Cusp Forms on $SL_3(\mathbf{Z})\backslash SL_3(\mathbf{R})/SO_3(\mathbf{R})$

Stephen David Miller

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
By the Department of
Mathematics

June 1997

Abstract

We discuss cusp forms occuring on the quotients $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$, focusing mainly on the case $n = 3$. We present an overview of spectral problems on these discrete quotients as well as automorphic L-functions and some of their analytic properties. We then address how small the eigenvalues of the Laplace-Beltrami operator can be on this quotient, and how many many eigenvalues exist. Our answer to the first question proves part of the archimedean Ramanujan-Selberg conjecture, and our answer to the second establishes part of a conjecture of Sarnak that Weyl's law holds for the cusp forms on $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$. The proof of the latter involves an analysis of Eisenstein series and so we describe their general properties. All techniques center on using the analytic properties of L-functions. We present applications to representation theory and to the group cohomology of $SL_n(\mathbf{Z})$.

Acknowledgements

I am very fortunate to not have enough room to thank everyone who has helped me. This thesis is dedicated to my parents, without whom it would be much shorter. I also want to thank my bubby Tillie Silverstein and my dearest friends, my brothers Marc and Alan. Were it not for all the fun we have had growing up together I could've finished this years earlier. Also, a number of good friends in the Philadelphia area, in particular the Levy family of Delaware, have really helped me out.

My advisor Peter Sarnak has been a wealth of mathematical wisdom, insight, and inspiration for the last three years. I'm sure most students aren't as fortunate to have had even one problem as good as the many he has posed to me. He is never short of ideas for challenging inquiries. I thank him for the opportunities he has presented me.

I am also grateful to many other mathematicians who have helped me and taught me. Zeév Rudnick hosted me for three months at Tel Aviv University, where I learned much background for this project and benefited from discussions with him. I also am privileged to have worked out some of the key early steps of this project inside the Knesset, in Alexander Lubotzky's office while he was in a committee meeting. I also want to thank Bill Duke for discussions about the positivity method. I benefited from reading a preprint of Armand Borel's new book [Borel 1974] on automorphic forms. And of course I want to thank Princeton and the National Science Foundation for financial support.

The Princeton mathematics department has a great many good friends who have made Fine Hall one of the nicest places on campus. Fred Almgren helped me a lot early on. I am delighted to have had tea-table discussions over the past few years with John Conway, Simon Kochen, Joe Kohn, Jeff McNeal, and Gerry Washnitzer.

Invariably these conversations started taking place across the street, at the Center for Jewish life, where I have made many good friends who made Princeton far more enjoyable than it possibly could have been otherwise. However, there are a lot of them and none of them will ever read a math dissertation anyway, so I can get away without mentioning them. I must, though, mention Glenn Kashan, who has been my most influential friend here and who persuaded me to do many new things I had avoided out of inertia. Also, I have really valued the friendships of Danny Fax, Jonny Levine, Moti Novick, and Charles Rue.

With love to my parents Joel and Elaine

Contents

Chapter 1

Introduction

The primary subject of this thesis is the existence of cusp forms on the quotient $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$. Let $G = SL_n(\mathbf{R})$, $K = SO_n(\mathbf{R})$ (a maximal compact subgroup), and $\mathcal{H}_n = G/K$, a rank-$(n-1)$ symmetric space. Cuspidal eigenfunctions are functions $\phi : \mathcal{H}_n \to \mathbf{R}$ satisfying four properties

1.0.1 (periodicity) $\phi(\gamma x) = \phi(x)$ *for all* $\gamma \in \Gamma = SL_n(\mathbf{Z})$

1.0.2 (growth) $\phi \in L^2(\Gamma\backslash\mathcal{H}_n)$.

1.0.3 (a Laplace eigenfunction) $\Delta\phi + \lambda\phi = 0$, *where* Δ *is the Laplace-Beltrami operator and* λ *is* ϕ*'s "eigenvalue."*

1.0.4 (cuspidality) *If* P *is a standard, proper parabolic subgroup of* G *and* N *is its unipotent radical,* ϕ*'s constant term*

$$\phi_P(x) = \int_{\Gamma\cap N\backslash\Gamma} \phi(nx)dn$$

vanishes identically.

We will investigate two basic questions in this thesis: how many cusp forms exist and how small can their eigenvalues be? We answer the first question by giving the asymptotic density of eigenvalues, Weyl's law. The second question was asked by Selberg, who proved a general lower bound on the cuspidal spectrum of the laplacian on $L^2(SL_2(\mathbf{Z})\backslash\mathbf{H})$ and related spaces, and conjectured an optimal bound. To tackle these problems we use techniques involving L-functions, which are dirichlet series formed from the Fourier coefficients of cusp forms. L-functions allow us to convert a spectral problem involving many variables to one of a single complex variable. We use the analytic properties of L-functions (particularly Rankin-Selberg L-functions) in a variety of circumstances for both proving that cusp forms exist in abundance and that they do not occur with certain eigenvalues. In the next two sections we will describe these two questions in more detail.

1.1 Weyl's law

Cusp forms are fundamental objects in number theory, geometry, and representation theory. However, their existence is very subtle. Selberg proved their abundance for $n = 2$ by giving an asymptotic count:

Theorem 1.1.1 *(Selberg) Let $N(T)$ be the number of discrete eigenvalues of the laplacian on $L^2(SL_2(\mathbf{Z})\backslash\mathbf{H})$ (with multiplicity) $\leq T$. Then as $T \to \infty$*

$$N(T) \sim \frac{T}{4\pi} vol(SL_2(\mathbf{Z})\backslash\mathcal{H}_2).$$

Remark 1.1.2 *It is known that, except for the constant function/zero-eigenvalue, the discrete spectrum consists of cusp forms. Thus Selberg addressed both by the spectral count.*

Remark 1.1.3 *Selberg ([Selberg 1956]) developed his famous trace formula to prove this theorem. In the ensuing analysis, the spectral count is influenced by the presence of Eisenstein series, which are present only for non-compact quotients* $\Gamma\backslash\mathcal{H}_n$. *For compact quotients one can easily recover the identical spectral count*

$$N(T) \sim \frac{T}{4\pi} vol(\Gamma\backslash\mathbf{H}) \text{ as } \mathbf{T} \to \infty,$$

which is Weyl's law. *We will hence refer to results of this type as Weyl's law for the particular space under consideration.*

Remark 1.1.4 *For the generic discrete subgroup* $\Gamma \subset SL_2(\mathbf{R})$ *Weyl's law is almost certainly false. Indeed, the investigations of Phillips and Sarnak have shown the Eisenstein series contribution mentioned above is the main term and that the discrete spectrum is probably finite.*

Remark 1.1.5 *Sarnak conjectured that, while Weyl's law and the existence of infinitely-many cusp forms occurs for only very special discrete subgroups* $\Gamma \subset SL_2(\mathbf{R})$, *it should hold for the cuspidal spectrum by itself on congruence quotients of* $\mathcal{H}_n$. *More precisely, the laplacian has both a continuous and a discrete spectrum on* $L^2(\Gamma\backslash\mathcal{H}_n)$ *for* $\Gamma = SL_n(\mathbf{Z})$ *or one of its congruence subgroups*[1], *and the discrete spectrum includes residues from the continuous spectrum in addition to cusp forms.*

[1] *If* N *is a positive integer, the* N-th principal congruence subgroup *is*

$$\Gamma(N) = \{g \in G \mid g \equiv I_{n\times n} \pmod N \text{ (element by element) } \},$$

the kernel of the projection

$$SL_n(\mathbf{Z}) \to SL_n(\mathbf{Z}/N\mathbf{Z}).$$

A congruence subgroup *of* $SL_n(\mathbf{Z})$ *is one which contains some principal congruence subgroup. There are finite-index subgroups of* $SL_2(\mathbf{Z})$ *which are not congruence subgroups, but every finite index subgroup of* $SL_n(\mathbf{Z})$ *for* $n \geq 3$ *is a congruence subgroup.*

Conjecture 1.1.6 *([Sarnak 1984]) Let* Γ *be a congruence subgroup of* $SL_n(\mathbf{Z})$*,*

$$N_{cusp}(T) = \#\{\lambda \leq T \mid \Delta\phi + \lambda\phi = 0 \text{ for some cusp form } \phi\}$$

count the eigenvalues of Δ *on* $L^2(\Gamma\backslash\mathcal{H}_n)$ *(with multiplicity), and let* $d = \dim \mathcal{H}_n = \frac{n(n+1)}{2} - 1$*. Then*

$$N_{cusp}(T) \sim \frac{T^{d/2}}{(4\pi)^{d/2}\Gamma(1+\frac{d}{2})} vol(\Gamma\backslash\mathcal{H}_n) \text{ as } T \to \infty$$

This is stronger than Weyl's law in that it holds for the cuspidal spectrum alone, though often (when n is prime) there are no residual eigenfunctions other than the constant functions (see [Jacquet],[MœWal]). We prove Sarnak's conjecture for $n = 3$ and $\Gamma = SL_3(\mathbf{Z})$, i.e. full level:

Theorem 1.1.7 *(= theorem 5.1.1) For* $n = 3$ *and* $\Gamma = SL_3(\mathbf{Z})$

$$N_{cusp}(T) \sim \frac{T^{5/2}}{(4\pi)^{5/2}\Gamma(\frac{7}{2})} vol(\Gamma\backslash\mathcal{H}_3) \text{ as } T \to \infty.$$

Of course since 3 is prime $N(T) = N_{\text{cusp}}(T) + 1$.

Remark 1.1.8 *This theorem establishes the existence of non-lifted cusp forms for* $SL_3(\mathbf{Z})\backslash\mathcal{H}_3$*. Infinitely many are known via the Gelbart-Jacquet lift (see [GelJac]). In particular, if* $N_\ell(T)$ *is the number of these lifted eigenvalues (counted with multiplicity) less or equal to* T*, then* $N_\ell(T) \sim \frac{T}{48}$*.*

1.2 Low-lying eigenvalues

The laplacian Δ has a continuous spectrum on $L^2(\mathbf{H})$ consisting of $[1/4, \infty)$. It persists to $L^2(\Gamma\backslash\mathbf{H})$ for $\Gamma = SL_2(\mathbf{Z})$ or a congruence subgroup. Selberg made the following conjecture to the effect that there are no discrete eigenvalues outside that range other than 0 (coming from the constants):

Conjecture 1.2.1 *(Selberg, 1965 (see [Selberg 1965])) No eigenvalue of a cusp form on $\Gamma\backslash\mathbf{H}$ is less than $\frac{1}{4}$.*

This conjecture has an interpretation in terms of (the Langlands) parameters associated to a cuspidal eigenfunction ϕ, $\mu = (\nu, -\nu)$. The eigenvalue $\lambda = \frac{1}{4} - \nu^2 \geq 0$, so ν is either purely imaginary (equivalently $\lambda \geq 1/4$) or real and $|\nu| \leq 1/2$ ($\lambda \leq 1/4$). So Selberg's conjecture is that ϕ's Langlands parameters μ are imaginary.

In higher rank, where $\mathcal{H}_n$ replaces $\mathbf{H} = \mathcal{H}_{\mathbf{2}}$, this is indeed the appropriate generalization (and in fact also at the finite places in any rank, where it subsumes the classical Ramanujan conjecture). There are many G-invariant differential operators on $\mathcal{H}_n = G/K$ other than the laplacian and the Langlands parameters describe ϕ's eigenvalue under any of them (see section 5.2). The archimedean Ramanujan-Selberg conjecture asserts that the Langlands parameters are all imaginary. In particular, if $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ are the Langlands parameters of a cusp form ϕ and $\Delta\phi + \lambda\phi = 0$, then

$$\lambda = \frac{n^3 - n}{24} - \frac{\mu_1^2 + \cdots \mu_n^2}{2}.$$

Here Δ is normalized to have the continuous spectrum $[\frac{n^3-n}{24}, \infty)$ on both $L^2(\mathcal{H}_n)$ and $L^2(\Gamma\backslash H_n)$, and an implication of the archimedean Ramanujan-Selberg conjectures is that all cuspidal eigenvalues of Δ on $\Gamma\backslash\mathcal{H}_n$ are $\geq \frac{n^3-n}{24}$, though we stress that this is weaker than the full conjecture for $n \geq 3$.

Theorem 1.2.2 *([Miller]) For $\Gamma = SL_n(\mathbf{Z})$ every cuspidal eigenvalue of the laplacian is greater than $\frac{n^3-n}{24}$, for all $n > 1$.*

This confirms the conjecture at full level, for merely Δ among the ring of invariant differential operators. Roelcke ([Roelcke]) and Selberg independently proved this for $n = 2$ in 1956. We will present a pictoral proof of this (and more) for $n = 3$ in section 4.2.

1.3 Background, methods, and related results

The next chapter, 2, is a short summary of spectral problems on $SL_2(\mathbf{Z})\backslash\mathbf{H}$. Though our primary focus is on higher rank spaces, this classical example motivates and has many features in common with the problems we are examining. We conclude chapter 2 with a list of the major solved and unsolved spectral problems on the upper-half plane.

The third chapter, 3, gives background information on the geometry of the symmetric space $\mathcal{H}_n$. We review the major results of the Langlands' theory of Eisenstein series, in particular focusing on their constant terms. The Eisenstein series are part of the spectral decomposition of the laplacian on $SL_n(\mathbf{Z})\backslash\mathcal{H}_n$, and their constant terms control their size.

Our techniques are all based on using the L-functions of cusp forms. In chapter 4 we will recall the basic properties of these and derive inequalities which can show certain L-functions and hence the cusp forms they are created from, do not exist. We will then present applications, including a computation of cuspidal Betti numbers of $SL_n(\mathbf{Z})\backslash\mathcal{H}_n$. Also, we can show that no cusp forms on $SL_3(\mathbf{Z})\backslash\mathcal{H}_3$ have certain Langlands parameters, which yields a lower bound on the discrete spectrum of Δ there, a special case of theorem 1.2.2.

In the last chapter, 5, we prove Weyl's law for $SL_3(\mathbf{Z})\backslash\mathcal{H}_3$ (thm 5.1.1) and recall some preliminaries of the Selberg trace formula and properties of Eisenstein series for various parabolic subgroups.

Chapter 2

The Upper Half Plane

2.1 Maass forms

Let $\mathbf{H}$ be the complex upper-half plane $\{z = x + iy | y > 0\}$. Any element $\gamma \in G = SL_2(\mathbf{R})$ acts on $\mathbf{H}$ as a fractional linear transformation

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

This action is transitive and the only elements which act trivially are $\{\pm I\}$. If Γ is a discrete subgroup of $SL_2(\mathbf{R})$, let $\bar{\Gamma}$ denote its image in $PSL_2(\mathbf{R}) = SL_2(\mathbf{R})/\{\pm I\}$. The quotient $\bar{\Gamma}\backslash\mathbf{H}$ is a hyperbolic manifold. For example, any Riemann surface of genus ≥ 2 can be covered by $\mathbf{H}$ and is conformal to $\Gamma\backslash\mathbf{H}$ for some $\Gamma \subset PSL_2(\mathbf{R})$.

We are mostly interested in a family of discrete subgroups called *congruence* subgroups. Firstly, the *principle congruence* subgroups are

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N | ad - bc = 1 \right\},$$

the kernels of the surjective homomorphisms $\Gamma(1) = SL_2(\mathbf{Z}) \to SL_2(\mathbf{Z}/N\mathbf{Z})$. A congruence subgroup is simply a subgroup of $SL_2(\mathbf{Z})$ which contains some $\Gamma(N)$, $N \geq 1$. The congruence subgroups are not co-compact (i.e. $\bar{\Gamma}\backslash\mathbf{H}$ is not compact), but co-finite ($\bar{\Gamma}\backslash\mathbf{H}$ has finite volume under the G-invariant measure $\frac{dxdy}{y^2}$).

These quotients $X = \bar{\Gamma}\backslash\mathbf{H}$ are called *modular curves* and inherit the G-invariant line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and laplacian

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

from $\mathbf{H}$. They are hyperbolic surfaces with singularities called *cusps*. For example, $PSL_2(\mathbf{Z})\backslash\mathbf{H}$ is a once-punctured sphere (with conic singularities). This is because the group $SL_2(\mathbf{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which correspond to the transformations of the upper-half plane $\mathbf{H}$ given by

$$z \mapsto z + 1 \text{ and } z \mapsto -1/z.$$

A fundamental domain is provided by

$$\mathcal{F} = \{z \in \mathbf{H} \mid |z| \geq 1, |x| \leq 1/2\}.$$

2.2 Analysis

Let us stick to the example of "full level," $\Gamma = PSL_2(\mathbf{Z})$. Good references for this section are [Terras], volume 1, [Borel 1974], [Bump], and [Sarnak 1995].

An *automorphic* function f is one which satisfies $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ and

$z \in \mathbf{H}$. The word "automorphic" is used because the fractional linear transformation $\gamma : \mathbf{H} \to \mathbf{H}$ is an invertible conformal mapping of the upper half plane to itself, i.e. a conformal automorphism.

An *automorphic form* for us will be an automorphic function ϕ which is an eigenfunction of the laplacian Δ and which satisfies the polynomial growth condition that for some $N > 0$

$$f(x+iy) = O(y^N)\ ,\ x+iy \in \mathcal{F}.$$

We call the *eigenvalue* of ϕ the number λ such that $\Delta\phi + \lambda\phi = 0$, the negative of what is usually called an eigenvalue. This is because Δ is negative semi-definite: integration by parts yields

$$\int_X f\Delta f \frac{dxdy}{y^2} = -\int_X |\nabla f|^2 \frac{dxdy}{y^2},$$

so the constant functions comprise the null-space of Δ in $L^2(X)$ and all other eigenvalues are positive. Their corresponding eigenfunctions are not only orthogonal to constants but in fact *cuspidal*:

$$\int_0^1 \phi(x+iy)dx = 0 \text{ for all } y > 0$$

as there is no residual spectrum in this case.

