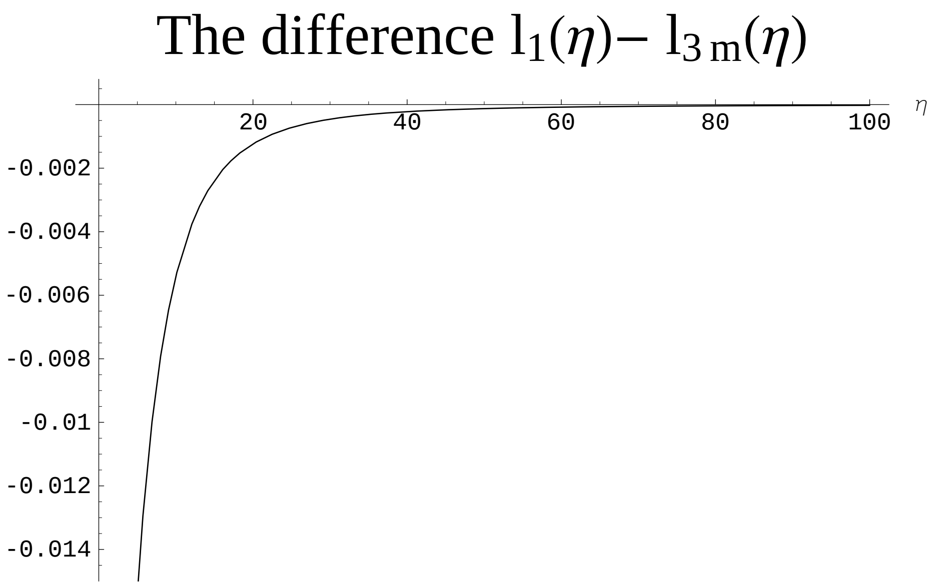

Figure 5: The difference between $l(\eta)$ and $l_{3m}(\eta)$

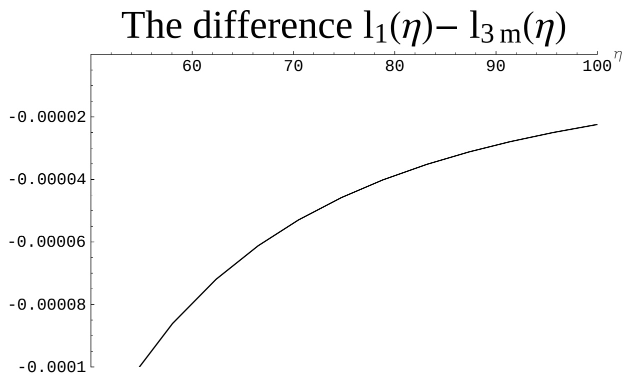

Figure 6: The difference between $l(\eta)$ and $l_{3m}(\eta)$, magnified.

and

$$\sum_{j=1}^{m} [l_1(\eta_j) - l_{2m}(\eta_j)] > 0$$

hold simultaneously. Partition the $\eta_j \in (-\frac{1}{2}, \infty)$ into 3 sets:

$$N = \{\eta_j \mid l_{2m}(\eta_j) \leq 0, l_1(\eta_j) - l_{2m}(\eta_j) \leq 0\} \quad = \quad (-\frac{1}{2}\ ,\ 5.4471\cdots],$$

$$S = \{\eta_j \mid l_{2m}(\eta_j) > 0, l_1(\eta_j) - l_{2m}(\eta_j) \leq 0\} \quad = \quad (5.4472\cdots\ ,\ 8.6553\cdots],$$

and

$$P = \{\eta_j \mid l_{2m}(\eta_j) > 0, l_1(\eta_j) - l_{2m}(\eta_j) > 0\} \quad = \quad (8.6553\cdots\ ,\ \infty).$$

Of course if

$$\sum_{j=1}^{m} l_{2m}(\eta_j) < 0,$$

then also

$$\sum_{\eta_j \in N \cup P} l_{2m}(\eta_j) < 0,$$

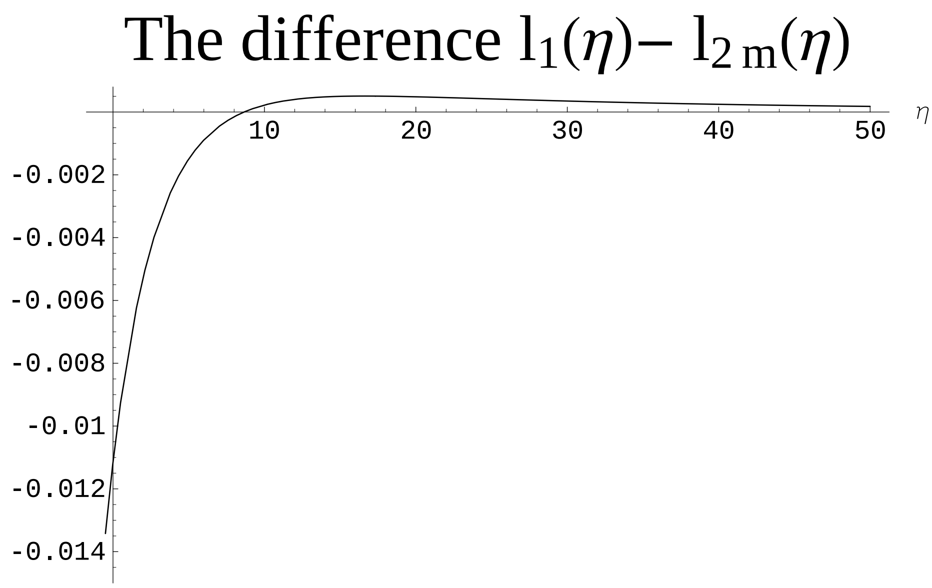

Figure 7: The difference between $l(\eta)$ and $l_{2m}(\eta)$.

and if

$$\sum_{j=1}^{m} [l_1(\eta_j) - l_{2m}(\eta_j)] > 0,$$

then

$$\sum_{\eta_j \in N \cup P} [l_1(\eta_j) - l_{2m}(\eta_j)] > 0$$

as well. Thus we need only consider the case where S is empty.

From computer investigations (see Figure 8) on the functions $l_1(\eta)$ and $l_{2m}(\eta)$ we can determine the following very precise information:

$$\eta_j \in N \Longrightarrow -.628291 \leq l_1(\eta_j) \leq 0 \quad , \quad l_1(\eta_j) - l_{2m}(\eta_j) \leq -.001201$$

and

$$\eta_j \in P \Longrightarrow l_1(\eta_j) \geq .187484 \quad , \quad 0 \leq l_1(\eta_j) - l_{2m}(\eta_j) \leq .0005801.$$

Thus

$$0 > \sum_{j=1}^{m} l_{2m}(\eta_j) = \sum_{\eta_j \in N} l_{2m}(\eta_j) + \sum_{\eta_j \in P} l_{2m}(\eta_j)$$

