Notes for Graduate Course “Graphs, Spectral Theory, and Modular Forms” Yale University, Spring 2001

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These are rough slides of the topics/arguments presented in my graduate course on "Graphs, Spectral Theory, and Modular Forms" at Yale University in the Spring of 2001.

The goal was to study the spectra of the graph laplacian on k-regular graphs, in particular certain Cayley graphs introduced by Audrey Terras. From these we deduced a non-trivial bound for Kloosterman sums:

$$\left|\sum_{x=1}^{p-1} e^{2\pi i(ax+bx^{-1})/p}\right| \leq 2^{1/4}p^{3/4}, \quad p \text{ prime}, p \nmid a, b.$$

This is enough to prove non-trivial bounds towards the Ramanujan and Selberg conjectures for $GL(2)$, which in turn have been used to construct expander graphs. Later we studied the connection, due to Selberg, between Kloosterman sums and spectral problems on hyperbolic surfaces. The course concluded with equidistribution of horocycles and constant terms of Eisenstein series (some notes on this can be found on my webpage).

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OUTLINE OF COURSE

1. Study Terras' "Euclidean Graphs."

2. Show from them

$$|K\ell(a,b;p)| \ll p^{3/4}, p \nmid a,b$$
$$K\ell(a,b;p) = \sum_{j\in(\mathbb{Z}/p)^*,\ j\bar{j}\equiv 1(p)} e^{\frac{2\pi i}{p}(aj+b\bar{j})}$$

for Kloosterman sums, thus circumventing Weil's deep work showing $|K\ell| \leq 2\sqrt{p}$, which is sharp. (Note the $p^{3/4}$ bound is elementary and was originally proven by Kloosterman in the 20's). Our proof is just a geometrical rearrangement. There already is a *DEEP* algebraic-geometric proof.

3. Prove Selberg's formula relating the poles of the "Kloosterman Zeta Function"

$$\sum_{c=1}^{\infty} \frac{K\ell(m,n,c)}{c^{2s}}$$

and relatives to the Laplace spectrum on a modular curve.

4. Deduce from 3 and 2 that

$$\lambda_1 \geq \frac{1}{4} - \frac{3}{8^2} = \frac{7}{64} > 0,$$

enough for Property τ & expansion.

Property τ

for $SL_2(\mathbb{Z})$ with respect to congruence subgroups:

$$\inf\{\lambda_1(\Gamma\backslash\mathbb{H}) \mid \Gamma \text{ congruence}\} > 0.$$

Definitions

$$SL_2(\mathbb{Z}) \supseteq \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; n|b, c, a-1, d-1 \right\}$$

$$= \text{“}N\text{-th principal congruence sgp”}$$

$$= \text{Kernel: } SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N),$$

hence normal

$$\Gamma(1) = SL_2(\mathbb{Z})$$

$\Gamma \subseteq \Gamma(1)$ is *congruence* if it contains *some* $\Gamma(N), N > 0$.

Not all finite index subgroups of $SL_2(\mathbb{Z})$ are congruence, but for $SL_n(\mathbb{Z}), n \geq 3$ they are.

Modular forms are forms/functions on the "modular curves"

$$X(N) = \Gamma(N)\backslash\mathbb{H}$$

$$\mathbb{H} = \{x + iy \mid y > 0\}$$

$$f(\gamma z) = \theta(\gamma)f(z)$$

$$\theta = ``multiplier/cocycle''$$

$$E.g. \quad \Delta(x+iy) = \Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}$$

Ramanujan conjectured

$$|\tau(p)| \leq 2p^{11/2}, p \text{ prime}$$

$$\text{for composite } n \quad |\tau(n)| \ll_\epsilon n^{11/2+\epsilon}, \quad \epsilon > 0$$

Example of Modularity:

$$\Delta(\frac{az+b}{cz+d}) = (cz+d)^{12}\Delta(z)$$

$\uparrow$

weight

Natural Ramanujan conjecture:

$\Sigma a(n)q^n$ holomorphic form of weight k

$$|a(p)| \leq 2p^{(k-1)/2}$$

e.g. for elliptic curves ($k = 2$) agrees with Hasse's bound $|a(p)| \leq 2\sqrt{p}$.

