Dear Martin,

You asked about the one-variable Rankin-Selberg integral. Actually, it involves a limit: It uses the fact that horocycles are equidistributed on $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

<u>Thm</u> Let f be a smooth function on $\mathbb{H}$ which is invariant under $SL_2(\mathbb{Z})$ & which decays rapidly ($O(y^{-N})$, any $N>0$) in the fundamental domain $\mathcal{F}=\{x+iy \mid -\frac{1}{2}\le x\le \frac{1}{2},\ |x+iy|\ge 1\}$. Then

$$\lim_{y\to 0} \frac{\pi}{3}\int_0^1 f(x+iy)\,dx = \int_{\mathcal{F}} f(x+iy)\frac{dx\,dy}{y^2}. \quad (*)$$

Weaker conditions on f may be used.

To prove this "horocyclic equidistribution" theorem, which is a first case of modern results of Margulis, Ratner, etc... on unipotent flows, we follow Weyl's method. Since $f\in L^2(SL_2(\mathbb{Z})\backslash\mathbb{H})$, f has a spectral expansion

$$f = \langle f,\varphi_0\rangle\varphi_0 + \sum_{j=1}^{\infty}\langle f,\varphi_j\rangle\varphi_j + \frac{1}{4\pi}\int_{\mathbb{R}}\langle f, E_{\frac{1}{2}+it}\rangle E_{\frac{1}{2}+it}\,dt,$$

where φ_0 = the constant function $3/\pi$

$\varphi_1,\varphi_2,\ldots$ are Maass cusp forms.

& $E_s(z) = \frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)=1}}\left(\frac{y}{|mz+n|^2}\right)^s$

is the standard nonholomorphic Eisenstein series.

It suffices to show (*) for each of these basis functions. First, (*) is obvious for constants (the $\pi/3$ is to normalize the measures, as $\int_{\mathcal{F}} dx\,dy/y^2 = \pi/3$). For the Maass cusp forms,

$$\int \varphi(x+iy)\,dx = 0 \quad \text{for all } y$$

by the definition of cuspidality. This is fortuitous, since these Maass forms are quite mysterious & transcendental, and allow us little to attack them with. Finally, the theory of the constant term tells us

$$\int_0^1 E_s(x+iy) = y^s + \frac{\pi^{-(1-s)/2}\,\Gamma(\frac{1-s}{2})\,\zeta(1-s)}{\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)}\, y^{1-s}$$

$$= O(y^{1/2}) \quad \text{for } \operatorname{Re} s = 1/2.$$

Thus the theorem is proved. Note that the only effect of $SL_2(\mathbb{Z})$ itself appears through $\zeta(s)$ in the constant term of the Eisenstein series. In fact, Sarnak & Zagier have explained how the reformulate RH as a statement about the rate of equidistribution.

Anyway, back to Rankin-Selberg. The L-function integral is given by

$$\int_{\mathcal{F}} H(x+iy)\,E_s(x+iy)\,\frac{dx\,dy}{y^2},$$

where

$$H(z) = f_1(z)\overline{f_2(z)}\,y^k,$$

if f_1 & f_2 are holom cusp forms of wt. k

or $= \psi_1 \psi_2$ if ψ_1, ψ_2 are Maass cusp forms.

(One should also assume H is not odd, $-H(z) \neq H(-z)$ for some z or else the integral vanishes.)

Now H is smooth & of rapid decay, & hence $H(x+iy)\,E_s(x+iy)$ is also, and since automorphic, satisfies our theorem.

The Rankin-Selberg integral is thus

$$\lim_{y \to 0} \frac{\pi}{3} \int_0^1 H(x+iy)\,E_s(x+iy)\,dx,$$

-- Stephen Miller

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