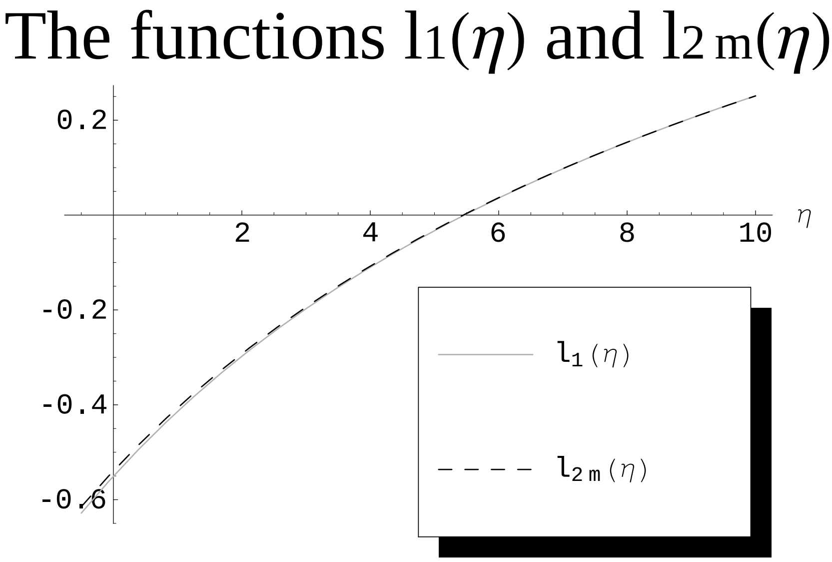

Figure 8: The graphs of $l(\eta)$ and $l_{2m}(\eta)$.

$$\geq |N|(-.628291) + |P|(.187484),$$

which implies

$$\frac{|N|}{|P|} < \frac{.187484}{.628291} = .298403.$$

On the other hand

$$0 < \sum_{j=1}^{m} [l_1(\eta_j) - l_{2m}(\eta_j)] = \sum_{\eta_j \in N} [l_1(\eta_j) - l_{2m}(\eta_j)] + \sum_{\eta_j \in P} [l_1(\eta_j) - l_{2m}(\eta_j)]$$

$$\leq |N|(-.001201) + |P|(.005801)$$

forces

$$\frac{|N|}{|P|} > \frac{.0005801}{.001201} = .483104,$$

a contradiction. $\square$

5 Low zeroes for modular form L-functions

In this section we prove that L-functions of cusp forms on GL_2 over $\mathbb{Q}$ have low zeroes:

Theorem 5.1 (=1.2). *Let π be a cuspidal automorphic representation of GL_2 over $\mathbb{Q}$ with a trivial central character. Then $L(s, \pi)$ has a* low zero.

Before giving the proof we shall give some background on the hardest case – Maass form L-functions. In particular we will precisely describe their completions, analytic continuations, and functional equations in some important cases.

5.1 Background on Maass forms on $\Gamma_0(p)\backslash\mathbb{H}$

It is known that if $D = p$ is a prime and π is a cuspidal automorphic representation not corresponding to a holomorphic modular form, then π instead corresponds to a Maass form ϕ on $\Gamma_0(p)\backslash\mathbb{H}$. The Laplace operator Δ, the Hecke operators $T_n, n \geq 0$, as well as the involutions

$$(T_{-1}f)(x+iy) = f(-x+iy)$$

$$(W_p f)(x+iy) = f\left(\frac{-1}{p(x+iy)}\right)$$

all commute. Thus, after diagonalizing, we may take a basis of Maass cusp forms on $\Gamma_0(p)\backslash\mathbb{H}$ which are joint eigenfunctions of Δ, T_n, T_{-1}, and W_p. Writing

$$\Delta\phi = \lambda\phi \quad , \quad \lambda = \frac{1}{4} - \nu^2,$$

ϕ has the Fourier expansion

$$\phi(x+iy) = \sum_{n\in\mathbb{Z}} c_n \sqrt{y} K_\nu(2\pi|n|y) e^{2\pi i n x},$$

where

$$K_\nu = K_{-\nu} = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^\nu \frac{dt}{t}$$

is the K-Bessel function of order ν. The cuspidality condition forces $c_0 = 0$; the involution T_{-1} interchanges c_n and c_{-n}.

There are four symmetry classes of Maass forms under the action of the involutions T_{-1} and W_p. The standard argument of Hecke and Maass to prove that the L-functions of cusp forms are entire also describes the functional equations of L-functions of Maass forms having various symmetries.

Proposition 5.2. *Suppose ϕ is a Maass form on $\Gamma_0(p)\backslash\mathbb{H}$ with*

$$T_{-1}\phi = (-1)^r\phi$$

and

$$W_p\phi = (-1)^\omega \phi \quad , \quad \tau, \omega = 0 \text{ or } 1.$$

Multiplicatively normalize the coefficients of ϕ *so that* $a_1 = 1$ *and*

$$\phi(x+iy) = \begin{cases} \sum_{n=1}^\infty a_n \sqrt{y} K_\nu(2\pi n y)\cos(2\pi n x), & \tau = 0 \\ \sum_{n=1}^\infty a_n \sqrt{y} K_\nu(2\pi n y)\sin(2\pi n x), & \tau = 1. \end{cases}$$

Then

$$\Lambda(s,\phi) = \Gamma_{\mathbb{R}}(s+\tau+\nu)\Gamma_{\mathbb{R}}(s+\tau-\nu)\sum_{n=1}^\infty \frac{a_n}{n^s}$$

satisfies the functional equation

$$\Lambda(s,\phi) = (-1)^{\tau+\omega} p^{1/2-s}\Lambda(1-s,\phi). \tag{5}$$

Proof: First consider the case $\tau = 0$. Then

$$\begin{aligned} \int_0^\infty \phi(iy) y^{s-1/2}\frac{dy}{y} &= \sum_{n=1}^\infty a_n \int_0^\infty K_\nu(2\pi n y) y^s \frac{dy}{y} \\ &= \sum_{n=1}^\infty a_n (2\pi n)^{-s}\left[\int_0^\infty K_\nu(y) y^s \frac{dy}{y}\right] \\ &= \sum_{n=1}^\infty a_n (2\pi n)^{-s}\left[2^{s-2}\Gamma(\frac{s+\nu}{2})\Gamma(\frac{s-\nu}{2})\right] \\ &= \frac{1}{4}\Lambda(s,\phi). \end{aligned}$$

The transformation property

$$\phi(iy) = (-1)^\omega \phi(\frac{i}{py})$$

gives

$$\begin{aligned} \Lambda(s,\phi) = 4\int_0^\infty \phi(iy) y^{s-1/2}\frac{dy}{y} &= 4(-1)^\omega \int_0^\infty \phi(\frac{i}{py}) y^{s-1/2}\frac{dy}{y} \\ &= 4(-1)^\omega \int_0^\infty \phi(\frac{iy}{p}) y^{1/2-s}\frac{dy}{y} \\ &= 4p^{1/2-s}(-1)^\omega \int_0^\infty \phi(iy) y^{1/2-s}\frac{dy}{y} \end{aligned}$$

$$= (-1)^{\omega} p^{1/2-s} \Lambda(1-s,\phi).$$

If instead $\tau = 1$ then actually $\phi(iy) = 0$ and we instead consider the derivative

$$\phi'(x+iy) := \frac{d}{dx}\phi(x+iy)$$

$$= \sum_{n=1}^{\infty} (2\pi n) a_n \sqrt{y} K_{\nu}(2\pi n y)\cos(2\pi n x).$$

The action under W_p now reads

$$\phi'(iy) = (-1)^{\omega}\phi'(\frac{i}{py^2})(\frac{-1}{py^2}).$$

We also have that

$$\int_0^{\infty} \phi'(iy) y^{s+1/2}\frac{dy}{y} = \sum_{n=1}^{\infty}(2\pi n)a_n \int_0^{\infty} K_{\nu}(2\pi n y) y^{s+1}\frac{dy}{y}$$

$$= \sum_{n=1}^{\infty} a_n (2\pi n)^{-s}\left[2^{s-1}\Gamma(\frac{s+1+\nu}{2})\Gamma(\frac{s+1-\nu}{2})\right],$$

and the functional equation for $\Lambda(s,\phi)$ follows as before. □

5.2 Low zeroes for Maass form L-functions

We will first prove Theorem 5.1 for Maass forms through a series of propositions.