- Trivial bound is $p^{k/2}$
- Any power p^β, $\beta < k/2$ is useful. This is analogous to Property τ, and can be established through the Kloosterman sum bounds we will derive.

Best situation - no multiplier

$$f(\gamma z) = f(z)$$

- Impossible for Holomorphic Forms, unless constant.

There are (mysteriously) lots of nonholomorphic examples (crucial for number theory)

$$\Delta = -y^2(dx^2 + dy^2) \text{ Laplacian}$$

acts on the space $L^2(\Gamma\backslash\mathbf{H})$

$$\lambda_0 = 0 \quad (\textit{ constant functions})$$
$$\lambda_1 > 0 \quad (\textit{ interesting values}).$$

Selberg Conjecture (1965)

$$\lambda_1(X(N)) \geq \frac{1}{4}$$

for any modular curve $X(N) = \Gamma(N)\backslash\mathbb{H}$

- more generally if Γ is a congruence subgroup
- $\frac{1}{4}$ does occur for certain N
- false if Γ not congruence (see result of Buser-Sarnak)
- Property τ is the weaker version where 1/4 is replaced by some positive quantity.
- Even τ is false for non-congruence subgroups.

Rayleigh Quotient

$$\Delta f(x) = 4 \lim_{r \to 0} \frac{1}{r^2} \frac{\int_{s_r} f}{\int_{s_r} 1}$$

S_r ball of radius r about x.

$$\int_{\text{closed Manifold } M \text{ (no bdy)}} f \Delta f = \int_M \|\Delta f\|^2 \geq 0$$

so $\Delta \geq 0$ and selfadjoint.

$$\int_M g \Delta f = \int_M f \Delta g$$

$$\lambda_1(M) = \inf_{f \perp 1} \frac{\int \|\Delta f\|^2}{\int |f|^2}$$

PICTURE

$\lambda_1 \to 0$ as neck shrinks.

So modular curves have few necks.

Margulis (1975):

first construction of expander graphs X w/similar *expansion* property

$$\frac{|\partial Y|}{|Y|}\text{large for } |Y| \leq \frac{|X|}{2}.$$

Then later Lubotzky-Phillips-Sarnak (1986) using modular forms theory.

This course more of a reverse:

special graphs $\rightarrow$ modular forms $\rightarrow$ prove τ, rather than use τ.

Graph Definitions

Graph Laplacian

$$\Gamma = \text{g}raph$$
$$A = \text{adjacency operator matrix}$$
$$A_{ij} = \begin{cases} 0 & \text{disconnected} \\ 1 & x_i \sim x_j \text{ adjacent} \end{cases}$$

Assume always k-regular connected, no loops, no double edges.

Degree=Valency=k=Number of edges at each vertex.

Laplacian:

$$\Delta = kI - A$$

$$(\Delta f)(x) = \sum_{y \sim x} [f(x) - f(y)]$$

(averages still)

- Δ kills constants (one-dimensional space)
- $< \Delta f, f > \;\; \geq \;\; 0$
- Since A symmetric, spec $A \subset [-k, k]$

$\lambda_1 = \lambda_1^{\Delta} =$ first pos eig of Δ

$= k$-(second largest eig of A).

Ramanujan Condition

$$\lambda_1^{\Delta} \geq k - 2\sqrt{k-1}$$

k-regular Tree spectral density

$$= \frac{k\sqrt{4(k-1) - x^2}}{2\pi(k^2 - x^2)}$$

PICTURE

Expansion Theorem:

Let $c = \min\{\frac{|\partial Y|}{|Y|}, |Y| \leq \frac{|X|}{2}\}$.