The cusp forms have Fourier expansions. Firstly, for fixed $y > 0$

$$\phi(x+iy) = \phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(x+iy)\right) = \phi(x+1+iy),$$

so

$$\phi(x+iy) = \sum_{n=-\infty}^{\infty} \alpha_n(y) e^{2\pi i n x}.$$

The cuspidality condition here is that $\alpha_n(y) = 0$. Secondly, the differential equation $\Delta\phi + \lambda\phi = 0$ gives two possible families of solutions for $\alpha_n(y)$, one of exponential growth in y and the other of exponential decay. Our polynomial growth assumption forces

$$\alpha_n(y) = \frac{a_n}{2}\sqrt{y} K_\nu(2\pi|n|y),$$

where a_n is a real number, $\lambda = \frac{1}{4} - \nu^2$, and K is the K-Bessel function. Thus, ϕ decays exponentially in the cusp, that is, as $\Im z \to \infty$ for $z \in \mathcal{F}$. One may separate the spectrum into odd and even parts under $z \mapsto -\bar{z}$ so that the even

$$\phi(x+iy) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_\nu(2\pi n y) \cos(2\pi n x)$$

and the odd

$$\phi(x+iy) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_\nu(2\pi n y) \sin(2\pi n x).$$

Note the odd eigenfunctions are automatically cuspidal.

If n is a positive integer, the *Hecke operator* $T_n : L^2(X) \to L^2(X)$ is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n, a>0 \\ b \bmod d}} f\left(\frac{az+b}{d}\right).$$

The Hecke operators are self-adjoint operators which commute with each other and the laplacian, and so we may further diagonalize the cusp forms by resolving Laplace eigenspaces of dimension greater than one by their Hecke eigenvalues. Thus, we now assume our orthonormal set

$$\frac{1}{\sqrt{\mathrm{vol}X}}, \phi_1, \phi_2, \ldots$$

consists of Hecke and Laplace eigenforms. Then

$$a_{1_i}(T_n\phi_i)(x) = a_{i_n}\phi_i(x),$$

where a_{i_n} is the n-th Fourier coefficient of $\phi_i(x)$. Renormalize and set $a_1 = 1$ so that the Hecke eigenvalues are also the coefficients.

The *standard L-function* formed from ϕ is defined to be

$$L(s,\phi) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \text{ prime}} (1 - a_p p^{-s} + p^{-2s})^{-1}, \Re s > 1.$$

It can be "completed" by multiplying it by a certain product of gamma functions: let $\epsilon = 1$ if ϕ is odd and $\epsilon = 0$ if ϕ is even. Then

$$\Lambda(s,\phi) = \pi^{-s}\Gamma(\frac{s+\epsilon+\nu}{2})\Gamma(\frac{s+\epsilon-\nu}{2})L(s,\phi)$$

can be analytically continued to the entire complex plane and has the functional equation $\Lambda(s,\phi) = (-1)^{\epsilon}\Lambda(1-s,\phi)$.

2.3 Fundamental facts and conjectures

In closing, here are some of the main facts and open questions about the cusp forms on $SL_2(\mathbf{Z})\backslash\mathbf{H}$.

Theorem 2.3.1 *(Selberg)*

$$N(T) = \#\{\lambda \leq T\} \sim \frac{T}{12}$$

as $T \to \infty$.

It was known by Roelcke earlier that the automatically-cuspidal odd eigenfunctions exist and the number of these between zero and T is $N_{odd}(T) \sim \frac{T}{24}$.

Theorem 2.3.2 *([Roelcke],Selberg) The parameter* $\nu = \sqrt{\frac{1}{4} - \lambda}$ *is always imaginary, so that* $\lambda \geq \frac{1}{4}$.

It is a famous unsolved conjecture of Selberg ([Selberg 1965]) that $\lambda \geq \frac{1}{4}$ for all congruence subgroups as well.

Conjecture 2.3.3 *([Cartier]) The multiplicity of any eigenvalue in the discrete spectrum of* Δ *on* $L^2(SL_2(\mathbf{Z})\backslash\mathbf{H})$ *is one.*

Conjecture 2.3.4 *(Generalized Ramanujan Conjecture) For p prime,* $|a_p| \leq 2$.

Conjecture 2.3.5 *(Sato-Tate) For a fixed cusp form* ϕ*, the* a_p*'s are distributed according to the* Sato-Tate *distribution. Write*

$$a_p = 2\cos\theta_p, 0 \leq \theta_p \leq \pi.$$

Then for any continuous function $f : [0, \pi] \to \mathbf{R}$

$$\frac{1}{\#\{p \leq X\}} \sum_{p \leq X} f(\theta_p) \to \int_0^\pi f(\theta) \frac{\sin^2(\theta)}{\pi/2} d\theta \text{ as } X \to \infty.$$

Chapter 3

Algebraic Groups

3.1 Lie group background

Lemma 3.1.1 *The group $G = SL_n(\mathbf{R})$ has the Iwasawa decomposition $G = NAK$, that is, each element $g \in G$ can be uniquely written as $g = nak$, with*

$$n \in N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \text{ (a nilpotent } \textit{radical}\text{)},$$

$$a \in A = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \mid a_j > 0 \right\} \text{ (a maximal torus)},$$

and

$$k \in SO_n(\mathbf{R}) \text{ (a maximal compact } \textit{subgroup}\text{)}.$$

Proof: Let $g \in G$. Then gg^t is positive-definite and symmetric and so there is an upper triangular matrix p with positive diagonal entries (i.e. $p \in NA$) such that

$gg^t = pp^t$. The matrix $k = p^{-1}g$ must lie in $K = SO_n(\mathbf{R})$ since

$$gg^t = pk(k^t p^t) = pp^t \Longrightarrow kk^t = I.$$

Moreover, if $p_1 k_1 = p_2 k_2$ then $p_1 \in K \cap NA = \{I\}$, which proves uniqueness. Finally, every element p of NA can be uniquely expressed as na for some $n \in N$ and $a \in A$ by letting a be the diagonal elements of p. □

The map $g \mapsto gg^t$ is actually an isometry of the symmetric space $\mathcal{H}_n = G/K$ with $\mathcal{P}_n$, the space of positive-definite symmetric matrices of determinant equal to one. We can use this to describe the geometry of $\mathcal{H}_n$ more easily in terms of the $\mathcal{P}_n$ coordinates, and conclude this section with a summary of it. First note that the action of G on $\mathcal{H}_n$ by left-multiplication

$$hK \overset{g}{\mapsto} ghK$$

becomes similarity in $\mathcal{P}_n$:

$$hh^t \overset{g}{\mapsto} g(hh^t)g^t.$$

Our coordinates on $\mathcal{P}_n$ will be expressed in terms of the symmetric matrices

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{jk} \\ \vdots & \ddots & \vdots \\ y_{kj} & \cdots & y_{nn} \end{pmatrix},$$

$$dY = \begin{pmatrix} dy_{11} & \cdots & dy_{jk} \\ \vdots & \ddots & \vdots \\ dy_{kj} & \cdots & dy_{nn} \end{pmatrix},$$

and

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_{11}} & \cdots & \frac{1}{2}\frac{\partial}{\partial y_{jk}} \\ \vdots & \ddots & \vdots \\ \frac{1}{2}\frac{\partial}{\partial y_{kj}} & \cdots & \frac{\partial}{\partial y_{nn}} \end{pmatrix}.$$

The invariant metric is

$$ds^2 = \text{trace}((Y^{-1}dY)^2),$$

for the similarity

$$Y \mapsto UYU^t$$

induces

$$dY \mapsto UdYU^t, Y^{-1} \mapsto U^{t^{-1}}Y^{-1}U^{-1},$$

so

$$(Y^{-1}dY)^2 \mapsto U^{t^{-1}}(Y^{-1}dY)^2U^t,$$

which have the same trace. The invariant volume is given by

$$dV = \det(Y)^{-\frac{n+1}{2}} \prod_{1\leq i<j\leq n} dy_{ij}.$$

By the same token

$$\text{trace}((Y^{-1}\frac{\partial}{\partial Y})^k), k = 2, \ldots, n$$

are G-invariant differential operators and in fact generate $\mathcal{D}$, the ring of G-invariant differential operators on $\mathcal{P}_n$. The Laplace-Beltrami operator here is

$$\Delta = \text{trace}((Y^{-1}\frac{\partial}{\partial Y})^2).$$

More detailed information can be found in [Terras], volume II, or [Maass].

3.1.1 Parabolic subgroups

Fix a partition $\pi = (n_1, n_2, \ldots, n_r)$ of $n = n_1 + n_2 + \cdots + n_r$. The *standard parabolic* associated to π is

$$P_\pi = \left\{ \begin{pmatrix} \overbrace{*}^{n_1} & \overbrace{*}^{n_2} & \cdots & \overbrace{*}^{n_r} \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\} = N_\pi M_\pi,$$

its nilpotent radical

$$N_\pi = \left\{ \begin{pmatrix} I_{n_1 \times n_1} & * & * & * \\ 0 & I_{n_2 \times n_2} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & I_{n_r \times n_r} \end{pmatrix} \right\}$$

and its Levi component

$$M_\pi = \left\{ \begin{pmatrix} \overbrace{*}^{n_1} & \overbrace{0}^{n_2} & \cdots & \overbrace{0}^{n_r} \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}.$$

Thus P_π consists of those elements of G for which the elements below the diagonal $n_1 \times n_1, n_2 \times n_2, \ldots, n_r \times n_r$ blocks are zero, N_π those elements of P_π whose diagonal blocks are identity matrices, and M_π those elements of P_π which are zero both above and below the diagonal blocks. The Levi component further decomposes into the direct product of A_π and $M'_\pi = M_\pi / A_\pi$, where A_π is the connected center of M_π and thus consists of diagonal matrices which are scalar multplies of the identity matrix in each $n_i \times n_i$ block. Every element m of M_π can be uniquely factored as $m = m_1 m_2 \cdots m_r$, with $m_i \in GL_{n_i}(\mathbf{R})$. Also, every element of $m' \in M'_\pi$ can be uniquely factored as $m' = m'_1 m'_2 \cdots m'_r$, with $m'_i \in GL^{\pm}_{n_i}(\mathbf{R})$ (determinant ± 1).

Let $\mathbf{a}_\pi$ be the Lie algebra of A_π, which is isomorphic to

$$\left\{ \begin{pmatrix} \overbrace{\begin{matrix} h_1 & & \\ & \ddots & \\ & & h_1 \end{matrix}}^{n_1} & & & & \\ & \overbrace{\begin{matrix} h_2 & & \\ & \ddots & \\ & & h_2 \end{matrix}}^{n_2} & & & \\ & & \ddots & & \\ & & & \overbrace{\begin{matrix} h_r & & \\ & \ddots & \\ & & h_r \end{matrix}}^{n_r} \end{pmatrix} \right\}$$

Each $\mathbf{a}_\pi$ can be embedded into $\mathbf{a}_0$: the vector $(n_1 h_1, n_2 h_2, \ldots, n_r h_r) \in \mathbf{a}_\pi$ with $h_1 + \cdots + h_r = 0$ is sent to the above matrix. This can be reversed by the surjection from $\mathbf{a}_0$ to $\mathbf{a}_\pi$ given by $(h_1, \ldots, h_n) \mapsto (h_1 + \cdots + h_{n_1}, \ldots, h_{n-n_r+1} + \cdots + h_n)$. There is a "logarithm" map H_π from G to $\mathbf{a}_\pi$ given as follows. Let $m(g) \in M_\pi$ be as in the Langlands decomposition and write $m(g) = m_1 m_2 \cdots m_r$ as described above. Then

$$H_\pi(g) = (\log|\det(m_1)|, \log|\det(m_2)|, \ldots, \log|\det(m_r)|).$$

3.1.2 Roots

There are $\binom{r}{2}$ linear functionals of the form

$$\alpha_{i,j}\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix} = a_i - a_j, i < j$$

on $\mathbf{a}_\pi$ and $\alpha_{i,j}$ is a *root* of (P_π, A_π) with multiplicity $n_i n_j$.

These occur as follows. The Lie algebra of N_π, $\mathbf{n}_\pi$, has a basis consisting of

$$\{e_{k,l} \mid n_1 + \cdots + n_{i-1} < k \leq n_1 + \cdots + n_i, n_1 + \cdots + n_{j-1} < l \leq n_1 + \cdots + n_j, i < j\}.$$

Here $e_{i,j}$ is shorthand for the $n \times n$ matrix with zeroes in every entry except for the i, j-th, which is one. The Lie algebra $\mathbf{n}_\pi$ acts on $\mathbf{a}_\pi$ by the "adjoint"

$$\mathrm{ad}(y) : x \mapsto [y, x] = yx - xy.$$

A simple calculation shows that

$$\mathrm{ad}(e_{k,l}) : a \mapsto (a_i - a_j)(e_{k,l}),$$

where $n_1+\cdots+n_{i-1} < k \leq n_1+\cdots+n_i, n_1+\cdots+n_{j-1} < l \leq n_1+\cdots+n_j$, and $i < j$. These are just the $\alpha_{i,j}$ from the previous paragraph, and thus occur with multiplicity $n_i n_j$. The *simple* roots among these are $\alpha_i = \alpha_{i,i+1}$. All these roots $\alpha_{i,j}$ are often called "positive" roots because $i < j$.

The roots $\alpha_{i,j}$ are generating elements of $\mathbf{a}^*$, the (real) dual of $\mathbf{a}$. We will sometimes use the *co-roots* $\alpha_{i,j}^\vee = (0, \ldots, 1, \ldots, -1, \ldots, 0)$, a vector which has 1 in the i-th entry,

-1 in the j-th, and zeroes everywhere else. A very important element of $\mathbf{a}_\pi^*$ is ρ_π, half the sum of the positive roots (with multiplicity). We will often write elements λ of $\mathbf{a}_\pi^*$ as vectors $(\lambda_1, \ldots, \lambda_r)$ if $\lambda(\alpha_{i,j}) = \lambda_i - \lambda_j$. The maps described at the end of the last section have obvious generalizations to $\mathbf{a}^*$ through this identification.

3.1.3 Haar measure

How does conjugation by an element of A_π affect N_π? Let

$$n = \begin{pmatrix} I_{n_1 \times n_1} & N_{12} & \cdots & N_{1r} \\ 0 & I_{n_2 \times n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & N_{rr-1} \\ 0 & \cdots & 0 & I_{n_r \times n_r} \end{pmatrix}$$

and

$$\left\{ \begin{pmatrix} \overbrace{a_1 \quad\quad}^{n_1} & & & & & & \\ & \ddots & & & & & \\ & & a_1 & & & & \\ & & & \overbrace{a_2 \quad\quad}^{n_2} & & & \\ & & & & \ddots & & \\ & & & & & a_2 & \\ & & & & & & \ddots \\ & & & & & & & \overbrace{a_r \quad\quad}^{n_r} \\ & & & & & & & & \ddots \\ & & & & & & & & & a_r \end{pmatrix} \right\}$$

be a typical element of A_π. Then

$$ana^{-1} = \begin{pmatrix} I_{n_1\times n_1} & N_{12}\frac{a_1}{a_2} & \cdots & N_{1r}\frac{a_1}{a_r} \\ 0 & I_{n_2\times n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & N_{rr-1}\frac{a_{r-1}}{a_r} \\ 0 & \cdots & 0 & I_{n_r\times n_r} \end{pmatrix}$$

and

$$N_{ij} \mapsto N_{ij}\frac{a_i}{a_j}.$$

The Haar measure on N_π is $\prod_{i<j} |dN_{ij}|$, where we use $|d(x_{ij})|$ to denote $\prod_{i,j} dx_{ij}$. Thus,

$$n \mapsto ana^{-1}$$

changes the volume by

$$dn \mapsto \prod_{i<j}(\frac{a_i}{a_j})^{n_i n_j} dn = e^{2\rho_\pi(H_\pi(a))} dn.$$

We can use this to describe the Haar measure on $G \simeq N_\pi \times M_\pi/K_\pi \times A_\pi \times K$ ($K_\pi = K \cap M_\pi$) from Haar measures on the four spaces (see [HarishChandra]), denoted $dn_\pi, dm_\pi, da_\pi, dk$. We normalize

$$\int_K dk = \int_{SL_n(\mathbf{Z})\cap N_\pi\backslash N_\pi} dn_\pi = 1.$$

Then

$$\int_G f(nmak)dg = \int_K \int_{A_\pi} \int_{M_\pi/K_\pi} \int_{N_\pi} f(nmak) e^{2\rho_\pi(H_\pi(a))} dn_\pi dm_\pi da_\pi dk.$$

3.1.4 Automorphic forms and constant terms

Our previous definition of automorphic form on $SL_2(\mathbf{R})$ was that of an automorphic function

$$\psi : G \to \mathbf{R}$$

such that

$$\psi(\gamma g k) = \psi(g) \text{ for all } \gamma \in \Gamma, k \in K, g \in G$$

which was an eigenfunction of the laplacian and all other invariant differential operators, and which satisfied a polynomial growth condition. This growth condition has a natural generalization in higher rank. If $\mathcal{F}_N$ is a compact fundamental domain for $\Gamma \cap N_0 \backslash N_0$, the set

$$S = \mathcal{F}_N \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} \mid a_j > 0, \frac{a_j}{a_{j+1}} > \frac{\sqrt{3}}{2}, j = 1, \ldots, n-1 \right\} K$$

is an example of a *Siegel set*. It can be shown that this particular choice of S contains a fundamental domain for $\Gamma \backslash G$ and intersects only finitely many other Γ translates of itself. Thus $\mathcal{F}$, a fundamental domain for $\Gamma \backslash \mathcal{H}_n$, is not compact and of finite volume (use the Haar measure from the previous section). In terms of the upper half plane coordinates, a Siegel domain is given by $\{x + iy | -\frac{1}{2} \leq x \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2}\}$. Our growth condition now is that there is a positive integer N such that

$$f(g) = O(e^{NH_0(g)}) \text{ for all } g \in S.$$

The *constant term* of an automorphic form ψ in the parabolic P is defined as

$$\psi_P(x) = \int_{\Gamma\cap N\backslash N} \psi(nx)dn.$$

A *cusp form* is an automorphic form whose constant terms in every standard proper parabolic subgroup vanish. As before, there are Hecke operators acting on $L^2(\Gamma\backslash\mathcal{H}_n)$ which are self-adjoint and commute with each other and the laplacian. We shall thus additionally require our automorphic forms to be Hecke eigenforms as well.