Proposition 5.3. *Every Maass form L-function whose conductor satisfies*

$$D \geq 3 \quad , \quad \textit{if} \quad T_{-1}\phi = \phi$$

or

$$D \geq 2 \quad , \quad \textit{if} \quad T_{-1}\phi = -\phi$$

has a low zero*.*

Proof: In these two symmetry classes the gamma factors of $\Lambda(s,\phi)$ are either

$$\Gamma_{\mathbb{R}}(s+\nu)\Gamma_{\mathbb{R}}(s-\nu)$$

or

$$\Gamma_{\mathbb{R}}(s+1+\nu)\Gamma_{\mathbb{R}}(s+1-\nu),$$

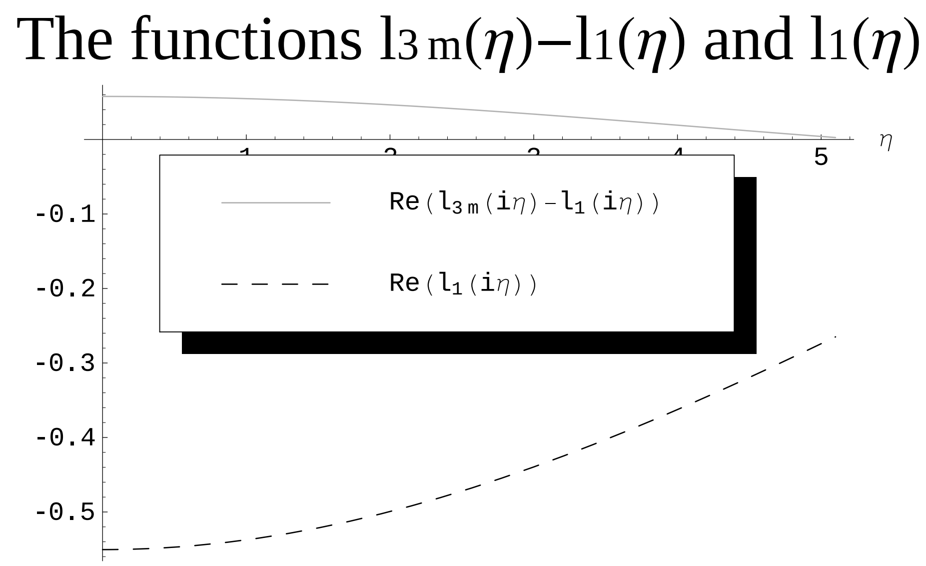

Figure 9: A plot showing that $\mathrm{Re}\, l_1(ir) + \frac{\log 3}{4} < 0$ and $\mathrm{Re}\ (l_{3m}(ir) - l_1(ir)) > 0$ for $-5.1 < r < 5.1$.

depending on whether ϕ is even or odd under T_{-1}. In each case we may assume the parameter ν is not real and hence purely imaginary, because Theorem 1.1 already covers the case of real archimedean type.

In the first case we have that

$$\mathrm{Re}\ (l_{3m}(ir) - l_1(ir)) > 0 \quad \text{if } -5.1 < r < 5.1,$$

a range in which $\mathrm{Re}\, l_1(ir) < -\frac{\log 3}{4} \approx -0.274653$ (see Figure 9).

In the second case

$$\mathrm{Re}\ (l_{3m}(1+ir) - l_1(1+ir)) > 0 \quad \text{if } -5.5 < r < 5.5,$$

where $\mathrm{Re}\, l_1(1+ir)$ and $\mathrm{Re}\, l_{3m}(1+ir)$ are both less than $-\frac{\log 2}{4}$ (see Figure 10). Criteria 2.2 thus shows there are *low zeroes* in either case. $\square$

This next proposition handles the case of Maass forms at full level (i.e. unramified for all primes $p < \infty$):

Proposition 5.4. *If ϕ is a Maass form on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ then $L(s, \phi)$ has a* low zero*.*

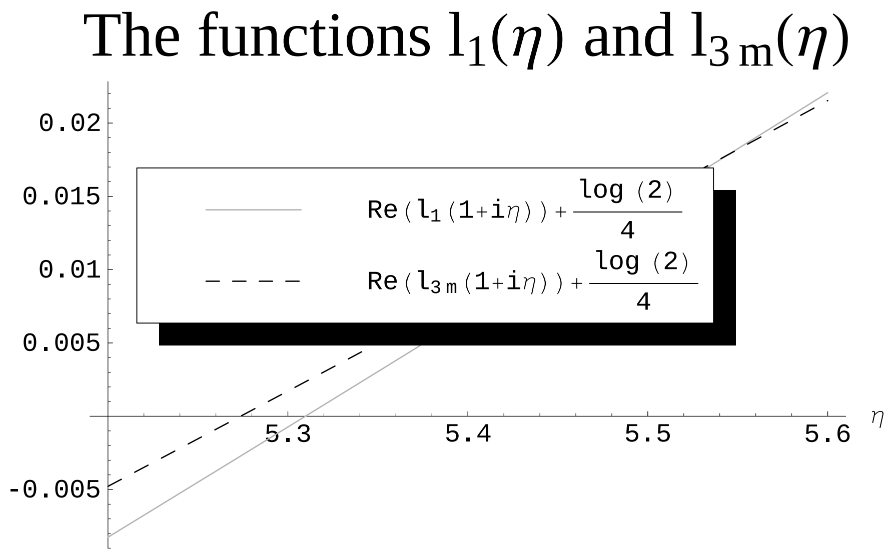

Figure 10: A plot showing that $\mathrm{Re}\, l_1(1+ir)+\frac{\log 2}{4}<0$ and $\mathrm{Re}\ (l_{3m}(1+ir)-l_1(1+ir))>0$ for $-5.5<r<5.5$.

Proof: We again break the proof up into two cases, according to whether ϕ is even or odd under T_{-1}. By Theorem 1.1 we need only consider the case $\mathrm{Re}\ \nu=0$.

If ϕ is even then the gamma factors of $L(s,\phi)$ are

$$\Gamma_{\mathbb{R}}(s+\nu)\Gamma_{\mathbb{R}}(s-\nu).$$

Figure 11 shows $\mathrm{Re}\ l_3(\nu)$ is negative when $\mathrm{Re}\ l_{1m}(\nu)$ is, which by Criteria 2.2 proves the proposition in this case.