Then

$$c \geq \frac{1 - \frac{\lambda_1^{\Delta}}{k}}{2}.$$

Proof. Rayleigh-Ritz

$$\lambda_1 = \max_{g \perp 1} \frac{< Ag, g >}{< g, g >} \text{ (} A \text{ symmetric).}$$

Let

$$f = \chi_W \qquad \text{(characteristic function)}$$

$$g = f - \bar{f}$$

so $< g, 1 >= 0$.

Then

$$\begin{aligned} &< Ag, g > \\ &\quad = < Ag, f > - < Ag, \bar{f} > \\ &\qquad\qquad \| \\ &\qquad\qquad 0 \\ &\qquad\qquad \text{since} \\ &\qquad\qquad < g, A\bar{f} > \\ &\qquad\qquad \| \\ &\qquad\qquad \text{const.} \\ &\quad = < f, Af > - < \bar{f}, Af > \\ &\quad = < f, Af > - \bar{f}^2 k |X| \end{aligned}$$

Now

$$2 < Af, f > = 2 \sum_{x \sim y} f(x) f(y)$$
$$= - \sum_{x \sim y} (f(x) - f(y))^2 + 2k \sum_{x \sim X} f(x)$$
$$\geq 2k|W| - 2k|\partial W|$$

Thus

$$\begin{array}{ccl} \lambda_1 < g, g > & \geq & < Ag, g > \\ \| & \geq & k(|W| - |\partial W|) - \dfrac{k|W|^2}{|X|} \end{array}$$

$$\lambda_1(|W| - \frac{|W|^2}{|X|})$$

since

$$< g, g > = < f - \bar{f}, f - \bar{f} >$$
$$= < f, f > -2 < f, \bar{f} > + < \bar{f}, \bar{f} >$$
$$= < f, f > -2\frac{|W|^2}{|X|} + \frac{|W|^2}{|X|}$$
$$= |W| - \frac{|W|^2}{|X|}$$

So

$$\Longrightarrow |\partial W| \geq (\frac{|W|^2}{|X|} - |W|)(1 - \frac{\lambda_1}{k})$$
$$\geq \frac{|W|}{2}(1 - \frac{\lambda_1}{k})$$

since

$$|W| \leq \frac{|X|}{2}.$$

$\square$

Certain Cayley Graphs formed from $SL_2(Z/N)$ are expanders because of Selberg's Theorem ($\lambda_1(\Gamma(N)) \geq \frac{3}{16}$)), and Conjecture ($\geq 1/4$).

Good lower bounds (e.g. Kim-Sarnak's $\lambda_1 \geq .23\cdots$) provide nice constructions, but the essential property is that λ_1 is bounded away from zero by a positive quantity.

Warning!: A family of expanders should have fixed degree.

The point of an expander is to construct an efficient network which is hard to sever.

One extreme would be to connect n points in a circle, which only requires n connections, but can be disconnected in two cuts.

Another would be the complete graph K_n on n vertices, formed by connecting all vertices to each other. This requires $\sim n^2/2$ connections, but is of course quite stable.

Expander graphs are as inexpensive as the first and as stable as the second.

However, if one uses the eigenvalue criteria alone for expansion, then complete graphs actually qualify as expanders!!!

The non-zero eigenvalues on Δ on K_n are all exactly n, exceeding the Ramanujan bound $k-2\sqrt{k-1}$, $k=n-1$.

Thus it is imperative that the degree be *fixed*.

Terras' Graphs

Cayley Graph

$$G = < S > \quad \text{finite group}$$
$$S \quad \text{generating set}$$
$$\text{with } S = S^{-1}$$

Γ: a graph on G with vertices connecting $g_1 \sim g_2$ if $g_1 = sg_2$, for some $s \in S$.

- Finite analog of Lie group/symmetric space.
- In abelian case, nicer.
- Action of adjacency matrix A is convoution with χ_S.

Theorem Characters of G (finite abelian) are the eigenfn's of $*\chi_S$ for any subset $S \subset G$.