3.1.5 Levi components and Eisenstein series

Fix a partition π as before and consider $M'_\pi = M_\pi/A_\pi$, which consists of r blocks on the diagonal. The *Langlands decomposition* of P_π is

$$P_\pi = N_\pi M_\pi = M_\pi N_\pi = N_\pi A_\pi M'_\pi.$$

An automorphic form ϕ on

$$(\Gamma \cap M'_\pi)\backslash M'_\pi/(K \cap M_\pi)$$

is really r automorphic forms ϕ_i on each block. Moreover, ϕ is cuspidal on the Levi M_π if and only if each ϕ_i is.

We are now ready for the definition of general Eisenstein series. Start with an discrete eigenfunction on M'_π/K_π ($K_\pi = M_\pi \cap K$) and extend it to $M_\pi = A_\pi M'_\pi = M'_\pi A_\pi$ by

$$\phi(m'(g))e^{(\lambda+\rho_\pi)(H(g))}, \lambda \in \mathbf{a}^*_{\mathbf{C}}.$$

(We will often write $\phi(m(g))$ for $\phi(m'(g))$.) This is unchanged by multiplying g by an element of N_π on the left, and the Eisenstein series is defined as the automorphized

$$E(P, x, \phi, \lambda) = \sum_{\gamma \in \Gamma \cap P_\pi \backslash \Gamma} \phi(m(\gamma x)) e^{(\lambda + \rho_\pi)(H(\gamma x))}$$

for λ where this series converges, which are when $\Re(\lambda - \rho_\pi)(\alpha_i^\vee) > 0$ for $i = 1, \ldots, r$.

3.2 Basic properties of Eisenstein series

The Eisenstein series can be meromorphically continued to $\lambda \in \mathbf{a}_{\mathbf{C}}^*$ ([Langlands 1976]) and have no poles for $\lambda \in i\mathbf{a}^*$, where they contribute to the continuous spectrum ([Arthur 1984]). In studying their analytic properties it is essential to consider not only the parabolic P but all associate parabolics P'.

Definition Two standard parabolics P_π and P'_π are considered to be *associate* to each other (denoted $P \sim P'$) if the partition π is a permutation of π'.

Of course both partitions must be of the same length. If $\pi = (n_1, n_2, \ldots, n_r)$ and $\pi' = (n'_1, n'_2, \ldots, n'_r)$ have associate parabolics, then there is an element $s \in S_r$ (the permutation group on r letters) such that $n'_i = n_{s(i)}, i = 1, \ldots, r$. The collection of such association permutations is called the *Weyl group* $\Omega(\pi, \pi')$. Of course, each element of the Weyl group[1] gives rise to an isomorphism of $\mathbf{a}_\pi \to \mathbf{a}_{\pi'}$. If $s \in \Omega(\pi, \pi')$ then s sends

$$h = (h_1, \ldots, h_r) \in \mathbf{a}_\pi$$

to

$$sh = (h_{s(1)}, \ldots, h_{s(r)}) \in \mathbf{a}_{\pi'}.$$

This is a group action if we follow the convention of multiplying permutation repre-

[1] Often we will refer to this as $\Omega(P, P')$ or $\Omega(\mathbf{a}_P, \mathbf{a}_{P'})$.

sentations from left to right. For example

$$(13)(12)(h_1, h_2, h_3) = (13)(h_2, h_1, h_3) = (h_3, h_1, h_2)$$

$$= (321)(h_1, h_2, h_3).$$

Every Weyl group transformation is given by conjugation of $a \in \mathbf{a}$ with a permutation matrix:

$$sh = w_s h w_s^{-1}$$

if w_s is the permutation matrix of $s \in S_r$. The above example is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & h_1 & 0 \\ 0 & 0 & h_2 \\ h_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} h_3 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_2 \end{pmatrix}.$$

We can actually force w_s to be an element of K with integral entries by fixing the sign of at most one entry. It is easy to see that this does not affect the action.

The Weyl group also acts naturally on the dual via the same action. Note that

$$\lambda(sh) = \sum_{i=1}^{r} \lambda_i h_{s(i)} = \sum_{i=1}^{r} \lambda_{s^{-1}(i)} h_i = (s^{-1}\lambda)(h).$$

We remark that the conjugation sends $e_{i,j} \mapsto \pm e_{s^{-1}(i),s^{-1}(j)}$, the sign depending on how we choose $w_s \in K(\mathbf{Z})$.

3.2.1 Examples: $n = 2$

Here there are only two partitions of 2: $\pi_{-1} = (2)$ and $\pi_0 = (1,1)$, corresponding to the parabolics $P_{-1} = G$ and $P_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, respectively. The Lie algebra of $\mathbf{a}_{-1} = \{0\}$, so there is nothing to describe. However, the minimal parabolic P_0 is associate to itself by the permutation $(12) \in S_2$. The corresponding map $\mathbf{a}_0 \to \mathbf{a}_0$ is just $(a, -a) \mapsto (-a, a)$, which is also given by conjugation:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}.$$

Examples: n=3

Here there are three partitions, which we will call

$$\pi_{-1} = (3), P_{(3)} = P_{-1} = G,$$

$$\pi_0 = (1,1,1), P_{(1,1,1)} = P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

$$\pi_1 = (2,1), P_{(2,1)} = P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\},$$

and

$$\pi_2 = (1,2), P_{(1,2)} = P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}.$$

Of these, the minimal parabolic P_0 is associate to itself ($\Omega(\pi_0) = S_3$) and the maximal parabolics $P_1 \sim P_2$ are associate ($\Omega(\pi_1, \pi_2) = \{(12)\}$).

3.2.2 Constant terms

If ϕ is a Hecke cusp form, the constant term of the Eisenstein series $E(P, x, \phi, \lambda)$ is zero in any higher rank parabolic, and also in parabolics of the same rank which are not associate (a theorem of Langlands [Langlands 1976]). The story for parabolics of higher rank is slightly more complicated.

If $P \sim P'$ are associate parabolics then

$$E_{P'}(P, x, \phi, \lambda) = \int_{\Gamma \cap N' \backslash N'} E(P, nx, \phi, \lambda) dn$$

is only a function of M' in the x variable and

$$E_{P'}(P, x, \phi, \lambda) = \sum_{s \in \Omega(\pi, \pi')} e^{(s\lambda + \rho)(H(m))} \left[M(s, \lambda)\phi \right](x).$$

The "scattering operators" $M(s, \lambda)$ are in general difficult to calculate but we will compute some examples in the next section. Here are some descriptions.

Examples: $n = 2, \pi_0 = (1, 1)$

The "cusp forms" on the 1×1 block are just the constant functions (since the cuspidality condition is vacuous – there are no proper parabolics). So the only type of

Eisenstein series here is

$$E(P_0, x, \lambda) = \sum_{\gamma \in \Gamma \cap P_0 \backslash \Gamma} e^{(\lambda+\rho_0)(H(\gamma x))}.$$

Since

$$\Omega(\pi_0) = S_2 = \{e, (12)\},$$

$$E(P_0, x, \lambda) = e^{(\lambda+\rho_0)(H(x))} + \frac{\zeta^*(\lambda_i - \lambda_j)}{\zeta^*(1 + \lambda_i - \lambda_j)} e^{(-\lambda+\rho_0)(H(x))},$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

Examples: $n = 3, \pi_0 = (1, 1, 1)$

Again, ϕ is identically 1 and

$$E_{P_0}(P_0, x, \lambda) = \sum_{s \in \Omega(\pi_0) = S_3} e^{(s\lambda+\rho_0)(H(x))} \prod_{\substack{1 \le i < j \le 3 \\ s(i) > s(j)}} \frac{\zeta^*(\lambda_i - \lambda_j)}{\zeta^*(1 + \lambda_i - \lambda_j)},$$

a sum of six terms.

Examples: $n = 3, \pi_1 = (1, 2)$

Starting with a $\Gamma \cap M \backslash M / K_\pi$-cusp form (i.e. a $GL_2(\mathbf{Z}) \backslash \mathbf{H}$-cusp form, or equivalently an even $SL_2(\mathbf{Z}) \backslash \mathbf{H}$-cusp form) ϕ_2 on the 2×2 block and $\phi_1 = 1$ on the 1×1 block,

$$E(P_1, x, \phi, \lambda) = \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} e^{(\lambda+\rho_1)(H(\gamma x))} \phi(m(\gamma x)).$$

By Langlands' theorem

$$E_{P_0}(P_1, x, \phi, \lambda) = 0$$

and

$$E_{P_1}(P_1,x,\phi,\lambda)=\sum_{s\in\Omega(\pi_1)} e^{(s\lambda+\rho_1)(H(x))}\phi(sm(x))\prod_{\substack{1\le i<j\le 2\\ s(i)>s(j)}}\frac{\Lambda(\lambda_1-\lambda_2,\phi_1\times\tilde{\phi_2})}{\Lambda(1+\lambda_1-\lambda_2,\phi_1\times\tilde{\phi_2})}$$

$$=e^{(\lambda+\rho_1)(H(x))}\phi(m(x))$$

since $\Omega(\pi_1)=\{e\}$. Here $\Lambda(s,\phi_i\times\tilde{\phi_j})$ is a Rankin-Selberg L-function, and in the case $\phi_i=1$ $\Lambda(s,1\times 1)=\zeta^*(s)$. In general, given two associate parabolics P_1 and P_2, if $\phi(m_1(x))=\phi_1(m_{1_1}(x))\cdots\phi_r(m_{1_r}(x))$, define a function on M_2 corresponding to ϕ by

$$\phi(sm_2(x))=\phi(w_s^{-1}m_2(x)w_s)=\phi_1(m_{2_{s(1)}}(x))\cdots\phi_r(m_{2_{s(r)}}(x)).$$

Then

$$E_{P_2}(P_1,x,\phi,\lambda)=\sum_{s\in\Omega(\pi_1,\pi_2)} e^{(s\lambda+\rho_2)(H(x))}\phi(sm(x))\prod_{\substack{1\le i<j\le 2\\ s(i)>s(j)}}\frac{\Lambda(\lambda_1-\lambda_2,\phi_1\times\tilde{\phi_2})}{\Lambda(1+\lambda_1-\lambda_2,\phi_1\times\tilde{\phi_2})}$$

$$=e^{(t\lambda+\rho_2)(H(x))}\phi(tm(x))\frac{\Lambda(\lambda_1-\lambda_2,\phi)}{\Lambda(1+\lambda_1-\lambda_2,\phi)},$$

where $\Omega(\pi_1,\pi_2)=\{t\}$ and $\Lambda(s,\phi)=\Lambda(s,\phi_2\times\tilde{\phi_2})$ is the completed standard L-function of the $GL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form ϕ (a degree-two L-function) (see chapter 4).

3.3 Why is the constant term formula so simple?

3.3.1 Langlands' adelic method

In his influential monograph "Euler Products" ([Langlands 1971]), Langlands introduced an adelic method to compute the action of $M(s,\lambda)$ on cusp forms. The idea uses strong approximation and the correspondance between Hecke eigenforms on $\Gamma\backslash G(\mathbf{R})$ (where Γ is a congruence subgroup of $SL_n(\mathbf{Z})$) and those on $G(\mathbf{Q})\backslash G(\mathbf{A})$. If, for λ fixed, $M(s,\lambda)$ acts as a scalar on ϕ adelically, it must act as multiplication by that same scalar classically. Adelically the computation breaks up into a product of local integrals at each place, which give the local factors for both the gamma factors (the infinite place) and the Euler product (the finite places) of the ratio. The reason the action is scalar has to do with the fact that we have already chosen our basis of cusp forms to consist of eigenfunctions of not only the laplacian and all other invariant differential operators, but also of the Hecke operators as well. We shall now present an outline of Langlands method.

Let $K_{\mathbf{A}} = SO_n(\mathbf{R}) \times \prod_{p<\infty} SL_n(\mathbf{Z}_p)$ be a maximal compact subgroup of $SL_n(\mathbf{A})$. The *strong approximation* theorem for $SL_n(\mathbf{A})$ (see [Humphreys]) asserts that

$$SL_n(\mathbf{A}) = SL_n(\mathbf{Q})SL_n(\mathbf{R})K_{\mathbf{A}}. \tag{3.1}$$

Any cusp form ϕ on $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$ corresponds to a cusp form $\tilde{\phi}$ on $SL_n(\mathbf{Q})\backslash SL_n(\mathbf{A})/K_{\mathbf{A}}$ via the decomposition (3.1) (which is not unique but $\tilde{\phi}$ is well-defined because ϕ is invariant under $SL_n(\mathbf{Z})$ on the left and $SO_n(\mathbf{R})$ on the right – see [Gelbart]). The cuspidality condition is that the constant term $\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \tilde{\phi}(ng)dn$ vanishes for any proper standard parabolic P. This process is reversible, i.e. automorphic forms on $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$ correspond to automorphic forms on $SL_n(\mathbf{Q})\backslash SL_n(\mathbf{A})/K_{\mathbf{A}}$.

Similarly the logarithm $H_\pi : SL_n(\mathbf{R}) \to \mathbf{a}_\pi$ can be extended to $SL_n(\mathbf{A})$ just as before:

$$H(g) = (\log|\det(m_1)|_{\mathbf{A}}, \ldots, \log|\det(m_r)|_{\mathbf{A}}).$$

And so we can extend $F(x, \phi, \lambda) = e^{(\lambda+\rho)(H(x))}\phi(m(x))$ to $SL_n(\mathbf{A})$. This function is still invariant under the maximal compact $K_{\mathbf{A}}$, and thus $F(x, \phi, \lambda)$ is actually a function of $x \in B(\mathbf{A})$, where $B = P_0$ is often called the "Borel" subgroup.

3.3.2 Narrowing down to the Weyl group

We will compute the constant term

$$\int_{N'(\mathbf{Q})\backslash N'(\mathbf{A})} E(P, ng, \phi, \lambda)dn$$

of the Eisenstein series

$$E(P, g, \phi, \lambda) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} F(\gamma g, \phi, \lambda).$$

For "large" λ where this sum converges absolutely we may interchange the order of summation and integration:

$$\int_{N'(\mathbf{Q})\backslash N'(\mathbf{A})} E(P, ng, \phi, \lambda)dn = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \int_{N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(\gamma ng, \phi, \lambda)dn$$

$$= \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})/N'(\mathbf{Q})} \int_{\gamma^{-1}P(\mathbf{Q})\gamma \cap N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(\gamma ng, \phi, \lambda)dn$$

because

$$\begin{aligned} P(\mathbf{Q})\gamma N'(\mathbf{Q})n = \quad P(\mathbf{Q})\gamma N'(\mathbf{Q})n' &\iff n' = N'(\mathbf{Q})\gamma^{-1}P(\mathbf{Q})\gamma N'(\mathbf{Q})n \\ &\iff n'n^{-1} \in N'(\mathbf{Q}) \cap \gamma^{-1}P(\mathbf{Q})\gamma. \qquad (3.2) \end{aligned}$$

Recall that if w_s denotes a permutation matrix which represents $s \in \Omega(\mathbf{a}_0)$ (so that $w_s a w_s^{-1}$ permutes $a \in A_0$ as s does), then the *Bruhat decomposition* (see [Humphreys]) is

$$G(\mathbf{Q}) = \coprod_{s \in \Omega(\mathbf{a}_0)} B(\mathbf{Q}) w_s B(\mathbf{Q}).$$

Actually, $B(\mathbf{Q})w_sB(\mathbf{Q}) = B(\mathbf{Q})w_sN_0(\mathbf{Q})$ since the Weyl group permutes the entries of the diagonal matrices. Since B is contained in any standard parabolic, $\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})/N'(\mathbf{Q})$ may be chosen to be of the form $w\gamma'$, $\gamma' \in N_0(\mathbf{Q})/N'(\mathbf{Q})$.

We shall whittle away the possible choices of w and γ'. A typical term in the sum is an integral

$$\int_{\gamma'^{-1}w^{-1}P(\mathbf{Q})w\gamma' \cap N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(w\gamma' ng, \phi, \lambda) dn. \tag{3.3}$$

Change variables $n \mapsto {\gamma'}^{-1} n \gamma'$, and the integral becomes

$$\int_{w^{-1}P(\mathbf{Q})w \cap N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(wn\gamma' g, \phi, \lambda) dn.$$

Define ${}^0G = M/A$ (which we called M' before, but we will adhere to Langlands' notation from [Langlands 1971] in this section). Let ${}^0P_0 = B \cap M$ be the minimal parabolic of ${}^0G$ and define the (perhaps non-standard) nilpotent radical

$${}^0N = {}^0G \cap wN'w^{-1}.$$

Its lie algebra is generated by matrices $e_{i,j}$ in the lie algebra of M such that the matrix $e_{s(i),s(j)}$ lies in the lie algebra of N'. Then the integral decomposes as

$$\int_{(w^{-1}P(\mathbf{Q})w \cap N'(\mathbf{Q}))(w^{-1}{}^0N(\mathbf{A})w)\backslash N'(\mathbf{A})} \left(\int_{{}^0N(\mathbf{Q})\backslash {}^0N(\mathbf{A})} F(n_1 wng, \phi, \lambda) dn_1 \right) dn. \tag{3.4}$$

So unless ${}^0N = \{I\}$ the inner integral

$$\int_{{}^0N(\mathbf{Q})\backslash {}^0N(\mathbf{A})} e^{(\lambda+\rho)(H(n_1 b))} \phi(m(n_1 b)) dn_1 \quad , \quad bk = wng$$

$$= e^{(\lambda+\rho)(H(b))} \int_{^0N(\mathbf{Q})\backslash ^0N(\mathbf{A})} \phi(m(n_1 b))dn_1 \quad = \quad 0 \tag{3.5}$$

since ϕ is a cusp form. (Note cuspidality implies this even for non-standard parabolics which can be conjugated into standard ones by represenatives of the Weyl group that are in $K \cap G(\mathbf{Z})$, like ^0N might be.)