If instead ϕ is odd the gamma factors are instead

$$\Gamma_{\mathbb{R}}(s+1+\nu)\Gamma_{\mathbb{R}}(s+1-\nu),$$

and similarly $\mathrm{Re}\ l_3(1+\nu)$ is negative when $\mathrm{Re}\ l_{1m}(1+\nu)$ is – see Figure 12. The proposition follows by invoking Criteria 2.2. $\square$

To handle the remaining case, of even Maass forms on $\Gamma_0(2)\backslash\mathbb{H}$, we will use a result about the smallest even eigenvalue of the laplacian there. Perhaps Proposition 5.6 below can be proven without such explicit information.

Proposition 5.5. *If ϕ is a Maass form on $\Gamma_0(2)\backslash\mathbb{H}$ which is even under both T_{-1} and W_2, then its Laplace eigenvalue exceeds $\frac{1}{4}+6.14^2$.*

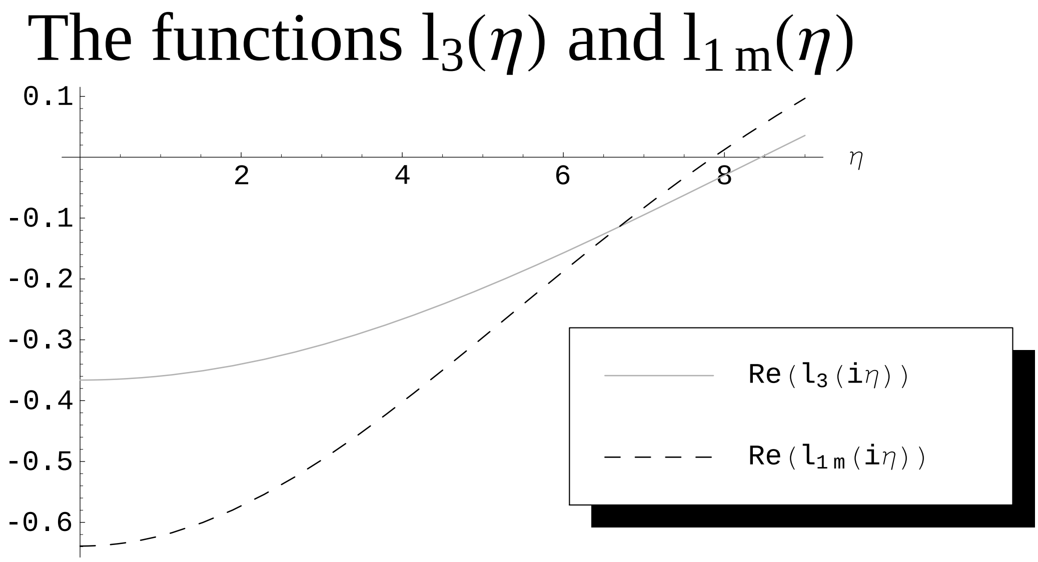

Figure 11: A plot showing that Re $l_3(\nu)$ is negative when Re $l_{1m}(\nu)$ is.

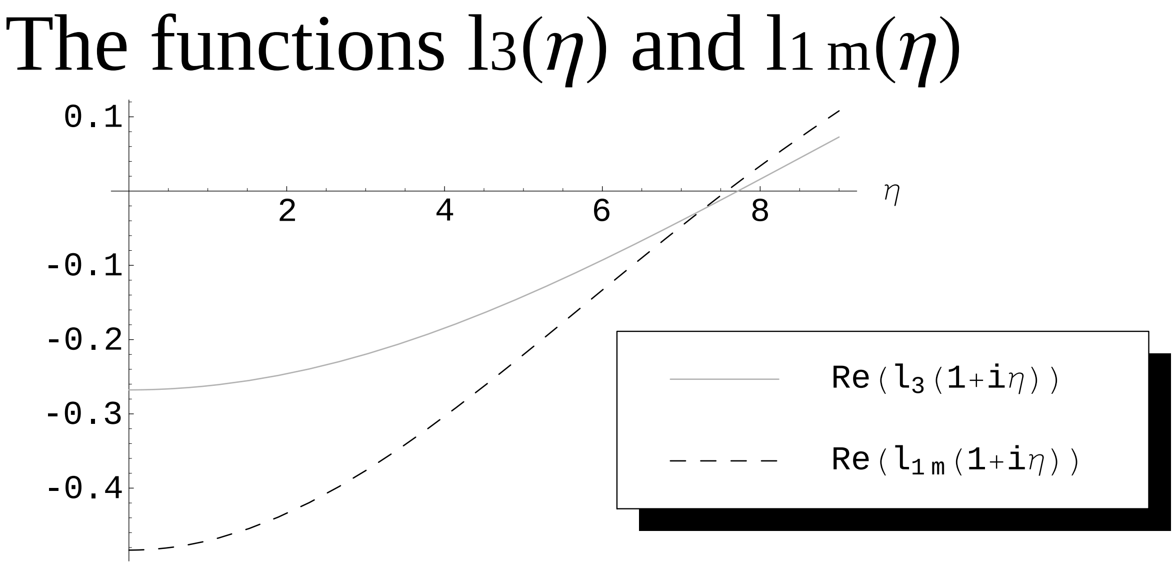

Figure 12: A plot showing that Re $l_3(1+\nu)$ is negative when Re $l_{1m}(1+\nu)$ is.

Hejhal [H] has numerically computed that the first such eigenvalue is $\approx \frac{1}{4}+8.922^2$. We present the following argument to demonstrate a technique.

Proof: First, Figure 13 shows that

$$\text{Re } l_1(ir) + \frac{\log 2}{4} < 0$$

for $-6.07 \le r \le 6.07$, so we need only consider the range $6.07 \le r \le 6.14$. Using the symmetries, set

$$f(t) = \phi\left(\frac{i}{\sqrt{2}}e^t\right) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} W_{ir}\left(\frac{n}{\sqrt{2}}e^t\right) = f(-t),$$

$$W_{ir}(y) = \sqrt{y} K_{ir}(2\pi y).$$

Thus f is an even function in t, and so

$$f'(0) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} W'_{ir}\left(\frac{n}{\sqrt{2}}\right)\frac{n}{\sqrt{2}} = 0 \tag{6}$$

and

$$f'''(0) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} \left[W'''_{ir}\left(\frac{n}{\sqrt{2}}\right)\frac{n^3}{2\sqrt{2}} + W''_{ir}\left(\frac{n}{\sqrt{2}}\right)\frac{3n^2}{2} + W'_{ir}\left(\frac{n}{\sqrt{2}}\right)\frac{n}{\sqrt{2}}\right]$$

$$=: \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} V_{ir}(n) = 0. \tag{7}$$

The terms in the Fourier expansion decay rapidly with n, and so we will use the first three terms as an approximation. Recall that we are focusing on the range $6.07 \le r \le 6.14$. We may assume that ϕ is a Hecke eigenform with $a_1 = 1$, and [BDHI] have proven that their coefficients satisfy the bound

$$|a_n| \le \tau(n) n^{5/28},$$

where $\tau(n)$ is the number of divisors of n. Using the crude bound $\tau(n) \le 2\sqrt{n}$ we can bound the tails

$$\left|\sum_{n=4}^{\infty} \frac{a_n}{\sqrt{n}} W'_{ir}\left(\frac{n}{\sqrt{2}}\right)\frac{n}{\sqrt{2}}\right| \le 1.14 \cdot 10^{-7} \tag{8}$$

and

$$\left|\sum_{n=4}^{\infty} \frac{a_n}{\sqrt{n}} V_{ir}(n)\right| \le 2.7 \cdot 10^{-5}. \tag{9}$$