Proof: Enough to show for $S = \{t\}, t \in G$ where it gives definition of a character. □

Terras - model Euclidean $\mathbf{R}^2$ by $\mathbb{Z}/p \times \mathbb{Z}/p$

and

$$\begin{aligned} S &= \{(x,y) \mid x^2+y^2 = 1 \pmod{p}\} \\ &\supseteq \{(0,\pm 1),(\pm,1,0)\} \text{ , so generates.} \\ S &= -S. \end{aligned}$$

So Δ averages ball like it should.

Characters are Eigenfunctions

$$e\left(\frac{ax+by}{p}\right), \quad 1 \le a,b \le p,$$

$$Eigs: \; \lambda_{(a,b)} = \sum_{x^2+y^2\equiv 1(p)} e\left(\frac{ax+by}{p}\right)$$

like Fourier Transform of char F'n of ball, which is a Bessel function. Here it should be a Kloosterman sum.

Degree/Valency of Graph is

$$k = |S| = p - (\frac{-1}{p}), \qquad\qquad p > 2$$

Eigenvalue Calculation (Stark): The eigenvalues are Kloosterman sums

$$\lambda_{(a,b)} = K\ell(1, \frac{a^2+b^2}{4})(\frac{-1}{p}).$$

Proof:

We first write

$$p\lambda_{(2a,2b)} = \sum_{r/p} B_r(a,b)e(-r),$$

where

$$\begin{aligned} B_r(a,b) = \sum_{x,y\in\mathbf{Z}/p} & e_p(2(ax+by)+r(x^2+y^2)) \\ & (= 0 \text{ if } r \equiv 0) \\ & = G_1^2 \cdot e_p(-\bar{r}(a^2+b^2)), r\bar{r} \equiv 1(p) \\ G_1 = & \sum_{y \mod p} e_p(y^2) = \text{Gauss Sum} \end{aligned}$$

since $G_1^2 = \sum e_p(r[(x+\frac{a}{r})^2 + (y+\frac{b}{r})^2])$

↑ signs cancel in squaring

The proof follows from the well-known fact due to Gauss that

$$G_1 = \sqrt{(\frac{-1}{p})p}. \qquad \square$$

SPECTRUM

$p \equiv 1(4)$

$(a,b) = (0,0) \quad \lambda_0 = k = p-1$

rest $K\ell(1,a), a \neq 0$

$mult\ p-1 = k$

$K\ell(1,0) = -1$ mult $2(p-1)$

$p \equiv 3(4)$

$\lambda_0 = k = p+1$

$-K\ell(1,a), a \neq 0$

mult $p+1 = k$

$(A^n)_{ij} = \#\ of\ paths\ connecting\ x_i \leftrightarrow x_j\ \ (includes\ backtracking)$

So $\sum \lambda_j^n = n$-th moment $= \#$ of closed paths on Γ of length $= n$ (includes backtracking).

Paths of Length 2 — PICTURE

kp^2 of them

$\nearrow \quad \nwarrow$

$valency \quad |\Gamma|$

Paths of Length 3

Triangles multiplication like $\mathbb{C}$

$$(a,b)\cdot(c.d) = (ac - bd, bc + ad) \text{ preserves } S$$

$$(a,b)\cdot(a_, -b) = (1,0)$$

so can "rotate" a triangle to the position

$$\begin{array}{ccc} & (a,b) & \\ \nearrow & & \nwarrow \\ (0,0) & \longrightarrow & (1,0) \end{array}$$

like the configuration of $\{0, 1, e^{\pi i/3}\}$ in $\mathbb{C}$.

Equations for triangle:

$$\begin{aligned} a^2 + b^2 &= 1 \\ (a-1)^2 + b^2 &= 1 \\ &\| \\ a^2 + b^2 + 1 - 2a &= 1 \end{aligned}$$

$$p \Rightarrow a = \frac{1}{2}, b = \pm \text{``}\frac{\sqrt{3}}{2}\text{''}$$

so no Δ's if $(\frac{3}{p}) \neq 1$ and $2p^2 \cdot k$ otherwise.