Hence we need only investigate those w for which $^0N = \{I\}$. This is trivially the case is P is minimal, for then M is diagonal. Also, it implies several cases of Langlands' famous theorem that the constant term is zero if P has lower rank than P' or if P and P' have the same rank but are not associate. In general, $^0B \cap wN'w^{-1} = \{I\} \Leftrightarrow wM'w^{-1} \supset M$. If P and P' are parabolics of the same rank then we in fact have equality, or equivalently that $wM' = Mw$. Then in our choice of represenative of $g = w\gamma'$ we have $\gamma' \in B(\mathbf{Q})/N'(\mathbf{Q})$ and thus can take $\gamma' \in M'(\mathbf{Q})$. But then $w\gamma' = \gamma'' w$ for some $\gamma'' \in M(\mathbf{Q}) \subset P(\mathbf{Q})$, which we can take to be the identity as we are dividing by this on the left. Further reduction of our options is often possible.

3.3.3 The constant term for associate parabolics

Assume P and P' are associate. Then the constant term is

$$\sum_{s \in \Omega(P',P)} \int_{w_s^{-1} P(\mathbf{Q}) w_s \cap N'(\mathbf{Q}) \backslash N'(\mathbf{A})} F(w_s n g, \phi, \lambda) dn$$

since $w_s M' w_s^{-1} = M$ and two elements of $\Omega(\mathbf{a}_0)$ correspond to the same map in $\Omega(P', P)$ if and only if they differ by an element of $\Omega(P)$ which stabilizes each block separately, and those permutation matrices are in $M(\mathbf{Q}) \subset P(\mathbf{Q})$. For the rest of this section (until the computations) s will be viewed as a permutation in S_r.

Let us examine

$$\int_{w_s^{-1}P(\mathbf{Q})w_s\cap N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(w_s ng, \phi, \lambda)dn.$$

Since we are integrating over $N'(\mathbf{A})$ we can take $g = m' \in M'$. The integral over $w_s^{-1}P(\mathbf{Q})w_s \cap N'(\mathbf{Q})\backslash N'(\mathbf{A})$ is over $\mathbf{Q}\backslash\mathbf{A}$ in some variables and over $\mathbf{A}$ in others. The former are those n_1 for which $w_s n_1 w_s^{-1} \in N(\mathbf{Q})$ (since we just eliminated the possibility of them lying in M) and $F((w_s n_1 w_s^{-1})(w_s nm'), \phi, \lambda) = F(w_s nm', \phi, \lambda)$ in these. The measure of $\mathbf{Q}\backslash\mathbf{A}$ is normalized to be one and so the integral

$$\int_{w_s^{-1}P(\mathbf{Q})w_s\cap N'(\mathbf{Q})\backslash N'(\mathbf{A})} F(w_s nm', \phi, \lambda)dn = \int_{N_w(\mathbf{A})} F(wnm', \phi, \lambda)dn,$$

where N_w denotes the other variables: those $n \in N'$ such that $w_s n w_s^{-1}$ is not in N. Its lie algebra is spanned by the elements of blocks $n_{i,j} \in \mathbf{n'}$ such that the block $n_{s^{-1}(i),s^{-1}(j)}$ is not in $\mathbf{n}$.

Change variables $n \mapsto m'nm'^{-1}$ to get

$$e^{2\rho_w(H(m'))}\int_{N_w} F(wm'n)dn.$$

Here ρ_w is half the sum of the positive roots of N_w. Since $w \in K$, if we set

$$m = wm'w^{-1} \in M, \bar{n} = wnw^{-1} \in \bar{N}_w = wN_w w^{-1}$$

then

$$F(wm'n, \phi, \lambda) = F(wm'w^{-1} \cdot wnw^{-1}, \phi, \lambda) = F(m\bar{n}, \phi, \lambda)$$

and the integral is

$$e^{2\rho_w(H(m'))}\int_{\bar{N}_w} F(m\bar{n})d\bar{n} = e^{(\lambda+\rho)(H(m))+2\rho_w(H(m'))}\int_{\bar{N}_w} e^{(\lambda+\rho)(H(\bar{n}))}\phi(m\cdot m(\bar{n}))d\bar{n}.$$

We will study the remaining integral in the next two sections and will finish this section by showing

$$(\lambda+\rho)(H(m)) + 2\rho_w(H(m')) = (s^{-1}\lambda+\rho')(H(m')). \tag{3.6}$$

Write $P = P_\pi$, $P' = P_{\pi'}$. Since these parabolics are associate, the partitions π and π' are permutations of each other. We already gave the condition that the i,j-th block of N' is in N_w if and only if the $s^{-1}(i), s^{-1}(j)$-th block is not in N, which can only happen for $s^{-1}(i) > s^{-1}(j)$. Thus

$$\begin{aligned} s^{-1}\rho &= \frac{1}{2}\sum_{\substack{i<j \\ s^{-1}(i)<s^{-1}(j)}} \rho'_{ij} - \frac{1}{2}\sum_{\substack{i<j \\ s^{-1}(i)>s^{-1}(j)}} \rho'_{ij} \\ &= \frac{1}{2}\sum_{i<j}\rho'_{ij} - \sum_{\substack{i<j \\ s^{-1}(i)>s^{-1}(j)}} \rho'_{ij} \\ &= \rho' - 2\rho_w \end{aligned}$$

(here we have used ρ'_{ij} to denote the sum of all the roots from the i,j-th block of N'). This completes the proof of (3.6) because $(\lambda+\rho)(H(wm')) = (s^{-1}\lambda + s^{-1}\rho)$. Since $\Omega(P,P') = \Omega(P',P)^{-1}$ the constant term is now a sum over the Weyl group:

$$\sum_{s\in\Omega(P,P')} e^{(s\lambda+\rho')(H(m'))}\int_{\bar{N}_{w_s^{-1}}} e^{(\lambda+\rho)(H(\bar{n}))}\phi((w_{s^{-1}}m'w_s)\cdot m(\bar{n}))d\bar{n}.$$

3.3.4 Example: $n = 2$ and the minimal parabolic

Let $P = P' = P_0$, the minimal parabolic of $SL(2)$. Here the Weyl group $\Omega(\mathbf{a}_0) = \{e, (12)\}$ and

$$\bar{N}_e = \{I\}, \bar{N}_{(12)} = \left\{ \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix} \right\}.$$

Here $\phi = 1$ and the integral splits as a product of local integrals. This next proposition determines its value as well as describing how to compute the local integrals which are used for determining the constant term in general.

Proposition 3.3.1 *(Langlands, see [Langlands 1989], pp.130-2)*

$$\int_{\bar{N}(\mathbf{A})} F(\bar{n}, \lambda) d\bar{n} = c(\lambda(\alpha^\vee)), \tag{3.7}$$

where

$$c(s) = \frac{\zeta^*(s)}{\zeta^*(s+1)}, \tag{3.8}$$

and

$$\zeta^*(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s). \tag{3.9}$$

Proof: If $p = \infty$ and

$$\bar{n}_p = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \tag{3.10}$$

then we want to find $a(\bar{n}_p)$, i.e. an $n \in N$ and $a \in A$ such that $\bar{n}_p \in naK_\infty$, $K_\infty = SO_2(\mathbf{R})$. Such an na must equal $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} \alpha & n/\alpha \\ 0 & 1/\alpha \end{pmatrix}$ and

$$\begin{pmatrix} \alpha & n/\alpha \\ 0 & 1/\alpha \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ n/\alpha & 1/\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\implies \tfrac{1}{\alpha^2} = x^2+1 \iff \alpha = \tfrac{1}{\sqrt{x^2+1}}. \tag{3.11}$$

Thus

$$a\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{\sqrt{x^2+1}} & 0 \\ 0 & \sqrt{x^2+1} \end{pmatrix}. \tag{3.12}$$

So our integrand

$$F\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = (\sqrt{x^2+1})^{-(\lambda+\rho)(\alpha^\vee)} \tag{3.13}$$

and

$$\int_{\bar{N}(\mathbf{R})} F\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) d\bar{n} = \int_{\mathbf{R}} (\sqrt{x^2+1})^{-\lambda(\alpha^\vee)-1} dx = \pi^{1/2} \frac{\Gamma\left(\frac{\lambda(\alpha^\vee)}{2}\right)}{\Gamma\left(\frac{\lambda(\alpha^\vee+1)}{2}\right)}. \tag{3.14}$$

For p finite, $K_p = SL_2(\mathbf{Z}_p)$ and we need to find an n and an a such that $\bar{n}_p \in naK_p$. Note that

$$\begin{aligned} &\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \\ = &\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & -x^{-1} \\ x & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix}. \end{aligned} \tag{3.15}$$

So if $x \in \mathbf{Z}_p$, $\bar{n}_p$ is already in K and $a(\bar{n}_p) = 1$; otherwise

$$a(\bar{n}_p) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}. \tag{3.16}$$

The integral is

$$\int_{\bar{N}(\mathbf{Q})} F(\bar{n}_p) d\bar{n}_p = \int_{\mathbf{Z}_p} dx + \int_{\mathbf{Q}_p - \mathbf{Z}_p} |x|^{-(\lambda+\rho)(\alpha^\vee)} dx$$

$$= 1 + \sum_{n=1}^{\infty} p^{-n((\lambda+\rho)(\alpha^\vee))}(p^n - p^{n-1}) = 1 + \left(1 - \frac{1}{p}\right) \sum_{n=1}^{\infty} p^{-n\lambda(\alpha^\vee)}$$

$$= 1 + \left(1 - \frac{1}{p}\right) \frac{p^{-\lambda(\alpha^\vee)}}{1 - p^{-\lambda(\alpha^\vee)}} = \frac{1 - p^{-\lambda(\alpha^\vee)} + p^{-\lambda(\alpha^\vee)} - p^{-\lambda(\alpha^\vee)-1}}{1 - p^{-\lambda(\alpha^\vee)}}$$

$$= \frac{1 - p^{-\lambda(\alpha^\vee)-1}}{1 - p^{-\lambda(\alpha^\vee)}}.$$

That is the p-th local factor of

$$\frac{\zeta(\lambda(\alpha^\vee))}{\zeta(\lambda(\alpha^\vee)+1)}. \tag{3.17}$$

□

This computation is a cornerstone of all the constant term calculations, for the L-function emerges from the local integrals. Our constant term is finally just

$$e^{(\lambda+\rho)(H(x))} + e^{((12)\lambda+\rho)(H(x))} c(\lambda(\alpha^\vee)).$$

3.3.5 Example: $n = 3$ and maximal parabolics

Let $P = P_1$, $P' = P_2$. There is just one Weyl group element, which we can take to be given by the permuation (13), and

$$\bar{N}_{(13)} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \star & \star & 1 \end{pmatrix} \right\}.$$

We will eventually compute

$$\int_{\bar{N}_{(13)}(\mathbf{Q}_p)} e^{(\lambda+\rho)(H(\bar{n}))}\phi(m \cdot m(\bar{n}_p))d\bar{n}_p$$

in terms of spherical functions by symmetrizing the integrand over

$$K_0 = G(\mathbf{Z}_p) \cap P(\mathbf{Q}_p)/G(\mathbf{Z}_p) \cap N(\mathbf{Q}_p) \simeq M(\mathbf{Z}_p)$$

for $p < \infty$ (for $p = \infty$ take just $M(\mathbf{R})/K_M(\mathbf{R})$). Define a measure $\mu_{M/K}$ on $M(\mathbf{Q}_p)/K_0$ by defining the measure of a set E in terms of the measure $\mu_{\bar{N}}$ on $\bar{N}$:

$$\mu_{M/K}(E) = \mu_{\bar{N}}\{\bar{n} \in \bar{N}(\mathbf{Q}_p) \mid m(\bar{n}) \in E\}.$$

Lemma 3.3.2 *([Langlands 1971]) The measure $\mu_{M/K}$ is left-invariant under K_0, i.e.*

$$\mu_{M/K}(kE) = \mu_{M/K}(E). \tag{3.18}$$

Proof: By definition

$$\mu_{M/K}(kE) = \mu_{\bar{N}}\{\bar{n} \in \bar{N}(\mathbf{Q}_p) \mid m(\bar{n}) \in kE\}. \tag{3.19}$$

Let $\bar{k}$ be an element of $P(\mathbf{Z}_p)$ which is in the coset k. Then

$$\bar{k}\bar{n}\bar{k}^{-1} = (\bar{k}n(\bar{n})\bar{k}^{-1})(\bar{k}m(\bar{n})\bar{k}^{-1})(\bar{k}k(\bar{n})\bar{k}^{-1}) \tag{3.20}$$

and since $\bar{k}k(\bar{n})\bar{k}^{-1}$ is clearly in K and $\bar{k}n(\bar{n})\bar{k}^{-1} \in N$ since $\bar{k} \in P(\mathbf{Z}_p)$,

$$m(\bar{k}\bar{n}\bar{k}^{-1}) = \bar{k}m(\bar{n})\bar{k}^{-1}. \tag{3.21}$$

Thus

$$\{\bar{n} \in \bar{N}(\mathbf{Q}_p) \mid m(\bar{k}^{-1}\bar{n}\bar{k}) \in E\} = \{\bar{k}\bar{n}\bar{k}^{-1} \in \bar{N}(\mathbf{Q}_p) \mid m(\bar{n}) \in E\}, \tag{3.22}$$

which has the same $\bar{N}$ measure as both E and kE. □

We can thus define a measure μ on M which is bi-K_0-invariant:

$$\mu(E) = \int_{M/K} \left(\int_{K_0} \chi_E(mk)dk \right) d\mu_{M/K}(m). \tag{3.23}$$

Suppose that $(\alpha_p, -\alpha_p)$ are the parameters of eigenform ϕ under the action of the Hecke algebra at the place p (which, if $p = \infty$, is the algebra of invariant differential operators described in section 5.2). Then the Hecke parameters of

$$e^{(\lambda+\rho)(H(\bar{n}))}\phi(mm(\bar{n}_p))$$

are

$$\mu_p = (\lambda_0 + 1 + \alpha_p, \lambda_0 - \alpha_p, -2\lambda_0 - 1),$$

if we write $\lambda = (\lambda_0, \lambda_0, -2\lambda_0)$. Using the bi-$K_0$-invariance

$$\int_{N(\mathbf{Q}_p)} e^{(\lambda+\rho)(H(\bar{n}_p))}\phi(mm(\bar{n}_p))$$

$$= \int_M e^{(\lambda+\rho)(H(m_1))}\phi(mm_1)d\mu(m_1)$$

$$= \phi(m) \int_M e^{(\lambda+\rho)(H(m_1))}\phi(m_1)d\mu(m_1)$$

$$= \phi(m) \int_{\bar{N}(\mathbf{Q}_p)} e^{\mu_p(H(\bar{n}_p))}d\bar{n}.$$

We can compute this integral as in the previous cases by rederiving the formula of Gindikin and Karpelevich (see [Langlands 1971]). First note that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}, \tag{3.24}$$

so

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} &\in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & n_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha_x} \end{pmatrix} K_p \\ &= \begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha_x} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y\alpha_x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{n_x}{\alpha_x^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} K_p. \end{aligned} \tag{3.25}$$

Observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & w & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & w & 1 \end{pmatrix} = \begin{pmatrix} 1 & -zw & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & w & 1 \end{pmatrix} \tag{3.26}$$

so modulo N_p and K_p, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix}$ is

$$\begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha_x} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y\alpha_x & 1 \end{pmatrix} = \begin{pmatrix} \alpha_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha_x} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha_{y\alpha_x} & 0 \\ 0 & 0 & \frac{1}{\alpha_{y\alpha_x}} \end{pmatrix}. \tag{3.27}$$

If we change variables by $y \mapsto \frac{y}{\alpha_x}$, $dy \mapsto \frac{dy}{\alpha_x}$ the integral is

$$\begin{aligned} \int_{\mathbf{Q}_p} \int_{\mathbf{Q}_p} \alpha_x^{\mu_1-\mu_3} \alpha_{y'}^{\mu_2-\mu_3} dy dx = & \quad \textstyle\int_{\mathbf{Q}_p} \alpha_x^{\mu_1-\mu_3-1} dx \int_{\mathbf{Q}_p} \alpha_y^{\mu_2-\mu_3} dy \\ = & \quad \textstyle\int_{\mathbf{Q}_p} \alpha_x^{3\lambda_0+\alpha_p-1} dx \int_{\mathbf{Q}_p} \alpha_y^{3\lambda_0-\alpha_p-1} dy. \end{aligned} \tag{3.28}$$

This gives, once performed at all places ($\phi(m)$ emerges after each integral),

$$\frac{\Lambda(3\lambda_0, \phi)}{\Lambda(3\lambda_0+1, \phi)} \phi(m), \tag{3.29}$$

where $\Lambda(s, \phi)$ is the completed standard L-function of ϕ, a degree-2 L-function. If ϕ_2 is the cusp form on M' which is ϕ on the 2×2 block, the overall constant term is

$$\frac{\Lambda(3\lambda_0, \phi)}{\Lambda(3\lambda_0+1, \phi)} e^{((13)\lambda+\rho)(H(g))} \phi_2(m_2(g)). \tag{3.30}$$

Chapter 4

Applications of Positivity to L-functions

In this chapter we describe certain L-functions and applications of their properties. Our techniques are based on the analysis of two types of L-functions that can be attatched to a cusp form ϕ. Recall that if $P = NM$ is a standard parabolic subgroup of G and if ϕ is an automorphic function on $\Gamma\backslash G$, ϕ's *constant term along* P is defined as

$$\phi_P(x) = \int_{\Gamma\cap N\backslash N} \phi(nx)dn.$$

A cusp form is an automorphic form ϕ for which $\phi_P(x) \equiv 0$ for all proper standard parabolic subgroups P.