Thus, (6) and (8) show

$$\left|W'_{ir}\left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}+\frac{a_2}{\sqrt{2}}W'_{ir}\left(\frac{2}{\sqrt{2}}\right)\frac{2}{\sqrt{2}}+\frac{a_3}{\sqrt{2}}W'_{ir}\left(\frac{3}{\sqrt{2}}\right)\frac{3}{\sqrt{2}}\right|\leq 1.14\cdot 10^{-7}$$

while (7) and (9) show

$$\left|V_{ir}(\frac{1}{\sqrt{2}})+\frac{a_2}{\sqrt{2}}V_{ir}(\frac{2}{\sqrt{2}})+\frac{a_3}{\sqrt{2}}V_{ir}(\frac{3}{\sqrt{2}})\right|\leq 2.7\cdot 10^{-5}.$$

Now, $|W'_{ir}(\frac{3}{\sqrt{2}})|\leq 2.5\cdot 10^{-6}$ in the range $6.07\leq r\leq 6.14$. Yet $W_{ir}(\frac{1}{\sqrt{2}})$ and $W_{ir}(\frac{2}{\sqrt{2}})\frac{2}{\sqrt{2}}$ are much larger, never smaller than $5.7\cdot 10^{-5}$ in magnitude. The ratio of

$$\frac{W'_{ir}(\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}}}{W'_{ir}(\frac{2}{\sqrt{2}})\frac{2}{\sqrt{2}}}$$

is smallest at $r=6.07$, where it is $\approx 1.475>1$. Thus, we must have that $\frac{a_2}{\sqrt{2}}>1$ for (6) to be valid.

At the same time, such a value of $\frac{a_2}{\sqrt{2}}$ is too large to achieve equality in (7). This is because it makes the second term much larger than the first and third terms could possibly be with the constraint that $a_3\leq 2\cdot 3^{5/28}$:

$$|V_{ir}(\frac{1}{\sqrt{2}})|+\frac{2\cdot 3^{5/28}}{\sqrt{3}}|V_{ir}(\frac{3}{\sqrt{2}})|<|V_{ir}(\frac{2}{\sqrt{2}})|\quad,\quad 6.07\leq r\leq 6.14.$$

So (6) and (7) cannot hold simultaneously. This contradiction shows every Maass form on $\Gamma_0(2)\backslash\mathbb{H}$ which is even under both T_{-1} and W_2 has Laplace eigenvalue greater than $\frac{1}{4}+6.14^2$. □

Proposition 5.6. *Maass form L-functions with conductor $D=2$ (which correspond to Maass forms on $\Gamma_0(2)\backslash\mathbb{H}$) have* low zeroes*.*

Proof: By Proposition 5.3 we need only consider the even Maass forms, where the gamma factors are

$$\Gamma_{\mathbb{R}}(s+\nu)\Gamma_{\mathbb{R}}(s-\nu).$$

In fact, by Proposition 5.2 we can assume that ϕ is even under both W_2 and T_{-1}; otherwise (5) dictates

$$\Lambda(\frac{1}{2},\phi)=-\Lambda(\frac{1}{2},\phi)=0.$$

The function Re $l_{1m}(ir)>-\frac{\log 2}{4}$ for $r>6.135$ (Figure 13), and Proposi-

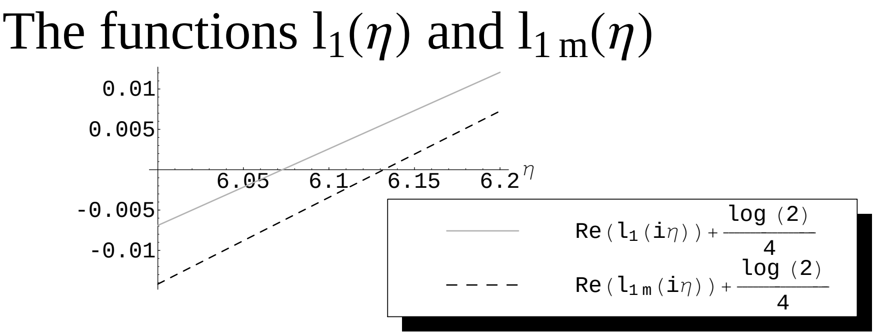

Figure 13: A plot showing that Re $l_{1m}(ir) > -\frac{\log 2}{4}$ for $r > 6.135$, while Re $l_1(ir) < -\frac{\log 2}{4}$ for $r < 6.07$

tion 5.5 shows all even eigenvalues are in that range. □

Proof of Theorem 5.1: Every cuspidal automorphic representation on GL_2 over $\mathbb{Q}$ comes from either a Maass form or a holomorphic modular form. Both holomorphic modular forms and non-tempered Maass forms (i.e. $\lambda < \frac{1}{4}$) have real archimedean type and are thus covered under Theorem 1.1. The rest of the Maass forms (the tempered ones) are covered by Propositions 5.3, 5.4, and 5.6. □

6 Cuspidal eigenvalue bounds

Now we move our focus completely towards automorphic representations rather than on their L-functions. In this section and in the next we will examine the discrete spectrum of the laplacian Δ on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$.[2] We normalize our laplacian so that its continuous spectrum on $L^2(SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ spans the interval from

$$\lambda_1(SL_n(\mathbb{R})/SO_n(\mathbb{R})) = \frac{n^3 - n}{24}$$

to ∞.

Because the ring of invariant differential operators $\mathcal{R}$ on $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ is commutative, we may take a basis of Laplace eigenfunctions which are

[2] Of course our methods carry over to some congruence covers but we will restrict our attention to full-level here.

also common eigenfunctions of the operators in $\mathcal{R}$. Thus, to each discrete eigenfunction $\phi \in L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ we can attach Langlands parameters $\mu_1, \ldots, \mu_n$. These describe ϕ's eigenvalues under the different operators in $\mathcal{R}$; in particular, the Laplace eigenvalue satisfies

$$\Delta\phi = \lambda\phi \quad , \quad \lambda = \frac{n^3 - n}{24} - \frac{\mu_1^2 + \cdots + \mu_n^2}{2}.$$

By the Jacquet-Shalika "trivial" bound [JS]

$$|\mathrm{Re}\ \mu_j| < \frac{1}{2} \quad , \quad j = 1, \ldots, n.$$

Thus,

$$\lambda > \frac{1}{2}\sum_{j=1}^{n}(\mathrm{Im}\ \mu_j)^2 + \frac{n^3 - \mathbf{4}n}{24}. \tag{10}$$

We will use (10) to bound λ from below.

Positivity Functions

Recall the function

$$g_{1,p} = \left((1 - \frac{|x|}{p})\cos\left(\frac{\pi x}{p}\right) + \frac{1}{\pi}\sin\left(\frac{\pi|x|}{p}\right)\right) / \cosh(x/2) \quad , \quad 0 < p \le \log 2.$$

Define

$$s(r) = \max \mathrm{Re}\ l_{1,\frac{1}{2}}(ir + \sigma),$$

where the maximum is taken over $\sigma \in [-\frac{1}{2}, \frac{3}{2}]$.