$\boxed{Squares.}$

First, paths of length 4 which involve backtracking.

$$k^2p^2 \quad PICTURE$$

$$k(k-1)p^2 \quad PICTURE$$

$$+_{\square}'s \text{ should be } p^2k(k-2) \text{ also}$$

$$\text{total } p^2(3k^2-3k)$$

Recall the spectrum is

$p \equiv 1(4)$	$p \equiv 3(4)$
$k = p-1$	$k = p+1$
$\lambda' s = k$	$\lambda' s = k$
$K\ell(1,a), a \neq 0$ mult k	$-K\ell(1,a), a \neq 0$
$mult\ 2k$	m$ult\ k$

Kloosterman's Thesis, 1926

Second Moment of Spectrum

$$kp^2$$

$$k^2 + k\sum_{a=1}^{p-1} K\ell(1,a)^2 \qquad k^2 + k\sum_{a=1}^{p-1} K\ell(1,a)^2$$

$$+ 2k$$

$$\sum_{a=1}^{p-1} K\ell(1,a)^2 = p^2 - 2 - k \qquad \sum_{a=1}^{p-1} K\ell(1,a)^2 = p^2 - k$$

$$= p^2 - 2 - p + 1 \qquad p^2 - p - 1$$

$$= p^2 - p - 1$$

Eitherway

$$\sum_{a=1}^{p-1} K\ell(1,a)^2 = p^2 - p - 1$$

Third Moment of Spectrum

$$\begin{cases} 0 & \text{if } (\frac{3}{p}) = -1 \\ 2p^2k & otherwise \end{cases}$$

$$k^3 + k\sum_{a=1}^{p-1} K\ell(1,a)^3 - 2k = 2p^2k \qquad 2p^2k = k^3 - k\sum_{a=1}^{p-1} K\ell(1,a)$$

or 0 or 0

$$\sum_{a=1}^{p-1} K\ell(1,a)^3 = 2p^2 + 2 - k^2 \qquad \sum_{a=1}^{p-1} K\ell(1,a)^3 = -2p^2 + k^2$$

or $2 - k^2$ or k^2

$$= \begin{pmatrix} 2p^2 \\ 0 \end{pmatrix} + 2 - p^2 + 2p - 1 \qquad \begin{pmatrix} -2p^2 \\ 0 \end{pmatrix} + p^2 + 2p + 1$$

$$\begin{pmatrix} 2p^2 \\ 0 \end{pmatrix} + 1 - p^2 + 2p$$

Either way

$$\sum_{a=1}^{p-1} K\ell(1,a)^3 = (-\frac{3}{p})p^2 + 2p + 1$$

is given by a congruence condition.

Fourth Moment

$$3p^2k(k-1)$$

$$k^4 + k\sum_{a=1}^{p-1} K\ell(1,a)^4 + 2k \qquad\qquad k^4 + k\sum_{a=1}^{p-1} K\ell(1,a)^4$$

$$\sum K\ell(1,a)^4 = 3p^2(k-1) - 2 - k^3 \qquad\qquad \text{or } 3p^2(k-1) - k^3$$

$$= 3p^2(p-2) - 2 - (p-1)^3 \qquad\qquad 3p^3 - (p+1)^3$$

$$= 3p^3 - 6p^2 - 2 - p^3 + 3p^2 - 3p + 1 \qquad\qquad 3p^3 - p^3 - 3p^2 - 3p - 1$$

$$= 2p^3 - 3p^2 - 3p - 1 \qquad\qquad 2p^3 - 3p^2 - 3p - 1$$

Thus

$$|K\ell(1,a)| \ll p^{3/4}$$

What about formulas for higher moments?