4.1 L-functions of automorphic forms

The first of these L-functions is the so-called "standard" L-function $L(s, \phi)$ which is a degree-n Euler product formed from the eigenvalues of the Hecke operators on ϕ:

$$L(s, \phi) = \prod_p (1 - \alpha_{p_1} p^{-s})^{-1} (1 - \alpha_{p_2} p^{-s})^{-1} \cdots (1 - \alpha_{p_n} p^{-s})^{-1}.$$

It can be made entire by "completing" it, i.e. multiplying it by an appropriate product of gamma functions:

$$\Lambda(s, \phi) = \prod_{j=1}^{n} \Gamma_{\mathbf{R}}(s + \nu_j) L(s, \phi),$$

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).$$

Then $\Lambda(s, \phi)$ is entire and satisfies the functional equation $\Lambda(1 - s, \phi) = \pm \Lambda(s, \bar{\phi})$.

The second type of L-function is the Rankin-Selberg convolution, which is formed from the coefficients of $L(s, \phi)$:

$$L(s, \phi \times \tilde{\phi}) = \prod_p \prod_{j=1}^{n} \prod_{k=1}^{n} (1 - \alpha_{p_j} \overline{\alpha_{p_k}} p^{-s})^{-1}.$$

It can also be completed by multiplying it by appropriate gamma factors:

$$\Lambda(s, \phi \times \tilde{\phi}) = \prod_{j=1}^{n} \prod_{k=1}^{n} \Gamma_{\mathbf{R}}(s + \mu_{jk}) L(s, \phi \times \tilde{\phi}).$$

Both the μ_{jk}'s and the ν_j's can be computed from the *Langlands parameters* of the cusp form ϕ, which we will call the μ_j's (see section 5.2). If ϕ is unramified at infinity, for example if it is spherical or almost-spherical – see section 4.2 (as is the case for an eigenfunction of the laplacian), then $\mu_{jk} = \mu_j + \overline{\mu_k}$. The completed Rankin-Selberg L-function has the following analytic properties:

4.1.1 *It can be meromorphically continued to the entire complex plane, with only simple poles at* $s = 0, 1$.

4.1.2 *It has the functional equation* $\Lambda(s, \phi \times \tilde{\phi}) = \pm\Lambda(1 - s, \phi \times \tilde{\phi})$.

Both the standard and Rankin-Selberg L-functions are holomorphic functions of order one. Let's be a little more general and consider a completed L-function of degree m of order one

$$\Lambda(s) = \prod_{j=1}^{m} \Gamma_{\mathbf{R}}(s + \eta_j) L(s),$$

to cover both cases. It has a Mittag-Leffler expansion since it is of order 1:

$$\frac{\Lambda'}{\Lambda}(s) = \sum_{j=1}^{m} \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s + \eta_j) + \frac{L'}{L}(s) = B + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right) + \delta \left(\frac{1}{s} + \frac{1}{s - 1} \right).$$

Here B is a complex constant, the ρ are the zeroes of $\Lambda(s)$, $\delta = 1$ if $\Lambda(s)$ has poles at $s = 0$ and 1, and $\delta = 0$ if not. It is known that all zeroes lie betweeen $0 < \Re(\rho) < 1$ (the *prime number theorem* – see [Shahidi]). The sum over the zeroes is actually only conditionally convergent – one should sum ρ together with $\bar{\rho}$. The functional equation reduces this to

$$\frac{\Lambda'}{\Lambda}(s) = \sum_{\rho} \frac{1}{s - \rho} + \delta \left(\frac{1}{s} + \frac{1}{s - 1} \right),$$

provided the sum is performed correctly. Note that since $L(s)$ is assumed to have an Euler product, $\frac{L'}{L}(s)$ also has a dirichlet series for $\Re s > 1$:

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} c_n n^{-s}.$$

There is a smoothed version of this, Riemann's explicit formula (see [Rud-Sar]). Let $\{\gamma\}$ be the zeroes of $L(\frac{1}{2} + is)$, $g \in C_c^\infty(\mathbf{R})$, and $h(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du$ be the Fourier transform of g. Weil's formula reads

$$\sum h(\gamma) =$$

$$\delta\left(h(-\frac{i}{2})+h(\frac{i}{2})\right)+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\sum_{j=1}^{m}\left(\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(\frac{1}{2}+\eta_j+ir)+\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(\frac{1}{2}+\overline{\eta_j}-ir)\right)dr$$

$$-\sum_{n=1}^{\infty}\left(\frac{c_n}{\sqrt{n}}g(\log n)+\frac{\overline{c_n}}{\sqrt{n}}g(-\log n)\right).$$

Lemma 4.1.3 *(Weil)*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}(\sigma+iar)dr=-\int_0^{\infty}\left(\frac{g(ax)e^{-\sigma x}}{1-e^{-x}}-g(0)\frac{e^{-x}}{x}\right)dx.$$

Since

$$\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s)=-\frac{1}{2}\log\pi+\frac{\Gamma'}{2\Gamma}(\frac{s}{2}),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(\frac{1}{2}+\eta_j+ir)dr$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\left(-\frac{\log\pi}{2}\right)dr+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{2\Gamma}(\frac{1}{4}+\frac{\eta_j}{2}+\frac{ir}{2})dr$$

$$=l(\eta_j):=-\frac{\log\pi}{2}g(0)-\frac{1}{2}\int_0^{\infty}\left(\frac{g(x/2)e^{-(\frac{1}{4}+\frac{\eta_j}{2})x}}{1-e^{-x}}-g(0)e^{-x}x\right)dx.$$

If both g and h are even and real on $\mathbf{R}$ then the explicit formula can be rewritten as

$$\sum h(\gamma)=\delta\left(h(-\frac{i}{2})+h(\frac{i}{2})\right)+2\Re\left(\sum_{j=1}^{m}l(\eta_j)-\sum_{n=1}^{\infty}\frac{c_n}{\sqrt{n}}g(\log n)\right). \tag{4.1}$$

4.2 The spherical unitary dual of $SL_3(\mathbf{R})$

The cuspidal eigenfunctions of the laplacian Δ on $X=SL_3(\mathbf{Z})\backslash SL_3(\mathbf{R})/SO_3(\mathbf{R})$, together with the constant functions, are the discrete eigenfunctions of Δ in $L^2(X)$.

These cuspidal eigenfunctions can be related to irreducible subspaces of $L^2(\Gamma\backslash G)$ under the action of the right-regular representation $(R_g f)(x) = f(xg)$. This is a unitary representation; for a description of the unitary dual see [Speh 1981]. Since Δ commutes with these isometries R_g, by Schur's lemma it acts as a scalar on the irreducible subspaces of the action. The cusp forms are related to the irreducible representations which are spherical or almost-spherical, that is equivalent to induced representations

$$Ind_B^{GL_3} \chi_{\mu_1,\mu_2,\mu_3}, \; B = \left\{ b = \begin{pmatrix} b_1 & * & * \\ 0 & b_2 & * \\ 0 & 0 & b_3 \end{pmatrix} \right\},$$

$$\chi(b) = (sgn(b_1))^{\epsilon_1} b_1^{\mu_1} (sgn(b_2))^{\epsilon_2} b_2^{\mu_2} (sgn(b_3))^{\epsilon_3} b_3^{\mu_3}, \epsilon_j = 0, 1.$$

The μ_j's parametrize the eigenvalues of ϕ with respect to all the invariant differential operators (see section 5.2). For example, if $\Delta\phi + \lambda\phi = 0$,

$$\lambda = 1 - \frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{2}.$$

Thus, the problem of describing the discrete, cuspidal spectrum of the invariant differential operators is equivalent to describing part of the unitary dual which occurs in the decomposition of R_g on $L^2(\Gamma\backslash G)$. We may assume that $\{\overline{\mu_j}\} = \{-\mu_j\}$ because we are in the unitary dual ([Vakhutinski]) and that $\mu_1 + \mu_2 + \mu_3 = 0$ because every element of G has determinant one. This gives two possibilities – either the μ_j's are all imaginary and add to zero:

tempered (satisfies full Ramanujan-Selberg)

$$\begin{array}{c}\text{Langlands}\\ \text{parameters}\end{array} = \begin{cases} \mu_1 = iy_1 \\ \mu_2 = iy_2 \\ \mu_3 = -iy_1 - iy_2 \end{cases} \tag{4.2}$$

$$\lambda = 1 + y_1^2 + y_1 y_2 + y_2^2$$

or two are not imaginary and the third is:

non-tempered (violates full Ramanujan-Selberg)

$$\begin{array}{c}\text{Langlands}\\ \text{parameters}\end{array} = \begin{cases} \mu_1 = x + iy \\ \mu_2 = -x + iy \\ \mu_3 = -2iy \end{cases} \tag{4.3}$$

$$\lambda = 1 + 3y^2 - x^2$$

The archimedean Ramanujan-Selberg conjecture asserts that only the first possibility occurs. An immediate consequence is that $\lambda \geq 1$; however, the converse is not true.

We will use the explicit formula to prove certain pieces of the unitary dual do not occur for X. A consequence of our investigations is that $\lambda > 80$ for X. We will use the functions

$$g_p(x) = \left((1 - |x/p|) \cos(\pi |x|/p) + \frac{\sin(\pi |x|/p)}{\pi} \right) / \cosh(x/2)$$

which are zero outside of $[-p, p]$, for $p > 0$ a real parameter, and the Rankin-Selberg L-functions. We know nothing about the zeroes or coefficients of these but if $p < \log 2$

the coefficients do not even enter. It is known ([Rud-Sar]) that for the Rankin-Selberg L-function the $c_n \geq 0$. Using the fact that $h_p \geq 0$ in the critical strip (see [Fermigier] and [Odlyzko]) we may drop these two terms from (4.1) and arrive at the inequality

$$\int_{-\infty}^{\infty} g(x) \left(e^{x/2} + e^{-x/2}\right) dx + 2\Re \sum_{j=1}^{m} \sum_{k=1}^{m} l(\mu_j + \mu_k) \geq 0. \qquad (4.4)$$

This is an example of the "positivity" technique of [Stark].

For fixed μ_1, μ_2, μ_3 this inequality may be numerically examined. If it is false the L-function and hence the cusp form it came from cannot exist. We did this for each of the two pieces of the unitary dual. Each is a two-dimensional fragment and we indicate on the figures below where the inequality (4.4) is false. It is immediate from these pictures that $\lambda > 80$ (figures 4.1, 4.2), and even > 1000 (figures 4.3, 4.4) on the non-tempered unitary dual (which conjecturally does not exist anyway).

4.3 Cuspidal cohomology of $SL_n(\mathbf{Z})$

Theorem 4.3.1 *For* $1 < n < 27$

$$H^p_{cusp}(SL_n(\mathbf{Z}); \mathbf{R}) = 0, p \geq 0.$$

This is to say there are no constant-coefficients cuspidal p-forms which give rise to cohomology on $SL_n(\mathbf{Z}) \backslash SL_n(\mathbf{Z}) / SO_n(\mathbf{R})$ for $1 < n < 27$ (see [Borel 1974] for definitions and background). We proved in [Miller] using the Mittag-Leffler expansion of a Rankin-Selberg L-function that this is true for $1 < n < 23$; Fermigier ([Fermigier]) proved a similar theorem for GL_n using the standard L-function and the smoothed, explicit formula. We will combine both methods here to go slightly further.

The only way to get cuspidal cohomlogy is from a certain induced representation.

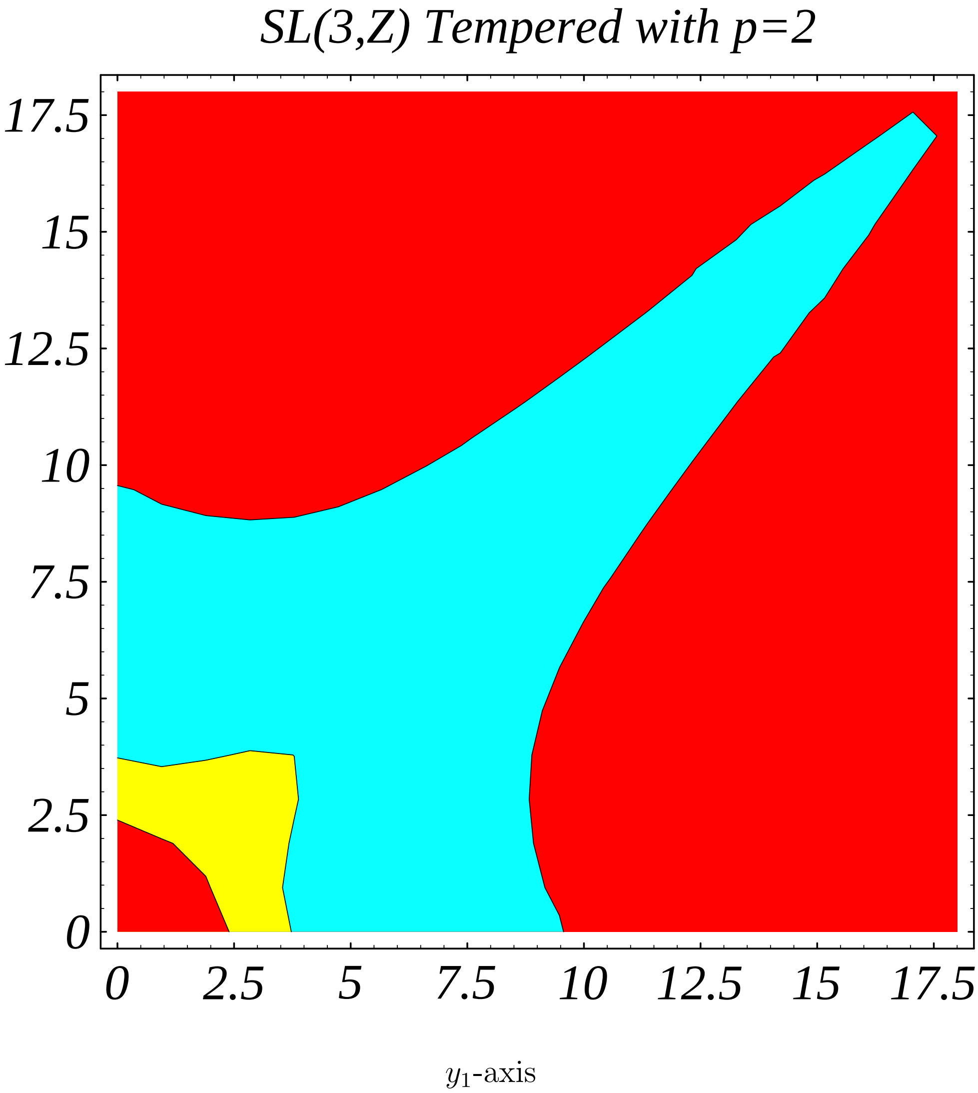

Figure 4.1: The tempered unitary dual ($p = 2$), with notation as in (4.2). The abcissa is the y_1-axis and the ordinate is the y_2-axis. Both y_1 and y_2 have been taken to be positive out of symmetry considerations. The shaded area starting from the upper left corner is where the inequality (4.4) is true; it is false in the other regions towards the lower left corner. The various shadings other than the darkest in the top left indicate how false the inequality is.

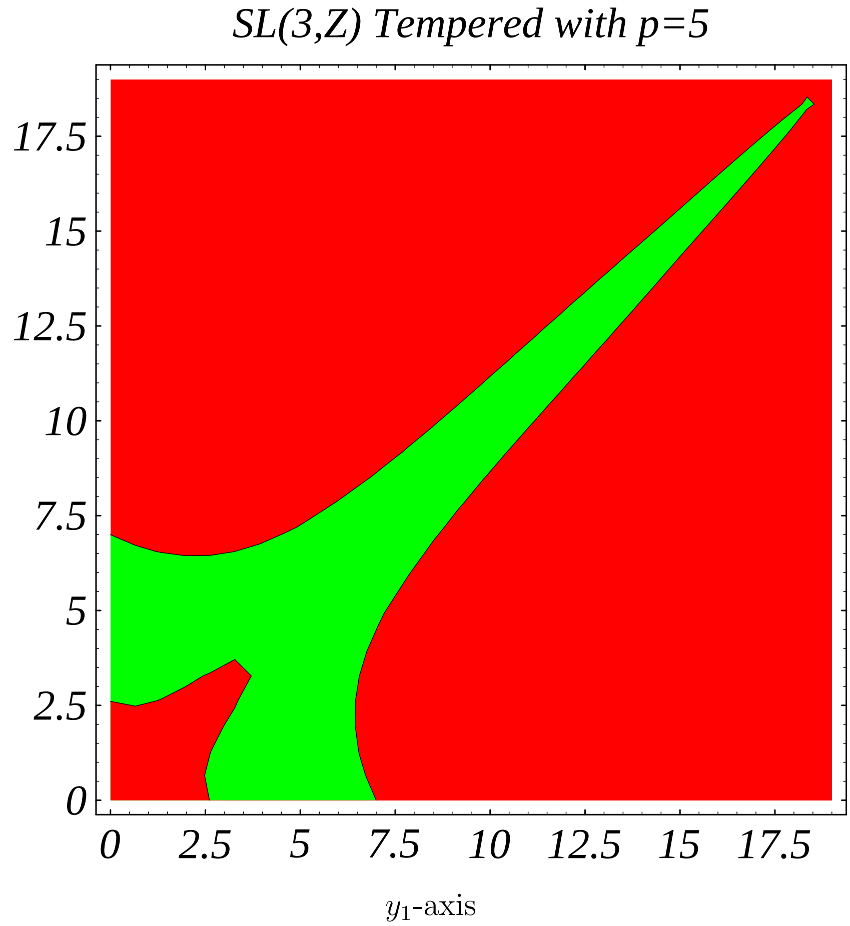

Figure 4.2: The tempered unitary dual ($p = 5$). Both x and y can be taken to be positive because of symmetry. Again, the inequality is true in the dark region in the upper left and lower right, but false in the other two regions which are closer to the lower left corner.