6.1 Criteria

If $\sum_{j=1}^{n} s(r_j) < 0$ then there is no cuspidal eigenfunction in $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ whose Langlands parameters $\mu_1, \ldots, \mu_n$ have $\mathrm{Im}\ \mu_j = r_j$.

If ϕ is a cusp form, then the *archimedean Ramanujan-Selberg conjectures* assert that π_∞ is tempered, i.e. $\mathrm{Re}\ \mu_j = 0$. A consequence is that $\lambda^{cusp} \ge \frac{n^3-n}{24}$. This was proven in [M] unconditionally using a similar positivity argument. Here we can derive some stronger results and different applications.

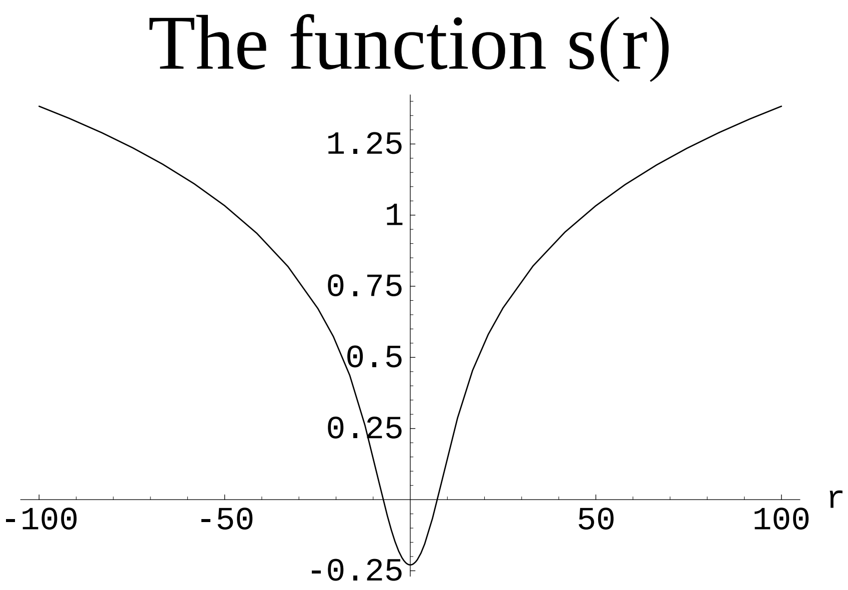

Figure 14: The graph of $s(r)$.

Proposition 6.1. *(A trivial bound) If ϕ is a cusp form in $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$, then with the above notation*

$$\sum_{j=1}^{n} r_j^2 > 51.84\left(1+\frac{1}{n-1}\right).$$

Proof: The plot shows $s(r) < 0$ for $|r| < 7.2$. Thus $\sum_{j=1}^{n} s(r_j) \geq 0$ only if at least one $|r_j| \geq 7.2$. Since the r_j are constrained to have $r_1+\cdots+r_n = 0$, this means

$$\sum_{j=1}^{n} r_j^2 \geq 7.2^2 + (n-1)\left(\frac{7.2}{n-1}\right)^2.$$

□

Theorem 1.4 follows immediately from Proposition 6.1 and (10).

6.2 Extreme values

Given d and the constraints $\sum r_j^2 = d, \sum r_j = 0$, if the largest value obtained by $\sum s(r_j)$ is negative, then Criteria 6.1 implies $\lambda > \frac{d}{2} + \frac{n^3-4n}{24}$.

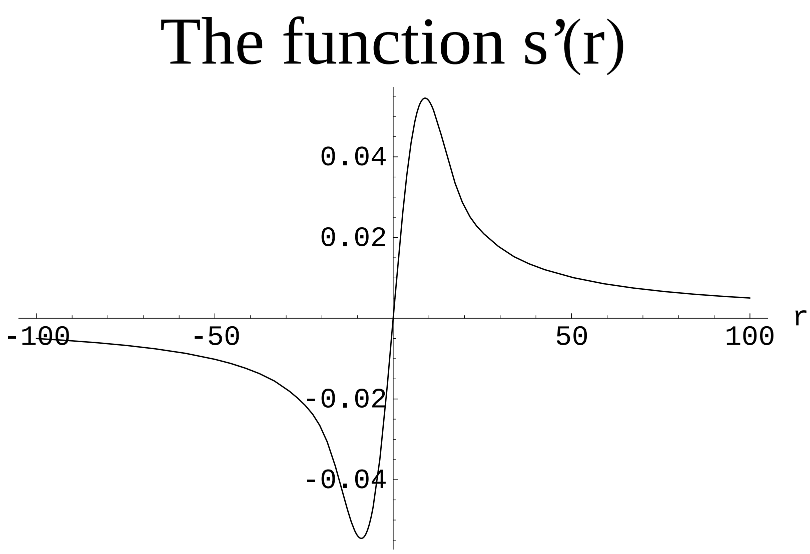

Figure 15: The graph of $s'(r)$.

Principle 6.2. [3] *If $(r_1, \ldots, r_n)$ is an extremal point of*

$$\sum_{j=1}^{n} s(r_j)$$

subject to the constraints

$$\sum_{j=1}^{n} r_j = 0 \quad , \quad \sum_{j=1}^{n} r_j^2 = d,$$

then the r_j assume at most three distinct values.

Proof: By Lagrange multipliers, there are real constants $c_1, c_2 \in \mathbb{R}$ such that

$$(s'(r_1), \ldots, s'(r_n)) = c_1(r_1, \ldots, r_n) + c_2(1, \ldots, 1),$$

i.e. the points $(r_j, s'(r_j))$ all lie on the intersection of some line and the graph of $y = s'(x)$.

But no line crosses this graph in more than three places. Even though Figure 15 only shows the range $|x| \leq 100$ it is legal to use this principle in this paper. For another crossing would give a value of r_j so large that it would not enter into our subsequent bounds. $\square$

[3]Some may not consider the justification to be a proof, but as we indicate, it can be verified in the applications we use it for.

Theorem 6.3. *We have the following bounds on the Laplace eigenvalue of a cuspidal eigenfunction of* Δ *in* $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$*:*

n	3	4	5	6	7	8
$\lambda \geq$	87.625	108.	140.875	167.	201.125	232.