Really k dominates - expect formula like

$$k^n + k \sum K\ell(1,a)^n + 2k = k^n + kp^{n/2}$$
$$\sim p^n + p^{n/2+1}$$

$\sqrt{}$-sized error term, hard to get

- Kloosterman, Davenport computed moments directly.
- An alternative, classical approach is through counting points on algebraic varieties
- Katz has gone furthest, showing the renormalized Kloosterman sums
$$\{\frac{K\ell(a,p)}{2\sqrt{p}} \mid 1 \le a \le p-1\}$$
are distributed according to the semi-circle law
$$\frac{2}{\pi}\sqrt{1-x^2}.$$
- Livne - deviation related to automorphic forms on $GL(n)$!!!
- In some sense, the varieties (and graphs) are "modular."

Selberg's Kloosterman Zeta Function

$$S(m,n;c) = \sum_{ad\equiv 1(\mod c)} e\left(\frac{ma+nd}{c}\right)$$

Focus on c not prime now.

Obviously

$$S(m,n;c) = S(n,m;c)$$
$$S(am,n;c) = S(m,an;c) \text{ if } (a,c)=1$$

Selberg's Property

$$S(m,n;c) = \sum_{d|(c,m,n)} d\, S\left(\frac{mn}{d^2},1;\frac{c}{d}\right)$$
$$= \sum_{d|(c,m,n)} d \sum_{ab\equiv 1(\frac{c}{d})} e\left(\frac{(\frac{mn}{d^2}a+b)d}{c}\right)$$

Twisted Multiplicativity

$$S(m,n,qr) = S(\bar{q}m,\bar{q}n,r)\cdot S(\bar{r}m,\bar{r}n;q)$$
$$(r,q)=1$$
$$q\bar{q}\equiv 1(r)$$
$$r\bar{r}\equiv 1(q)$$

Proof: RHS is

$$\sum_{ab\equiv 1(r),cd\equiv 1(q)} e(\frac{\bar{q}ma+\bar{q}nb}{r}+\frac{\bar{r}mc+\bar{r}nd}{q})$$

Note that we need only show

$$(q\bar{q}a+cr\bar{r})(q\bar{q}b+dr\bar{r})\equiv 1(qr).$$

But this easily follows $\mod q$ and $\mod r$. $\square$

Upshot: Bounds on $S(m,n;p^r)$ are all that's needed to bound $S(m,n;c)$, when c is composite.

We treat only $p>2$.

Useful Facts. For any $p > 2$, $\exists g$ such that g generates *all* $(\mathbb{Z}/p^k)$, any $k > 0$.

Gauss Sum. If $(c, 2n) = 1$, $c > 0$

$$\sum_{t \in \mathbf{Z}/c} e(\frac{nt^2}{c}) = (\frac{n}{c})\sqrt{c}\epsilon_c$$

$$\epsilon_c = \begin{cases} 1 & c \equiv 1 \mod 4, \\ i & c \equiv -1 \mod 4. \end{cases}$$

1936. Salie's Evaluation of Kloosterman Sums to prime power moduli.

Assume $(n, m, c) = 1$, where c is a prime power $c = p^\beta$, $\beta > 1, p > 2$.

1) $K\ell(n, m, c) = 0$ unless

$$n \equiv \ell^2 m(c) \text{ for } \quad \text{some } (\ell, c) = 1.$$

2) Otherwise $K\ell(n, \ell^2 m, c)$

$$= K\ell(n, n, c) = 2(\frac{n}{c})\ \sqrt{c}\ Re\,[\epsilon_i\ e(2n/c)]$$

Proof. for 1), first write

$$K\ell(n,m;p^{\beta+1}) = \sum_{(x,p)=1, x \mod p^{\beta+1}} e(\frac{nx+mx^{-1}}{p^{\beta+1}})$$

$$= \sum_{x=0}^{p-1} \sum_{y\in(\mathbf{Z}/p^{\beta})^*} e_{p^{\beta+1}}(n(p^{\beta}x+y) + \frac{m}{p^{\beta}x+y})$$

$$(np^{\beta}x + ny - \frac{mp^{\beta}x}{y^2} + \frac{my}{y^2})$$

$$\sum_{y\in(\mathbf{Z}/p^{\beta})^*} e(ny+my^{-1}) \sum_{x=0}^{p-1} e(\frac{x}{p}(n-my^{-2}))$$

$$0 \text{ if } \begin{cases} n \equiv m\Box & \mod p \\ \Updownarrow & \\ n \equiv m\Box & \mod p^{\beta+1}. \end{cases}$$