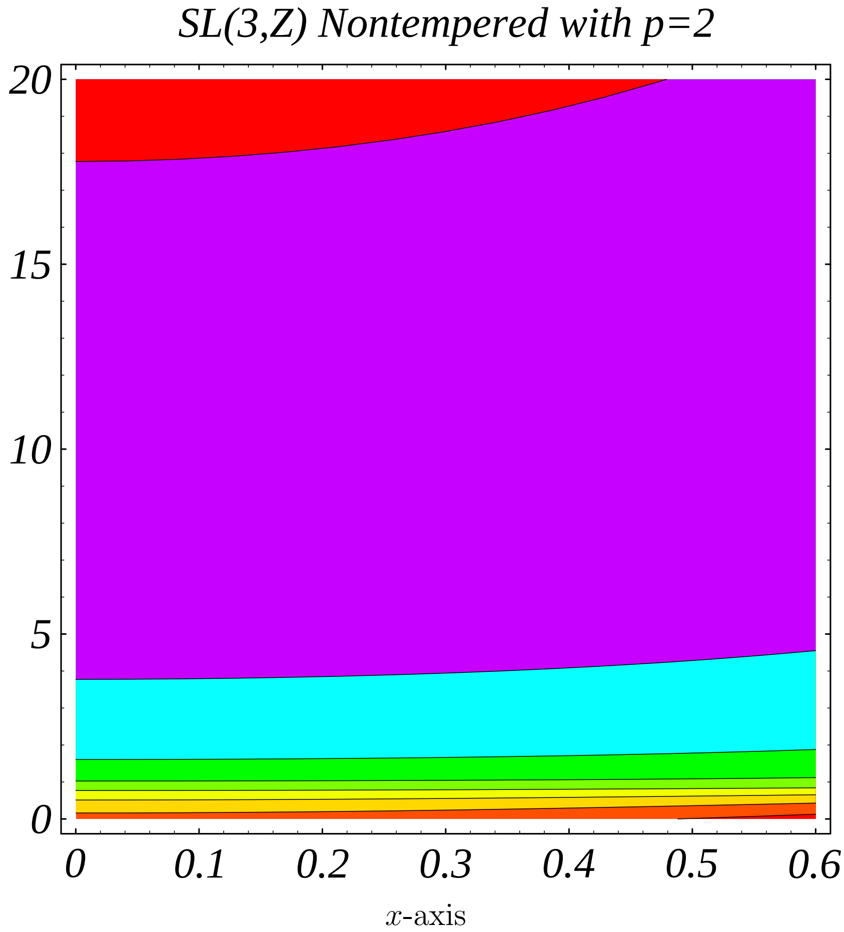

Figure 4.3: The non-tempered unitary dual ($p = 2$), with notation as in (4.3). The inequality is true in the upper left and false elsewhere.

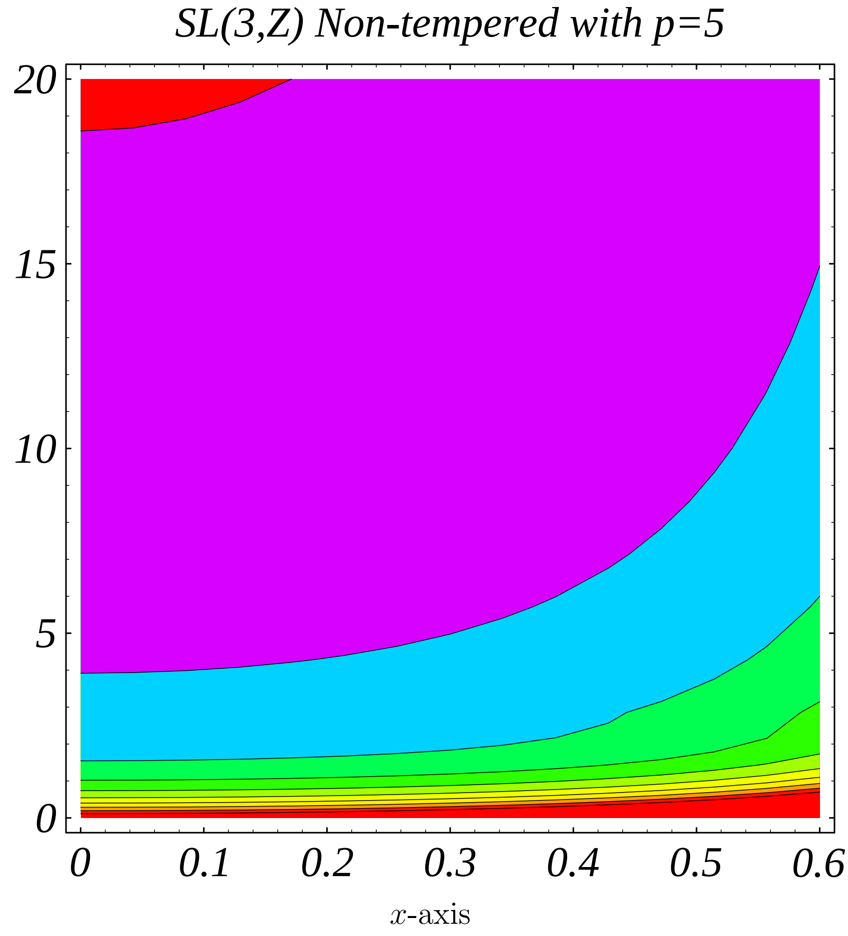

Figure 4.4: The non-tempered unitary dual ($p = 5$).

Write $n = 2m + t$, where $t = 0$ or 1 depending on the parity of n. If D_k denotes the k-th discrete series on GL_2 (corresponding to weight-k holomorphic forms), then this representation is

$$\mathrm{Ind}_{P_{(2,2,\ldots,2)}}^{GL_n}(D_2, D_4, \ldots, D_n)$$

if n is even and is

$$\mathrm{Ind}_{P_{(1,2,2,\ldots,2)}}^{GL_n}(1, D_3, D_5, \ldots, D_n)$$

if n is odd. The Rankin-Selberg parameters μ_{jk} of this representation can be easily computed via the procedure summarized in [Rud-Sar] and as multisets

$$\{\mu_{j,k}\} = \{t + j + k, t - 1 + j + k, |k - j|, 1 + |k - j| \mid 1 \leq j, k \leq m\}$$

$$\cup \overbrace{\{0, j, j, j + 1, j + 1 \mid 1 \leq j \leq m\}}^{\text{omit if } t=0}.$$

We can numerically evaluate the inequality (4.4) (which still applies here) and show that, for various choices of p, it is in fact false for $1 < n < 27$ (see table 4.1). It is unknown whether or not this vanishing persists, i.e. whether or not for some $n \geq 27$

$$H^p_{cusp}(SL_n(\mathbf{Z}); \mathbf{R}) \neq 0.$$

4.4 The winding number of a degree-2 L-function

This section uses the Mittag-Leffler expansion and hence we will present it here, even though it will not be needed until the next chapter.

n	p *(parameter for test function)*	*How much the inequaity (4.4) holds by*
2	2.	-3.46794
3	2.	-10.4516
4	2.	-17.9502
5	2.	-26.9594
6	2.	-35.5884
7	2.	-44.9868
8	2.	-53.3743
9	2.	-61.9835
10	2.	-69.0981
11	2.	-76.0017
12	2.	-81.0194
13	2.	-85.469
14	2.	-87.7046
15	2.	-89.0686
16	2.	-87.936
17	2.	-85.6679
18	2.	-80.6555
19	2.	-74.274
20	2.	-64.9277
21	2.	-54.0033
22	2.	-39.915
23	2.	-24.0592
24	2.	-4.85866
25	3.	-7.95176
26	5.	-2.10588

Table 4.1: The numerical proof of the cohomology theorem.

Proposition 4.4.1 *Let ϕ be a $SL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form with $\Delta\phi + (\frac{1}{4}+\nu^2)\phi = 0$. Its completed standard L-function is*

$$\Lambda(s,\phi) = \Gamma_{\mathbf{R}}(s+i\nu)\Gamma_{\mathbf{R}}(s-i\nu)\prod_p (1-\alpha_p p^{-s})^{-1}(1-\overline{\alpha_p}p^{-s})^{-1}.$$

Recall that $\nu > 0$. Then

$$\int_{1-iT}^{1+iT} \frac{\Lambda'}{\Lambda}(s)ds \ll T\log(T+\nu).$$

Proof: From the Mittag-Leffler expansion

$$\frac{\Lambda'}{\Lambda}(s) = \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s+i\nu) + \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s-i\nu) - \sum_{n=1}^{\infty} c_n n^{-s} = \sum_\rho \frac{1}{s-\rho}.$$

Let $s = 2+it$ and take $t > 0$ without loss of generality. Then since

$$\frac{L'}{L}(s) = -\sum_{n=1}^{\infty} c_n n^{-s} = -2\Re\sum_p \frac{\alpha_p p^{-s}\log p}{1-\alpha_p p^{-s}} = -2\Re\sum_p\sum_{n=1}^{\infty}(\alpha_p p^{-s})^n \log p,$$

the trivial bound $|a_p| \le 2\sqrt{p}$ implies $|\frac{L'}{L}(2+it)|$ is uniformly bounded in both t and ν. By Stirling's formula

$$\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(s) = -\frac{1}{2}\log\pi + \frac{1}{2}\log s + O(1/|s|)$$

and

$$\frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(2+it+i\nu) + \frac{\Gamma'_{\mathbf{R}}}{\Gamma_{\mathbf{R}}}(2+it-i\nu) \le \log(t+\nu) + O(1).$$

Thus,

$$\sum_\rho \frac{1}{1+|t-\gamma|^2} \ll \sum_\rho \frac{1}{s-\rho} \le \log(t+\nu) + O(1).$$

It follows that there are no more than $O(T\log(T+\nu))$ zeroes with $\gamma \in [-T, T]$ and the integral

$$\int_{1-iT}^{1+iT} \frac{\Lambda'}{\Lambda}(s)ds = \int_{1-iT}^{1+iT} \sum_{\rho} \frac{ds}{s-\rho} = \int_{1-iT}^{1+iT} \sum_{|\gamma|\leq T+1} \frac{ds}{s-\rho} + \int_{1-iT}^{1+iT} \sum_{|\gamma|>T+1} \frac{ds}{s-\rho}$$

$$\ll T\log(T+\nu).$$

Here we are using the principal value of the integrals

$$\int_{1-iT}^{1+iT} \frac{ds}{s-\rho} \leq \pi$$

for those zeroes with $\Re\rho = 1$, though there actually are not any because of the prime number theorem (which states that $0 < \Re\rho < 1$ for all zeroes of $\Lambda(s, \phi)$). $\square$

Chapter 5

Weyl's Law for $SL_3(\mathbf{Z})\backslash SL_3(\mathbf{R})/SO_3(\mathbf{R})$

5.1 Introduction

Let $G = SL_3(\mathbf{R})$, $K = SO_3(\mathbf{R})$, a maximal compact subgroup, and $\mathcal{H}$ = the rank-2 symmetric space G/K. If $\Gamma = SL_3(\mathbf{Z})$, a discrete group of isometries, then $X = \Gamma\backslash\mathcal{H}$ is non-compact (it has one cusp) but has finite volume under the Haar measure of G (we have normalized the Haar measure on K to be 1). The Laplace-Beltrami operator Δ on $\mathcal{H}$ is invariant under the isometries of G acting on X via left-multplication, and so Δ is defined on $L^2(\Gamma\backslash\mathcal{H})$. It has both a continuous spectrum and a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots,$$

and an orthonormal set

$$\phi_0 = \frac{1}{\sqrt{\mathrm{vol}(X)}}, \phi_1, \phi_2, \ldots,$$

with

$$\Delta\phi_j + \lambda_j\phi_j = 0.$$

The zero eigenvalue corresponds to the constant functions whereas the remaining, positive eigenvalues are from cusp forms $\phi_1, \phi_2, \ldots$. The cusp forms are also eigenvalues of all invariant differential operators, as well as the Hecke operators. It is known that the discrete spectrum is infinite because any $SL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form can be Gelbart-Jacquet ([GelJac]) lifted to X, and there are infinitely many of these (a theorem of Selberg).

We will establish the existence of non-lifted eigenvalues by proving Weyl's law for the laplacian Δ on X.

Theorem 5.1.1 *Let $N(T) = \#\{\lambda_j \leq T\}$. As $T \to \infty$*

$$N(T) \sim \frac{T^{5/2} vol(X)}{(4\pi)^{(5/2)}\Gamma(7/2)}.$$

Remark 5.1.2 *Weyl's law is true for compact manifolds X for any $n > 1$, but for the generic Γ not co-compact it is probably false (cf. the work of Phillips-Sarnak for general non-compact quotients of the hyperbolic plane $\mathbf{H}$). Our theorem confirms a conjecture of Sarnak (conjecture 1.1.6) which predicts that Weyl's law holds for any congruence cover of $SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R})/SO_n(\mathbf{R})$,$n > 1$.*

Remark 5.1.3 *The upper bound*

$$\limsup \frac{N(T)}{T^{5/2}} = \frac{vol(X)}{(4\pi)^{5/2}\Gamma(7/2)}$$

was proven by [Donnelly] for this quotient and more generally for the cuspidal spectrum of the laplacian on finite volume quotients of locally symmetric spaces. It is the *lower bound, and hence the existence, which we will address.*

Remark 5.1.4 *As alluded to above, Selberg proved Weyl's law for $SL_2(\mathbf{Z})\backslash\mathbf{H}$ in the mid-50's, via his seminal trace formula ([Selberg 1956]) (which he in fact developed for this application). However, there are difficulties of non-compactness which make this approach difficult in higher rank. We will use the spectral expansion of an integral operator a la Selberg, but only take a partial trace over a compact subset of X. This is of course insufficient to derive the full trace formula but is strong enough for our application.*

In the next few sections we will discuss the spectral expansion and the truncated fundamental domain that we will integrate over. We will conclude with the estimates on the continuous spectrum needed to complete the proof.

5.2 Integral operators on X

Let $g \in C_c^\infty(K\backslash G/K)$ (a smooth, bi-K-invariant function of compact support) act by convolution on $L^2(G/K)$:

$$(L_g f)(x) = \int_{G/K} f(y) g(y^{-1}x) dy.$$

The convolution operator L_g also acts on $f \in L^2(\Gamma\backslash G/K)$ by

$$(L_g f)(x) = \int_{G/K} f(y) g(y^{-1}x) dy = \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G/K} f(\gamma^{-1}y) g(y^{-1}\gamma x) dy$$

$$= \int_{\Gamma\backslash G/K} f(y) K(x,y) dy,$$

where the kernel

$$K(x,y) = \sum_{\gamma \in \Gamma} g(y^{-1}\gamma x).$$

Suppose that $g(x) = g(x^{-1})$ is real; then L_g is a self-adjoint operator on $L^2(\Gamma \backslash G/K)$ which commutes with the laplacian and all other invariant differential operators. Thus any eigenfunction of Δ on G/K is not only an eigenfunction of all invariant differential operators, but also of L_g. There is thus a function $\hat{g}$ which gives its eigenvalue:

$$(L_g\phi)(x) = \hat{g}(\phi)\phi(x).$$

The eigenvalue $\hat{g}(\phi)$ in fact only depends on a parameter associated to common eigenfunctions of the ring of invariant differential operators $\mathcal{D}$. Consider $D_1, D_2 \in \mathcal{D}$ with

$$D_1\phi = \lambda_1\phi \text{ and } D_2\phi = \lambda_2\phi.$$

Then $(D_1 \circ D_2)(\phi) = \lambda_1\lambda_2\phi$, so ϕ induces a homomorphism of $\mathcal{D}$ to $\mathbf{C}$. All such characters come from the common eigenfunctions

$$\phi_\lambda(x) = e^{(\lambda+\rho)(H(x))},$$

where $\lambda \in \mathbf{a}_{0\mathbf{C}}^*$, ρ_0 is half the sum of the positive roots, and $H_0(x)$ is the "logarithm" from $G \to \mathbf{a}_0$. *Selberg's uniqueness principle* asserts that $\hat{g}$ depends merely on the parameter λ, i.e. on the eigenvalues of ϕ on a set of generators of $\mathcal{D}$ (see chapter 3). This enables us to compute $\hat{g}(\lambda)$ explicitly by the example provided by $\phi_\lambda(x)$. For since $(L_g\phi_\lambda)(x) = \hat{g}(\lambda)\phi(x)$,

$$\hat{g}(\lambda) = (L_g\phi_\lambda)(1) = \int_{G/K} g(x)e^{(\lambda+\rho)(H(x))}dx.$$

This is the so-called Selberg/Helgason/Harish-Chandra spherical Fourier transform.

5.3 Spectral resolution

The spectrum of the laplacian Δ on $L^2(SL_3(\mathbf{Z})\backslash\mathcal{H}_3)$ consists of a discrete spectrum and a continuous spectrum. The latter is special to non-compact spaces and we shall see that it affects the size of the discrete spectrum. We have already discussed the spectral decomposition but reintroduce it here to reinforce notation. A good reference is [Arthur 1984].

5.3.1 Discrete spectrum

This has an orthonormal basis $\mathcal{B} = \{\phi_0, \phi_1, \phi_2, \ldots\}$ where $\Delta\phi_j + \lambda_j\phi_j = 0$ for some eigenvalue $\lambda_j \geq 0$. The first eigenfunction ϕ_0 is a constant (corresponding to $\lambda_0 = 0$) and all other eigenfunctions $\phi_1, \phi_2, \ldots$ are cusp forms and have positive eigenvalues. These are the objects that we want to show exist in abundance, and since the constant function ϕ_0 is the only non-cusp form among the ϕ_j's it is irrelevant in our asymptotics whether or not we include it in our spectral count.

5.3.2 Continuous spectrum

This has one- and two-dimensional parts. The two-dimensional part comes from the minimal parabolic Eisenstein series

$$E(P_0, x, \lambda) = \sum_{\gamma \in \Gamma \cap P_0 \backslash \Gamma} e^{(\lambda+\rho)(H(\gamma x))},$$

where λ ranges over $i\mathbf{a}_0^*$, a two-dimensional family.

The one-dimensional spectrum consists of infinitely-many pieces, each from a maximal parabolic Eisenstein series induced from a $GL_2(\mathbf{Z})\backslash\mathbf{H}$ eigenfunction ϕ on the Levi component M of P:

$$E(P,x,\phi,\lambda) = \sum_{\gamma\in\Gamma\cap P\backslash\Gamma} e^{(\lambda+\rho)(H(\gamma x))}\phi(m(\gamma x)).$$

Here $m(g)$ is the Levi factor in the Langlands decomposition $G = NMK$ and λ ranges over $i\mathbf{a}_P*$, a one-dimensional family. In summary, for each $GL_2(\mathbf{Z})\backslash\mathbf{H}$ eigenfunction on either of the two maximal parabolics there corresponds a one-dimensional spectrum. The Eisenstein series from different parabolics but which arise from the same $GL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form are related by a unitary transformation (part of the Langlands theory – see [Langlands 1976]).