Proof: By the last proposition, we need only consider the case where there are Ar_1's, Br_2's, and Cr_3's, with $r_1, r_2, r_3 \in \mathbb{R}$,

$$Ar_1 + Br_2 + Cr_3 = 0 \quad , \quad Ar_1^2 + Br_2^2 + Cr_3^2 = d,$$

and then try to find a large value of d such that

$$As(r_1) + Bs(r_2) + Cs(r_3)$$

is always negative. Since A, B, and C are all positive integers which sum to n, this is a finite calculation. We will take $A, B, C > 0$ by allowing some of the values of r_1, r_2, and r_3 to coincide. Then in terms of the parameter r_3, either

$$r_1 = -\frac{ACr_3 + \sqrt{AB\left(-C^2r_3^2 + A(D-C) + B(Dr_3^2 - Cr_3^2)\right)}}{A\,(A+B)}$$

and

$$r_2 = \frac{-\left(BCr_3\right) + \sqrt{AB\left(-C^2r_3^2 + A(D-Cr_3^2) + B(D-Cr_3^2)\right)}}{B\,(A+B)}$$

or instead

$$r_1 = \frac{-ACr_3 + \sqrt{AB\left(-C^2r_3^2 + A(D-C) + B(Dr_3^2 - Cr_3^2)\right)}}{A\,(A+B)}$$

and

$$r_2 = -\frac{\left(BCr_3\right) + \sqrt{AB\left(-C^2r_3^2 + A(D-Cr_3^2) + B(D-Cr_3^2)\right)}}{B\,(A+B)}.$$

Actually, the second set of solutions and the first are interchanged upon $r_3 \leftrightarrow -r_3$, so they take the same values. For a given n, we need only enumerate the integer triples of A, B, C with $A \geq B \geq C > 0$ and plot

$$As\left(-\frac{ACr_3 + \sqrt{AB\left(-C^2r_3^2 + A(D-C) + B(Dr_3^2 - Cr_3^2)\right)}}{A\,(A+B)}\right) +$$

$$Bs\left(\frac{-(BCr_3)+\sqrt{AB\left(-C^2r_3^2+A(D-Cr_3^2)+B(D-Cr_3^2)\right)}}{B\,(A+B)}\right)+Cs(r3)$$

over the range

$$-\frac{\sqrt{A+B}\sqrt{D}}{\sqrt{C}\sqrt{A+B+C}}\leq r_3\leq\frac{\sqrt{A+B}\sqrt{D}}{\sqrt{C}\sqrt{A+B+C}}.$$

One finds the following values of d work:

n	3	4	5	6	7	8
d	174	212	273	318	376	424

□

Remark 6.4. *Theorem 1.3 shows there exists a positive constant* $c > 0$ *such that*

$$\lambda^{cusp}-\frac{n^3-n}{24}>cn \quad , \quad n=1,2,\ldots.$$

The argument above gives a much better constant.

6.3 Some open problems about $\lambda_1^{cusp}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$

Conjecture 6.5. *Fix* $k = 1, 2, \ldots$ *and denote the* k*-th cuspidal eigenvalue of* Δ *on* $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ *as* $\lambda_k^{cusp}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$. *Then the sequence*

$$\left\{\frac{\lambda_k^{cusp}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))-\frac{n^3-n}{24}}{n}\mid n=1,2,\ldots\right\} \tag{11}$$

has a limiting distribution.

Questions 6.6. *Is the sequence in (11) also bounded from above as well as from below?*

7 Bounds on non-cuspidal eigenvalues

Lubotzky asked if the bound

$$\lambda\geq\frac{n^3-n}{24}$$

could also hold for the entire non-zero discrete spectrum of Δ on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$, i.e. not just for cusp forms alone. Although from the point of view of automorphic forms the cusp forms are most essential, the entire discrete spectrum enters into considerations in differential geometry. In fact, there are non-constant, non-cuspidal, square-integrable residues of Eisenstein series on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$ which are discrete Laplace eigenfunctions, and they are never tempered (that is, they violate Re $\mu_j = 0$). The first example of one on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ violating $\lambda \geq \frac{n^3-n}{24}$ occurs for $n = 68$:

Theorem 7.1. *There exists a discrete Laplace eigenfunction*

$$\phi \in L^2(SL_{68}(\mathbb{Z})\backslash SL_{68}(\mathbb{R})/SO_{68}(\mathbb{R}))$$

such that

$$\Delta\phi = \lambda_\phi \phi \quad , \quad \lambda_\phi \approx 12916.6 < \frac{68^3 - 68}{24} = 13098.5.$$

Yet for $n \leq 67$ the bound

$$\lambda \geq \frac{n^3 - n}{24}$$

is valid for every non-zero discrete eigenvalue of Δ on $L^2(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R}))$.

Of course the failure of $\lambda \geq \frac{n^3-n}{24}$ at $n = 68$ is the typical case for large n.

The key idea here is the classification of the discrete spectrum in terms of cusp forms. It was first conjectured by Jacquet [J] and later proven by Mœglin-Waldspurger [MW]. Let us now describe how discrete eigenfunctions can be constructed. Factor $n = ra$ and let ϕ be a cusp form on $SL_a(\mathbb{Z})\backslash SL_a(\mathbb{R})/SO_a(\mathbb{R})$. The group $SL_n(\mathbb{R})$ has a rank $r-1$ parabolic subgroup P of type $(a, a, \ldots, a)$ whose Levi component is

$$L = GL_a(\mathbb{R})^r \cap SL_n(\mathbb{R}).$$

The cusp form ϕ extends as a product to the r copies of $SL_a(\mathbb{R})$ in L in the obvious way. Given $h = (h_1, \ldots, h_r) \in \mathbb{C}^r$ such that $h_1 + \cdots + h_r = 0$, we can form a character of the split Levi component A of P, and the Eisenstein series $E(P, g, \phi, h)$. If ϕ has Langlands parameters $\mu_1, \ldots, \mu_a$, then $E(P, g, \phi, h)$ has Langlands parameters

$$\underbrace{(\overbrace{\mu_1 + h_1, \mu_2 + h_1, \ldots, \mu_a + h_1}^{a}, \mu_1 + h_2, \ldots\ldots\ldots, \overbrace{\mu_1 + h_r, \ldots, \mu_a + h_r}^{a})}_{ra=n}.$$

Furthermore, $E(P, g, \phi, h)$ has a pole of order $r-1$ at $h = (\frac{r-1}{2}, \frac{r-3}{2}, \ldots, -\frac{r-1}{2})$ and its $r-1$st iterated residue there is a discrete, L^2 eigenfunction of Δ.

Moreover, all of them arise this way. For example, if $r = 1$ these are just cusp forms, and if $a = 1$, constant functions.

We compute that the residue's Laplace eigenvalue is

$$2\lambda - \frac{n^3 - n}{12} = -\sum_{j=1}^{a}\sum_{k=1}^{r}\left(\mu_j + \frac{r-1}{2} - k\right)^2$$
$$= -r\sum_{j=1}^{a}\mu_j^2 - a\frac{r^3 - r}{12}.$$

Incidentally, Maass forms with Laplace eigenvalue $\frac{1}{4}$ are known to exist on congruence quotients of $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Using this procedure one may already construct a discrete residue on a congruence quotient of $SL_4(\mathbb{R})/SO_4(\mathbb{R})$ which violates the $\lambda \geq \frac{4^3-4}{24}$ bound.

Proof of Theorem 7.1: Firstly, Hejhal (see [H]) has computed that $\lambda_1^{cusp}(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})) = 91.1413\cdots$, corresponding to $\mu_1 = -\mu_2 \approx 9.534i$. Thus the Laplace eigenvalue of a residue formed from Hejhal's Maass form has

$$2\lambda - \frac{n^3 - n}{12} \approx r(181.8) - 2\frac{r^3 - r}{12},$$

and this difference is positive for

$$r < \sqrt{6 \cdot 181.8 + 1} \approx 33.04.$$

For $r = 34$ we have

$$\lambda \approx 12916.6 < \frac{68^3 - 68}{24} = 13098.5.$$

If in fact there was an example of a residue for $n = ra < 68$, with

$$-r\sum_{j=1}^{a}\mu_j^2 - a\frac{r^3 - r}{12} < 0,$$

we would necessarily have

$$r > \sqrt{-\frac{12}{a}\sum_{j=1}^{a}\mu_j^2 + 1}. \tag{12}$$

We already know that $r > 1$ since cusp forms obey the $\lambda \geq \frac{n^3-n}{24}$ bound. Thus we can restrict to the cases $r \geq 2, a = 1, \ldots, 34$. Using our pre-existing bounds for $\sum \mu_j^2$ we conclude $ra \geq 68$ – see Table 1 for details.