For 2), first assume $c = p^{2\alpha}$

$$K\ell(n,n;p^{2\alpha}) = \sum_{(x,p)=1, x \mod p^{2\alpha}} e_c(n(x+x^{-1})).$$

Write

$$x \Rightarrow x(1+yp^{\alpha}), \qquad x \in (\mathbf{Z}/p^{2\alpha}), \quad y \in \mathbf{Z}/p^{\alpha},$$

which covers all values p^{α} times.

Inverse becomes: $x^{-1}(1-yp^{\alpha})$ (mod $p)^{2\alpha}$

Sum is

$$= \frac{1}{p^{\alpha}} \sum_{x\in(\mathbf{Z}/p^{2\alpha})^*, y\in\mathbf{Z}/p^{\alpha}} e(\frac{n}{p^{2\alpha}}(x+x^{-1}) + \frac{ny}{p^{2\alpha}}(x-x^{-1})),$$

the sum $\nearrow$ over y makes $x \equiv x^{-1}$

which in turn forces

$x^2 \equiv 1(p^{\alpha})$

$$x = \pm 1 + tp^{\alpha}, \quad x^{-1} = \pm 1 - tp^{\alpha}, \quad t \in \mathbf{Z}/p^{\alpha},$$

and the sum is

$$= \sum_{t\in\mathbf{Z}/p^{\alpha}} (e(\frac{2n}{p^{2\alpha}}) + e(-\frac{2n}{p^{\alpha}})) = 2p^{\alpha} Re\ e(\frac{2n}{c}).$$

The Odd Case is similar, using

$$x(1+yp^{\alpha+1})$$

$$sum = \sum_{x\in(\mathbf{Z}/p^{2\alpha+1})^*, x\equiv x^{-1} (\mod p^\alpha)} e(\frac{n(x+x^{-1})}{p^{2\alpha+1}})$$

Write

$$x = \pm 1 + tp^\alpha, \quad x^{-1} = \pm 1 - tp^\alpha, \quad t \in \mathbb{Z}/p^{\alpha+1}$$

$$K\ell(n, n; p^{2\alpha+1}) = 2 \; Re \sum_{t\in\mathbf{Z}(p^{\alpha+1})} e(\frac{n}{p^{2\alpha+1}}(1+t^2p^{2\alpha}))$$

$\rightarrow$ get Gauss sum

For m fixed

$$S(m, n, c) \le d(c) \cdot c^{1/2}(m, n, c)^{1/2} \ll_\epsilon c^{1/2+\epsilon},$$

where $d(n) = \#$ of divisors of n.

Selberg's 1965 paper "On the estimation of Fourier coefficients of modular forms"

The condition

$$ad \equiv 1 \pmod{n}$$

suggests $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$

Generalized Kloosterman sum:

$$S(m,n,c,\Gamma) = \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N), 0 \le a < Nc, 0 \le d < Nc} e\left(\frac{ma+nd}{Nc}\right)$$

(before was $N = 1$) ...

Selberg's Kloosterman Zeta Function

$$Z(s,m,n;\Gamma) = \sum_{c=1}^{\infty} \frac{S(m,n,c,\Gamma)}{|c|^{2s}}$$

has poles at nonconst eigs of Laplacian.