5.4 Partial trace

The kernel of the integral operator L_g, $K(x,y) = \sum_{\gamma\in\Gamma} g(y^{-1}\gamma x)$ has a spectral expansion in terms of the discrete spectrum and continuous spectrum as ([Arthur 1984])

$$K(x,y) = \sum_{\phi_j\in\mathcal{B}_{(3)}} \hat{g}(\lambda_{\phi_j})\phi_j(x)\phi_j(y) + \frac{1}{3}\int_{i\mathbf{a}_0^*}\int \hat{g}(\lambda)E(P_0,x,\lambda)\overline{E(P_0,y,\lambda)}d\lambda$$

$$+\frac{1}{2}\sum_{P \text{ maximal}} \sum_{\substack{\phi_j \text{ a } GL_2(\mathbf{Z})\backslash\mathbf{H} \\ \text{discrete eigenfunction}}} \int_{i\mathbf{a}_P*} \hat{g}(\lambda+\lambda_{\phi_j})E(P,x,\phi_j,\lambda)\overline{E(P,y,\phi_j,\lambda)}d\lambda.$$

We have used the notation $\mathcal{B}_{(3)}$ to distinguish between eigenfunctions on $SL_3(\mathbf{R})$ and on $SL_2(\mathbf{R})$. The measure $d\lambda$ on $i\mathbf{a}^*$ is taken (see [Arthur 1984]) to be dual (i.e. for the Fourier transform) to the Lebesgue measure on the vector space $\mathbf{a}$ with the basis $\{\alpha_i^\vee \mid i = 1,\ldots,r-1\}$.

The Selberg trace formula involves taking the trace of this integral operator, i.e. integrating $K(x,x)$ over x in a fundamental domain $\mathcal{F}$ for $\Gamma\backslash G/K$. However, this integral does not converge because the Eisenstein series become too large in the cusp (and of course the sum on the left thus diverges as well there). We shall thus take a partial trace, only over a fixed compact subset $\mathcal{F}_C$ of $\mathcal{F}$:

$$\int_{\mathcal{F}_C} K(x,x)dx = \int_{\mathcal{F}_c} \sum_{\gamma\in\Gamma} g(x^{-1}\gamma x) = \int_{\mathcal{F}_C} \sum_{\phi_j\in\mathcal{B}_{(3)}} \hat{g}(\lambda_{\phi_j})|\phi_j(x)|^2 dx$$

$$+\frac{1}{3}\int_{\mathcal{F}_C}\int\int_{i\mathbf{a}_0*} \hat{g}(\lambda)|E(P_0,x,\lambda)|^2 d\lambda dx$$

$$+\frac{1}{2}\sum_{P \text{ maximal}}\int_{\mathcal{F}_C} \sum_{\substack{\phi\ a\ GL_2(\mathbf{Z})\backslash\mathbf{H} \\ \text{discrete eigenfunction}}} \int_{i\mathbf{a}_P*} \hat{g}(\lambda+\lambda_\phi)|E(P,x,\phi,\lambda)|^2 d\lambda dx. \quad (5.1)$$

We can restrict our choices of $g \geq 0$ to those which have $\hat{g}(\lambda)$ positive for $\Re\lambda = 0$ and that decay rapidly at infinity. This, for example, can be arranged by convolving g with itself. That is, let $g_1 \in C_c^\infty(K\backslash G/K)$ and recall we are assuming that $g_1(x^{-1}) = g_1(x) \geq 0$. Now let

$$g(x) = (L_{g_1}g_1)(x) = \int_{G/K} g_1(y)g_1(y^{-1}x)dy$$

and observe

$$\hat{g}(\lambda) = \int_{G/K} g(x)e^{(\lambda+\rho)(H(x))}dx = \int_{G/K}\left[\int_{G/K} g_1(y)g_1(y^{-1}x)dy\right]e^{(\lambda+\rho)(H(x))}dx$$

$$= \int_{G/K}\int_{G/K} g_1(y)e^{(\lambda+\rho)(H(y))}g_1(y^{-1}x)e^{(\lambda+\rho)(H(x)-H(y))}dxdy$$

(we will be thinking of G/K as N_0A_0). Change variables by letting $x = yx'$ so that

$H(x) = H(y) + H(x')$, and the integral splits as $\hat{g}(\lambda) = \hat{g}_1(\lambda)^2$. Since

$$\overline{\hat{g}_1(\lambda)} = \int_{G/K} \overline{g_1(x)e^{(\lambda+\rho)(H(x))}}dx = \int_{G/K} g_1(x)e^{(-\lambda+\rho)(H(x))}dx$$

$$= \int_{G/K} g_1(x^{-1})e^{(-\lambda+\rho)(H(x^{-1}))}e^{2\rho(H(x))}dx$$

$$= \int_{G/K} g_1(x)e^{(\lambda+\rho)(H(x))}dx$$

$$= \hat{g}_1(\lambda),$$

$\hat{g}(\lambda) \geq 0$. Thus, there are some functions in $C_c^\infty(K\backslash G/K)$ which have a positive spherical transform.

We can thus create an inequality out of the partial trace (5.1) by dropping various terms. Firstly, for the discrete spectrum terms,

$$\sum_{j=0}^{\infty} \hat{g}(\lambda_{\phi_j}) \int_{\mathcal{F}_C} |\phi_j(x)|^2 dx \leq \sum_{j=0}^{\infty}{}' \int_{\mathcal{F}_C} \hat{g}(\lambda_{\phi_j})|\phi_j(x)|^2 dx.$$

The $\sum'$ means we only include those cusp forms for which $\Re\hat{g}(\lambda_{\phi_j}) \geq 0$. Since

$$\int_{\mathcal{F}_C} |\phi_j(x)|^2 dx \leq \int_{\mathcal{F}} |\phi_j(x)|^2 = 1,$$

the remaining positive terms are

$$\leq \sum_{j=0}^{\infty}{}' \hat{g}(\lambda_{\phi_j}).$$

The *orbital integrals* are the summands in

$$\int_{\mathcal{F}_C} K(x,x)dx = \sum_{\gamma\in\Gamma} \int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx.$$

Since g is positive,

$$\int_{\mathcal{F}_C} K(x,x)dx \geq \int_{\mathcal{F}_C} g(x^{-1}Ix)dx = g(I)\mathrm{vol}\mathcal{F}_C.$$

This will in fact be the main term because if g has small support, only $\gamma \in \Gamma$ for which there is a point $x \in \mathcal{F}_C$ with $x^{-1}\gamma x$ near the identity (i.e. in the support of g) will have non-zero orbital integrals. By reduction theory, X has a fundamental domain $\mathcal{F}$ with one cusp and a finite polygonal boundary which touches only finitely many translates of $\mathcal{F}$. Since $\mathcal{F}_C$ is compact it avoids the cusp and hence all the elements $\gamma \in \Gamma$ with non-zero orbital integrals must map some boundary point of $\mathcal{F}_C$ to another. Thus only finitely many elements $\gamma \in \Gamma$ have non-zero orbital integrals, and these γ all have fixed points in the boundary of $\mathcal{F}_C$.

Let us select our family of test functions as

$$g_T(x) = T^5 g(x^T),$$

with g as before. Then for T large $g_T(x)$ has small support which shrinks to the identity as $T \to \infty$. Among $\Re\lambda = 0$, the spherical transform $\hat{g}_T(\lambda)$ is concentrated in $\{\lambda \mid \|\lambda\| = O(T)\}$. Also, $g_T(I) = T^5 g(I)$; we will show that the orbital integrals $\int g_T(x^{-1}\gamma x)dx = o(T^5)$, if $\gamma \neq I$, and hence that the identity orbital integral is the main term.

Lemma 5.4.1 *If $\gamma \neq I$, then*

$$\int_{G/K} g_T(x^{-1}\gamma x)dx = o(T^5).$$

Proof: We already pointed out that γ must fix a point x_0, so that γ is locally a rotation about x_0. By changing variables we see the integrand still has compact support in x around x_0, and so the integral is

$$\leq \max_{x \in \text{supp} g_T} g_T(x) \text{vol}(\text{supp} \widetilde{g_T}) \ll \left(T^5 \max g\right) o(1) = o(T^5),$$

where $\widetilde{g_T}(x) = g_T(x^{-1}\gamma x)$ (note that to get our constant above we are using the fact that there are only finitely many elements $\gamma \in \Gamma$ involved). $\square$.

Remark 5.4.2 *Were Γ co-compact then our spectral expansion would not include the contributions of the continuous spectrum (as it is special to non-compact quotients) and we could take the full trace by letting $\mathcal{F}_C = \mathcal{F}$. We would conclude*

$$T^5 g(I) vol(\mathcal{F}) \sim \sum_{j=1}^{\infty}{}' \hat{g}_T(\lambda_{\phi_j}) \ \text{as} \ T \to \infty.$$

Weyl's law holds for the compact case, and since g and $\hat{g}$ can be taken from such a broad range of functions, it is equivalent to that last asymptotic expression. We will use this fact instead of rederiving the tauberian argument which shows this and the exact constants involved, though were we to do this the preceding analysis of the orbital integrals would have been unneccesary.

Proof of Weyl's Law (theorem 5.1.1): For our case, where $\Gamma = SL_3(\mathbf{Z})$ is not co-compact, the contributions of the continuous spectra have the same sign as the spectral count $\sum' \hat{g}_T(\lambda_\phi)$. In general it is impossible to decouple the two, but we will here by showing the continuous spectra terms contribute $O_\epsilon(T^{2+\epsilon})$. Thus our partial trace inequality becomes

$$T^5 g(I) \text{vol}(\mathcal{F}_C) \leq \sum_{j=0}^{\infty}{}' \hat{g}_T(\lambda_{\phi_j}) + o(T^5), \tag{5.2}$$

or equivalently

$$\liminf \frac{N(T)}{T^{5/2}} \geq \frac{\mathrm{vol}\mathcal{F}_C}{(4\pi)^{5/2}\Gamma(7/2)}.$$

Choosing a sequence of compact sets $\mathcal{F}_{C_i}$ whose union is $\mathcal{F}$ the right-hand side can be replaced by

$$\frac{\mathrm{vol}\mathcal{F}}{(4\pi)^{5/2}\Gamma(7/2)}.$$

The same upper bound for $\limsup \frac{N(T)}{T^{5/2}}$ was proven by Donnelly and so we conclude

$$N(T) \sim \frac{\mathrm{vol}(SL_3(\mathbf{Z})\backslash SL_3(\mathbf{R})/SO_3(\mathbf{R}))}{(4\pi)^{5/2}\Gamma(7/2)} T^{5/2}.$$

□

5.5 The truncated fundamental domain $\mathcal{F}_C$

Let us now describe the choice of truncated $\mathcal{F}_C$ we will integrate over to bound the continuous spectrum terms. The rank-2 group G has 2 positive simple roots $\Sigma = \{\alpha_1, \alpha_2\} \subset \mathbf{a}_0^*$,

$$\alpha_1 \left(\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \right) = h_1 - h_2 \ , \ \alpha_2 \left(\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \right) = h_2 - h_3.$$

There is a third positive root $\alpha_3 = \alpha_1 + \alpha_2 = \rho_0 = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)$. Let $c > 0$ be a large positive parameter and

$$C = c\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{1-n}{2}\right) \in \mathbf{a}_0.$$

Define

$$\mathcal{F}_C = \{x \in \mathcal{F} \mid (2\alpha_1 + \alpha_2)(H(x) - C), (\alpha_1 + 2\alpha_2)(H(x) - C) \leq 0\}.$$

The reason for this somewhat-odd defination is that it matches some defintions in the next section.

The main difficulty of the trace formula in higher rank for non-compact quotients is that the Eisenstein series are not square-integrable. We will modify them by subtracting their constant terms from various parabolics

$$E_P(x) = \int_{\Gamma \cap N \backslash N} E(nx, \phi, \lambda) dn$$

near infinity so as to make $E(x, \phi, \lambda)$ square-integrable for $\Re\lambda(\alpha_i^\vee) \geq 0$.[1] The modified Eisenstein series $(\Lambda^C E)(x, \lambda)$ will agree with $E(x, \lambda)$ in $\mathcal{F}_C$ and so

$$\int_{\mathcal{F}_C} |E(x, \lambda)|^2 dx \leq \int_{\mathcal{F}} |(\Lambda^C E)(x, \lambda)|^2 dx,$$

the latter of which we will be able to explicitly calculate via the Langlands inner-product formula. Note that we can bound the integral of $|E(x, \lambda)|^2$ over any compact set, as any of these is contained in some $\mathcal{F}_C$ for c large enough. Thus our overall argument does not depend on the actual choice or shape of $\mathcal{F}_C$.

[1]Of course we can only truncate $E(x, \phi, \lambda)$ for values of λ where it is defined, i.e. away from the poles.

5.6 The truncated Eisenstein series $\Lambda^C E$

For any standard parabolic P_π let

$$\Delta_P = \Delta_\pi = \{\alpha_i \in \Sigma \mid \alpha_i \neq 0 \text{ on } \mathbf{a}_\pi\}.$$

Recall that $\{\alpha_1^\vee, \alpha_2^\vee\}$ is a basis of $\mathbf{a}_0$.

Let $\hat{\tau}_P(x)$ be the characteristic function of

$$\{x = c_1\alpha_1^\vee + c_2\alpha_2^\vee \in \mathbf{a}_0 \mid c_i > 0, \forall \alpha_i \in \Delta_P\}.$$

That is, $\hat{\tau}_{P_0}$ is the characteristic function of

$$\{x = c_1\alpha_1^\vee + c_2\alpha_2^\vee \in \mathbf{a}_0 \mid c_1, c_2 > 0\},$$

$$\widehat{\tau_{P_1}} \text{ of } \{x = c_1\alpha_1^\vee + c_2\alpha_2^\vee \in \mathbf{a}_0 \mid c_2 > 0\},$$

$$\widehat{\tau_{P_2}} \text{ of } \{x = c_1\alpha_1^\vee + c_2\alpha_2^\vee \in \mathbf{a}_0 \mid c_1 > 0\},$$

and

$$\widehat{\tau_G} \text{ of } \mathbf{a}_0.$$

The truncation of an automorphic form ψ is a sum over all parabolic subgroups

$$(\Lambda^C\psi)(x) := \sum_P (-1)^{\dim A} \sum_{\gamma\in\Gamma\cap P\backslash\Gamma} \hat{\tau}_P(H(\gamma x) - C) \int_{\Gamma\cap N\backslash N} \psi(n\gamma x)dn$$

$$= \sum_P (-1)^{\dim A} \sum_{\gamma\in\Gamma\cap P\backslash\Gamma} \hat{\tau}_P(H(\gamma x) - C)\psi_P(\gamma x),$$

which itself is clearly an automorphic form. It can be proven that it also decays rapidly in the cusp because of the way its constant terms have been removed (see [Arthur 1980]). Note that if ψ is a cusp form to begin with, all its constant terms $\psi_P \equiv 0$ in proper parabolics and thus $\Lambda^C\psi = \psi$.

Theorem 5.6.1 *Langlands inner product formula ([Langlands 1966], [Arthur 1980])*

$$\int_{\Gamma\backslash G/K} \left[(\Lambda^C E)(P,x,\phi,\lambda_1)\right]\left[(\Lambda^C E)(P,x,\phi,\lambda_2)\right] dx$$

$$= \sum_{P\sim P' \ associate} vol(\mathbf{a}'/<\alpha^\vee \mid \alpha\in\Delta_{P'}>)\times$$

$$\sum_{s_1,s_2\in\Omega(\mathbf{a},\mathbf{a}')} \frac{e^{(s_1\lambda_1+s_2\lambda_2)(C)}}{\prod_{\alpha\in\Delta_P}(s_1\lambda_1+s_2\lambda_2)(\alpha^\vee)}(M(s_1,\lambda_1)\phi, M(s_2,\lambda_2)\phi).$$

The last expression (ψ,ψ') *is an inner product over* $\Gamma\cap M'\backslash M'$.

For our purposes we will only need to deal with standard parabolic subgroups of $SL_3(\mathbf{R})$, and we already described the operators $M(s,\lambda)$. Hence up to the lattice-area constant, the inner product is

$$\sum_{P\sim P'}\sum_{s_1,s_2\in\Omega(\mathbf{a},\mathbf{a}')} \frac{e^{(s_1\lambda_1+s_2\lambda_2)(C)}}{\prod_{\alpha\in\Delta_P}(s_1\lambda_1+s_2\lambda_2)(\alpha^\vee)}\times$$

$$\prod_{\substack{1\le i<j\le r\\ s(i)>s(j)}} \frac{L(\lambda_{1_i}-\lambda_{1_j},\phi_i\times\tilde{\phi_j})}{L(1+\lambda_{1_i}-\lambda_{1_j},\phi_i\times\tilde{\phi_j})} \prod_{\substack{1\le k<l\le r\\ s(k)>s(l)}} \frac{L(\lambda_{2_k}-\lambda_{2_l},\phi_k\times\tilde{\phi_l})}{L(1+\lambda_{2_k}-\lambda_{2_l},\phi_k\times\tilde{\phi_l})}.$$

This is a generalization of the Maass-Selberg relations, which are for the the minimal parabolic Eisenstein series on $SL_2(\mathbf{R})$ – various approaches to them can be found in [Borel 1974] and [Terras], vol. I.