□

a	$r \leq [\frac{68}{a}]$	Lower bound for $-\sum_{j=1}^{a} \mu_j^2$	Upper bound for $-\sum_{j=1}^{a} \mu_j^2$
3	22	171.25	120.75
4	17	211	96.00
5	13	271.75	70.
6	11	316.5	60.
7	9	374.25	46.66
8	8	422	42.00
9	7	56.07	36.
10	6	55.10	29.16
11	6	54.27	32.08
12	5	53.55	24.
13	5	52.91	26.
14	4	52.32	17.5
15	4	51.79	18.75
16	4	51.29	20.
17	4	50.83	21.25
18	3	50.38	12.
19	3	49.96	12.66
20	3	49.56	13.33
21	3	49.18	14.
22	3	48.80	14.66
23	2	48.44	5.75
24	2	48.09	6.
25	2	47.75	6.25
26	2	47.41	6.5
27	2	47.08	6.75
28	2	46.76	7.
29	2	46.44	7.25
30	2	46.12	7.5
31	2	45.81	7.75
32	2	45.51	8.
33	2	45.20	8.25
34	2	44.91	8.5

Table 1: This table completes the proof of Theorem 7.1. Suppose a residue of a cusp form on GL_a occurred on some $GL_n, n = ra < 68$ with Laplace eigenvalue $\leq \frac{n^3-n}{24}$. The second column gives the upper bound $r \leq [\frac{68}{a}]$. The third column gives a lower bound for $-\sum_{j=1}^{a} \mu_j^2$ (from Proposition 6.1), but the fourth gives an upper bound on $-\sum_{j=1}^{a} \mu_j^2$ that would be satisfied by such a residue with low eigenvalue (as derived in (12) in the proof of Theorem 7.1). The inconsistency of these two inequalities is a contradiction which shows that the discrete Laplace spectrum on $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})/SO_n(\mathbb{R})$ is contained in $\{0\} \cup [\frac{n^3-n}{24}, \infty)$ for $n < 68$.

8 Cuspidal cohomology

The positivity inequality can be applied to products of L-functions which have poles, for example Rankin-Selberg L-functions $L(s, \pi \otimes \tilde{\pi})$ of cuspidal automorphic forms π on GL_n. If $\{\mu_{jk}\}_{j=1,k=1}^{m}$ are the archimedean $\Gamma_{\mathbb{R}}$ parameters, the inequality reads

$$\int_{\mathbb{R}} g(x) \left(e^{x/2} + e^{-x/2}\right) dx + 2\mathrm{Re} \sum_{j=1}^{m} \sum_{k=1}^{m} l(\mu_{jk}) + g(0) \log D \geq 0. \qquad (13)$$

The new term in (13) as compared to (4) comes from the poles of $L(s, \pi \otimes \tilde{\pi})$. Also, here we have simply dropped the coefficients entirely because $\frac{L'}{L}(s, \phi)$ has a Dirichlet series with non-positive coefficients (see [RS] for a verification of this) and so there is no restriction on the support of g.

If $\pi = \otimes_{p \leq \infty} \pi_p$ comes from a constant-coefficients cohomological cusp form on $GL_n(\mathbb{A}_{\mathbb{Q}})$ then π_∞ is of either the form

$$\pi_\infty = \mathrm{Ind}_{P_{(2,2,\ldots,2)}}^{GL_n} (D_2, D_4, \ldots, D_n), \ n \text{ even},$$

or

$$\pi_\infty = \mathrm{Ind}_{P_{(1,2,2,\ldots,2)}}^{GL_n} (\mathrm{sgn}(\cdot)^\epsilon, D_3, D_5, \ldots, D_n), \ n \text{ odd}.$$

(sgn is the sign character, $\epsilon = 0$ or 1, and D_k denotes the k-th discrete series on GL_2, corresponding to weight k holomorphic forms.) Thus, if n is written as $2m + t$, $t = 0$ or 1, the archimedean μ_{jk} can be computed via the recipe summarized in [RS] and are the following multisets:

$$\begin{aligned} \{\mu_{j,k}\} \ &= \ \{t + j + k, t - 1 + j + k, |k - j|, 1 + |k - j| \mid 1 \leq j, k \leq m\} \\ \cup \ & \qquad \underbrace{\{0, j, j, j + 1, j + 1 \mid 1 \leq j \leq m\}}_{\text{omit if } t = 0}. \end{aligned} \qquad (14)$$

Theorem 8.1.

$$H^{\cdot}_{cusp}(SL_n(\mathbb{Z}); \mathbb{R}) = 0, \ 1 < n < 27.$$

Proof: Let

$$g_p(x) = \left((1 - \frac{|x|}{p}) \cos(\frac{\pi x}{p}) + \frac{1}{\pi} \sin(\frac{\pi |x|}{p}) \right) / \cosh(x/2).$$

Then $h_{1,p}(r) = \hat{g}_{1,p}(r)$ is positive in the critical strip $|\mathrm{Im}\ r| < \frac{1}{2}$. For our cohomological forms $D = 1$ at full level and with the μ_{jk}'s as above we arrive at a contradiction to the positivity inequality (see Table 2). $\square$

n	t	LHS(13)	n	t	LHS(13)	n	t	LHS(13)	n	t	LHS(13)
2	3.	-2.821	15	6.	-111.4	9	6.	-71.43	22	6.	-77.30
3	6.	-8.113	16	6.	-112.1	10	6.	-80.27	23	6.	-64.06
4	6.	-17.02	17	6.	-112.4	11	6.	-89.68	24	6.	-46.70
5	6.	-28.30	18	6.	-109.2	12	6.	-96.45	25	6.	-28.18
6	6.	-38.51	19	6.	-105.4	13	6.	-103.4	26	6.	-5.388
7	6.	-50.30	20	6.	-97.87	14	6.	-107.5			

Table 2: The (numerical) proof of the cohomology theorem. The left-hand side of (13), $\int_{\mathbb{R}} g(x)\left(e^{x/2}+e^{-x/2}\right)dx + 2\mathrm{Re}\ \sum_{j=1}^{m}\sum_{k=1}^{m} l(\mu_{jk}) + g(0)\log D$ must be positive if the cusp form exists, and this table shows it is negative for $n < 27$.

Remarks

We proved this for $n < 23$ in [M] with the Rankin-Selberg L-functions but without Weil's formula (instead using the Mittag-Leffler expansion). Fermigier [F] proved a weaker result using Weil's formula but with the standard L-function. The above theorem surpasses both.

Nothing is known about these cuspidal Betti numbers for $n \geq 27$, let alone if they ever non-zero.

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