Selberg introduced non-holomorphic Poincaré series for positive integers m

$$P_m(z,s) = \sum_{\gamma \in \begin{pmatrix} 1 & N\mathbb{Z} \\ & 1 \end{pmatrix} \backslash \Gamma(N)} e(\frac{m}{N}\gamma(z))|\text{Im } \gamma(z)|^s$$

$$= \sum_{\gamma \in \begin{pmatrix} 1 & N\mathbb{Z} \\ & 1 \end{pmatrix} \backslash \Gamma(N)} = e(\frac{m}{N}\frac{az+b}{cz+d})\frac{y^s}{|cz+d|^{2s}}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Differential Equation

$$P_m(z,s) = -\frac{4\pi ms}{N}(\Delta + s(1-s))^{-1}P_m(z,s+1),$$

from $\quad \Delta(y^s e(mz)) + s(1-s)(y^s e(mz))$

$$= -4\pi ms(y^{s+1}e(mz))$$

Inner Product

$$\int_{\Gamma(N)\backslash\mathbb{H}} U_m(z,s)\overline{U_n(z,w)}\frac{dxdy}{y^2}$$

$$= \int_0^\infty \int_0^N \frac{dxdy}{y^2} e(\frac{m}{N}z)y^s\overline{U_n(z,w)}$$

like a Fourier Coefficient, in the classical theory of Poincaré series.

Via Poisson Summation

$$= \delta_{m,n}(4\pi n)^{1-s-\bar{w}}\Gamma(s+\bar{w}-1)$$
$$+\sum \frac{K(m,n,c)}{|c|^{2s}} \int_0^\infty \int_{-\infty}^\infty \frac{y^{\bar{w}-s}}{(x^2+1)^s} e[\frac{-m}{yc^2(x+1)} - n(xy-cy)] \frac{dxdy}{y}.$$

This uses Poincaré Series idea e.g., Holom Forms of wt k for $\Gamma(1)$

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^k} e(m\gamma(z))$$

$$P_m(z) - e(mz) = \sum_{c=1}^\infty \sum_{d \in (\mathbf{Z}/C)^*} \sum_{r \in \mathbb{Z}} \frac{1}{(cz+d+cr)^k} e(m \frac{az+b}{cz+d+cr})$$

$$= \sum_{c=1}^\infty \sum_{d \in (\mathbf{Z}/C)^*} \sum_{t \in \mathbf{Z}} \int_{\mathbf{R}} \frac{e(-rt)dr}{(cz+d+cr)^k} e(m[\frac{q}{c} - \frac{1}{c(cz+d+cr)}])$$

$$= \sum_{c=1}^\infty \sum_{d \in (\mathbf{Z}/C)^*} \sum_{t \in \mathbf{Z}} e(\frac{ma+td}{c}) \int_{\mathbf{R}} \frac{e(-rt + \frac{m}{c(cz+cr)})}{[c(z+r)]^k} dr.$$

The sum over d is a Kloosterman sum, and the integral is a Bessel function. Adelically, these arise on equal footing.

Selberg's Theorem (1965) Let

$$Z_{N,m,n}(s) = \sum_{N|c>0} \frac{S(m,n,c)}{c^{2s}}, \quad \text{Re } s > 1, \ N \geq 1.$$

Then $Z_{N,m,n}(s)$ has a meromorphic continuation to $\mathbb{C}$ with poles at $s = 1/2 + ir$, where $\lambda = 1/4 + r^2$ is a non-zero eigenvalue of the Laplacian Δ on $\Gamma_0(N)\backslash\mathbb{H}$.

Weil bound $|S(m,n,c)| \leq \tau(c)c^{1/2}(c,n)^{1/2}$ implies $\lambda \geq 3/16$.

The $c^{3/4}$ bound we derived from the graphs implies $\lambda \geq 7/64$.

Bounds for Fourier Coefficients

Using classical Poincaré series $P_{m,k}(z)$ and Poisson sum as before, find

$$P_{m,k}(z) = e(mz) + \sum_{n=1}^{\infty} e(nz) c_n,$$

$$c_n = \sum_{c=1}^{\infty} S(m,n,c) \frac{2\pi}{i^k c} (n/m)^{(k-1)/2} J_{k-1}(4\pi\sqrt{mn}/c).$$

The Poincaré series $\{P_{m,k}(z) | m \geq 1\}$ span the space of cusp forms.

Similar analysis uses Kloosterman sum bounds to obtain non-trivial bounds towards the Ramanujan conjectures on the sizes of their Fourier coefficients.