5.7 The minimal parabolic Eisenstein series

Recall that these are

$$E(P_0, x, \lambda) = \sum_{\gamma \in \Gamma \cap P_0 \backslash \Gamma} e^{(\lambda+\rho)(H(\gamma x))}, \lambda \in i\mathbf{a}_0^*.$$

The L-functions involved in the right-hand side of the Langlands inner product formula are (completed) Riemann zeta functions $\zeta^*(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Let

$$c(s) = \frac{\zeta^*(s)}{\zeta^*(s+1)} = \frac{\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)}{\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})\zeta(s+1)}$$

so that the inner product (up to the volume constant) is

$$\sum_{s_1, s_2 \in \Omega(\mathbf{a}_0)} \frac{e^{(s_1\lambda_1+s_2\lambda_2)((c,0,-c))}}{(s_1\lambda_1 + s_2\lambda_2)((1,-1,0))(s_1\lambda_1 + s_2\lambda_2)((0,1,-1))} \times \\ \prod_{\substack{1 \le i < j \le 3 \\ s_1(i) > s_1(j)}} c(\lambda_{1_i} - \lambda_{1_j}) \prod_{\substack{1 \le k < l \le 3 \\ s_1(k) > s_1(l)}} c(\lambda_{2_k} - \lambda_{2_l}). \tag{5.3}$$

Now take $\lambda_1 = (it_1, it_2, it_3) + (\epsilon, 0, -\epsilon) = \overline{\lambda_2}$ with t_i real and $t_1 + t_2 + t_3 = 0$. This expression is a constant multiple of an integral which converges for $\epsilon \ge 0$. Thus, by the dominated convergence theorem it too must have a limiting value when $\epsilon \to 0$ or at any other singularity – i.e. all its singularities in $\epsilon \ge 0$ are removable[2].

Note that the numerator of each term is unitary when $\epsilon = 0$ by the functional equation $\zeta^*(s) = \zeta^*(1-s)$ and the fact that $\zeta^*(s)$ is real for $s \in \mathbf{R}$:

$$|c(it)| = \left|\frac{\zeta^*(it)}{\zeta^*(1+it)}\right| = \left|\frac{\zeta^*(it)}{\zeta^*(-it)}\right| = \left|\frac{\zeta^*(it)}{\overline{\zeta^*}(it)}\right| = 1.$$

We can also control the derivatives:

[2]Again one must stay away from the pole of $c(s)$ at $s = 1$.

Lemma 5.7.1 *For any $n \geq 0$ and $\epsilon > 0$, $\frac{d^n}{dt^n} c(it) = O_\epsilon(t^\epsilon)$.*

Proof: Since $|c(it)| = 1$ it suffices to bound the derivates of $\log(c(it))$. For example,

$$-\frac{c'}{c}(s) = -\log \pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)\frac{1}{2} + \frac{\Gamma'}{\Gamma}\left(\frac{1+s}{2}\right) + \frac{\zeta'}{\zeta}(1-s) + \frac{\zeta'}{\zeta}(1+s).$$

Because of the quotient rule

$$\left(\frac{f^{(n)}}{f}\right)' = \frac{f^{(n+1)}}{f} - \frac{f^{(n)}}{f}\frac{f'}{f},$$

any repeated derivate of $\log(f)$ is a polynomial in $\frac{f'}{f}, \cdots, \frac{f^{(n)}}{f}$. We have that

$$\frac{\zeta^{(n)}}{\zeta}(1+it) = O_\epsilon(t^\epsilon)$$

(see [Titchmarsh] for an upper bound on the derivates and a lower bound on $\zeta(1+it)$), and

$$\frac{\Gamma^{(n)}}{\Gamma}(\frac{1}{2} + it) = O_\epsilon(t^\epsilon),$$

which is immediate from the expansion

$$\frac{\Gamma'}{\Gamma}(s) = -\frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n}\right) - \gamma$$

(γ is Euler's constant $0.57721566490153286061\cdots$). □

For the generic choice of λ, (5.3) has singularities only involving ϵ's in the denominators. Since (5.3) has a limiting value at $\epsilon = 0$, it is a constant multiple of a derivative of the singular terms involved at $\epsilon = 0$. All these derivates are $O_\epsilon(T^\epsilon)$ for $\|\lambda\| \leq T$ by the lemma. Keep in mind that C is a fixed constant and thus irrelevant. For the singular values of $\lambda \in i\mathbf{a}_0$ the same is also true. At most two derivatives ever need to be taken, and a third to interpolate nearby the singularities (when the denominators are large, the expression is trivially small since the numerators have modulus one). Thus, the inner product is $O_\epsilon(T^\epsilon)$.

Finally, to estimate the contribution of the two-dimensional continuous spectrum in the partial trace (5.1)

$$\int_{i\mathbf{a}_0*}\int \hat{g_T}(\lambda)\int_{\mathcal{F}_C}|E(P_0,x,\lambda)|^2dxd\lambda$$

as $T\to\infty$ it is sufficient (by a tauberian argument) to estimate

$$\int_{\alpha_1(\lambda)=-iT}^{iT}\int_{\alpha_2(\lambda)=-iT}^{iT}\int_{\mathcal{F}_C}|E(P_0,x,\lambda)|^2dxd\lambda \ll_\epsilon \int_{-T}^{T}\int_{-T}^{T}T^\epsilon d\lambda \ll_\epsilon T^{2+\epsilon}.$$

5.8 The maximal parabolic Eisenstein series induced from cusp forms

Recall that $G=SL_3(\mathbf{R})$ has 2 standard maximal parabolics:

$$P_1=\left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}\right\} \text{ and } P_2=\left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\}$$

with Levi components

$$M_1=\left\{\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}\right\} \text{ and } M_2=\left\{\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}\right\},$$

each isomorphic to $GL_2(\mathbf{R})$. If we divide these by their split components A we get $M' = M/A \simeq GL_2^{\pm}(\mathbf{R})$. The maximal parabolic Eisenstein series are induced from $\Gamma \cap M' \backslash M'/(M' \cap K)$ cusp forms, which are just $GL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp forms. These are precisely the even $SL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp forms. For each $GL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form ϕ we can make eigenfunctions ϕ_1 and ϕ_2 on the Levi subgroups M_1' and M_2', respectively, and Eisenstein series

$$E(P_1, \phi_1, x, \lambda_1) = \sum_{\gamma \in \Gamma \cap P_1 \backslash \Gamma} \phi_1(m_1(\gamma x)) e^{(\lambda_1 + \rho_1)(H_1(\gamma x))},$$

and

$$E(P_2, \phi_2, x, \lambda_2) = \sum_{\gamma \in \Gamma \cap P_2 \backslash \Gamma} \phi_2(m_2(\gamma x)) e^{(\lambda_2 + \rho_2)(H_2(\gamma x))}.$$

Here $m_i(g) \in M_i$ is from the Langlands decomposition $G = NMAK$ and H_i is the "logarithm" to $\mathbf{a}_i$. Also $\lambda_i \in i\mathbf{a}_i^*$ and ρ_i is half the sum of the positive roots of $\mathbf{a}_i$.

There is a functional equation due to Langlands relating the two Eisenstein series on different parabolics induced from the same $GL_2(\mathbf{Z})\backslash\mathbf{H}$-cusp form. If t is the unique element in $\Omega(\mathbf{a}_1, \mathbf{a}_2)$ (which associates P_1 to P_2), then

$$E(P_1, \phi_1, x, \lambda) = M(t, \phi, \lambda) E(P_2, \phi_2, tx, t\lambda)$$

where $|M(t, \phi, \lambda)| = 1$ for $\lambda \in i\mathbf{a}_1*$ and tx and $t\lambda$ are images under the association map induced by t. So in the partial trace (5.1)

$$\frac{1}{2} \sum_P \sum_\phi \int_{i\mathbf{a}_P*} \hat{g}(\lambda_\phi) \int_{\mathcal{F}_C} |E(P, x, \phi, \lambda)|^2 dx d\lambda = \sum_\phi \int_{i\mathbf{a}_1*} \hat{g}(\lambda_\phi) \int_{\mathcal{F}_C} |E(P_1, x, \phi, \lambda)|^2 dx d\lambda$$

and we only need to deal with the P_1 Eisenstein series, though our argument also works for P_2 just as easily.

We will prove

Proposition 5.8.1 *Let α be the root $\alpha_{1,2} \in \mathbf{a}_1^*$. Then*

$$\int_{\alpha(\lambda)=-iT}^{iT} \int_{\mathcal{F}_C} |E(P_1, x, \phi, \lambda)|^2 dx d\lambda \ll_\epsilon (T + |\lambda_\phi|)^{1+\epsilon}.$$

By a tauberian argument, this will imply the total contribution

$$\sum_\phi \int_{i\mathbf{a}_1*} \hat{g}(\lambda_\phi) \int_{\mathcal{F}_C} |E(P_1, x, \phi, \lambda)|^2 dx d\lambda \ll_\epsilon \sum_{|\lambda_\phi| \leq T} \int_{\alpha(\lambda)=-iT}^{iT} \int_{\mathcal{F}_C} |E(P_1, x, \phi, \lambda)|^2 dx d\lambda$$

$$\ll_\epsilon \sum_{|\lambda_\phi| \leq T} (T + |\lambda_\phi|)^{(1+\epsilon)} \ll T^{(2+\epsilon)}.$$

Here we have used the Weyl upper bound for $GL_2(\mathbf{Z}) \backslash \mathbf{H}$-cusp forms $N(T) = O(T^2)$. (Note $\Delta\phi + (\frac{1}{4} + \frac{\|\lambda_\phi\|^2}{2})\phi = 0$.)

Proposition 5.8.2 *Let α be as in the previous proposition and $M(t, \phi, \lambda)$ be the ratio of completed L-functions*

$$\frac{\Lambda(\alpha(\lambda), \phi)}{\Lambda(1 + \alpha(\lambda), \phi)}.$$

Note that $M(t, \phi, \lambda)M(t, \phi, -\lambda) = 1$ by the functional equation of $\Lambda(s, \phi)$. Then

$$\int_{\mathcal{F}_C} |E(P_1, x, \phi, \lambda)|^2 dx = \frac{M'}{M}(t, \phi, \lambda) + O(c)$$

up to a constant.

This proposition implies the previous one by lemma 4.4.1, where we bounded the winding number of M. It itself is an application of the Langlands inner product formula

$$\frac{e^{(\lambda+\bar{\lambda})(C)}}{(\lambda + \bar{\lambda})(\alpha^\vee)} + \frac{e^{(t\lambda+t\bar{\lambda})(C)}}{(t\lambda + t\bar{\lambda})(\alpha^\vee)} |M(t, \phi, \lambda)|^2 = \frac{e^{2\Re\lambda(C)} - e^{-2\Re\lambda(C)}|M(t, \phi, \lambda)|^2}{2\Re\lambda(\alpha^\vee)}.$$

The denominator will have a singularity when $\Re\lambda = 0$, so this expression is thus a derivative of the numerator, up to a constant (keep in mind that M is unitary when $\Re\lambda = 0$).

5.9 The maximal parabolic Eisenstein series induced from the constant function

We still need to examine the contribution of

$$E(P_1, x, 1, \lambda) = \sum_{\gamma \in \Gamma \cap P_1 \backslash \Gamma} e^{(\lambda+\rho_1)(H_1(\gamma x))} \ , \ \lambda \in i\mathbf{a}_1^*$$

to the partial trace formula. We will demonstrate

Proposition 5.9.1 *Let $\alpha = \alpha_{1,2} \in \mathbf{a}_1^*$. For T large*

$$\int_{\lambda(\alpha^\vee)=-iT}^{\lambda(\alpha^\vee)=iT} \int_{\mathcal{F}_C} |E(P_1, x, 1, \lambda)|^2 dx d\lambda = O(T \log T).$$

This, again by a tauberian argument, is sufficient and is the last step needed to prove Weyl's law. We will prove the proposition by expressing $E(P_1, x, 1, \lambda)$ as a residue of the minimal parabolic Eisenstein series $E(P_0, x, \lambda)$ on $SL_3(\mathbf{R})$. After all, the constant function on $SL_2(\mathbf{R})$ is itself a residue of the minimal parabolic Eisenstein series $E(P_0, x, \lambda)$ at the pole $\lambda = \alpha$, the unique positive root of $\mathbf{a}_0$. We have used different normalizations arising from the volumes of $GL_2(\mathbf{Z})\backslash\mathbf{H}$ and $SL_3(\mathbf{Z})\backslash\mathcal{H}_3$, but these are irrelevant for our theorem as all we need is an order-of-magnitude estimation.

5.9.1 Notation

Decompose any $H \in \mathbf{a}_0$ as

$$H = H_1 + H_\perp,$$

where $\alpha_1(H_1) = 0$, i.e. $H \in \mathbf{a}_1$. Similarly, write any $\lambda \in \mathbf{a}_0^*$ as

$$\lambda = \lambda_1 + \lambda_\perp, \lambda_1(\alpha_1^\vee) = 0 \ (\lambda \in \mathbf{a}_1^*).$$

If $H = (h_1, h_2, h_3)$ then $H_1 = (\frac{h_1+h_2}{2}, \frac{h_1+h_2}{2}, h_3)$ and $H_\perp = (\frac{h_1-h_2}{2}, \frac{h_2-h_1}{2}, 0)$. Also, $\rho = \rho_1 + \frac{1}{2}\alpha_1$. Let $P_0^{(2)}$ be the minimal standard parabolic of $SL_2(\mathbf{R})$ and recall $m_1'(x) \in M_1'$ denotes the Levi component from the Langlands decomposition $P = NM'A$.

Lemma 5.9.2 *The minimal parabolic Eisenstein series can be written as a maximal parabolic Eisenstein series induced from the minimal parabolic Eisenstein series on* $SL_2(\mathbf{R})$*:*

$$E(P_0, x, \lambda) = E(P_1, x, E(P_0^{(2)}, m_1'(\cdot), \lambda_\perp), \lambda_1).$$

Proof: This is an unfolding argument:

$$E(P_0, x, \lambda) = \sum_{\gamma \in \Gamma \cap P_0 \backslash \Gamma} e^{(\lambda+\rho_0)(H_0(\gamma x))}$$

$$= \sum_{\gamma \in \Gamma \cap P_1 \backslash \Gamma} \sum_{\delta \in \Gamma \cap P_0 \backslash \Gamma \cap P_1} e^{(\lambda+\rho_0)(H_0(\delta\gamma x))}.$$

Note that for any $y \in G$

$$(\lambda + \rho_0)(H_0(\delta y)) = (\lambda_1 + \lambda_\perp + \rho_1 + \frac{1}{2}\alpha_1)(H_1(\delta y) + H_\perp(\delta y))$$

$$= (\lambda_1 + \rho_1)(H_1(\delta y)) + (\lambda_\perp + \frac{1}{2}\alpha_1)(H_\perp(\delta y))$$

and that for $\delta \in P_1$, $H_1(\delta y) = H_1(y)$. Thus

$$E(P_0, x, \lambda) = \sum_{\gamma \in \Gamma \cap P_1 \backslash \Gamma} e^{(\lambda_1+\rho_1)(H(\gamma x))} \sum_{\delta \in \Gamma \cap P_0 \backslash \Gamma \cap P_1} e^{(\lambda_\perp + \frac{1}{2}\alpha_1)(H_\perp(\delta\gamma x))},$$

which is

$$E(P_1, x, E(P_0^{(2)}, m_1'(\cdot), \lambda_\perp), \lambda_1)$$

since

$$\Gamma \cap P_0 \backslash \Gamma \cap P_1 \simeq \Gamma_\infty \backslash SL_2(\mathbf{Z}).$$

□

Corollary 5.9.3 *Up to a constant*

$$E(P_1, x, 1, \lambda) = Res_{\lambda_\perp = \frac{1}{2}\alpha_1} E(P_0, x, \lambda)$$

and

$$(\Lambda^C E)(P_1, x, 1, \lambda) = Res_{\lambda_\perp = \frac{1}{2}\alpha_1} (\Lambda^C E)(P_0, x, \lambda).$$

We can use the Maass-Selberg relations for $E(P_0, x, \lambda)$ to conclude that we need only bound the residue of

$$\sum_{s_1, s_2 \in \Omega(\mathbf{a}_0)} \frac{e^{(s_1\lambda_1 + s_2\lambda_2)(C)}}{\prod_{i=1}^{2}(s_1\lambda_1 + s_2\lambda_2)(\alpha_i^\vee)} \prod_{\substack{1 \le i < j \le 3 \\ 1 \le k < l \le 3 \\ s_1(i) > s_1(j) \\ s_2(k) > s_2(l)}} c(\lambda_{1_i} - \lambda_{1_j}) c(\lambda_{2_k} - \lambda_{2_l})$$

when $\lambda_1 - \frac{1}{2}\alpha_1$, $\lambda_2 - \frac{1}{2}\alpha_1 \in i\mathbf{a}_1$.

5.9.2 Analysis of the polar terms

Let us take

$$\lambda_1 = \left(it + \epsilon + \frac{1}{2} + \frac{i\delta}{2}, it + \epsilon - \frac{1}{2} - \frac{i\delta}{2}, -2it - 2\epsilon\right) = \overline{\lambda_2}.$$

Poles at $\delta = 0$ can only occur in terms which have

$$(s_1\lambda_1 + s_2\lambda_2)(\alpha_1^\vee) = 0\ ,\ (s_1\lambda_1 + s_2\lambda_2)(\alpha_2^\vee) = 0,$$

or $c(s)$ evaluated at $s = 1$. We will of course only be interested in terms which have double poles, since we are investigating the integral of the square of the residue of $E(P_0, x, \lambda)$.

We shall require the following bounds on $c(s)$ and its derivative:

Lemma 5.9.4 *For* $t \in \mathbf{R}$

$$(i) : c(\frac{1}{2} + it) = O(1).$$

$$(ii) : \frac{d}{d\epsilon}\left(\frac{\zeta^*(\frac{3}{2} + it + \epsilon)}{\zeta^*(\frac{3}{2} + it - \epsilon)}\right)|_{\epsilon=0} = O(\log t).$$

Proof: (i) Recall

$$c(s) = \sqrt{\pi}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{\zeta(s)}{\zeta(s+1)}.$$

By Stirling's formula

$$\left|\frac{\Gamma(\frac{1}{4} + \frac{it}{2})}{\Gamma(\frac{3}{4} + \frac{it}{2})}\right| \sim \sqrt{\frac{2}{t}} \text{ as } t \to \infty$$

and by the convexity bound $\zeta(\frac{1}{2} + it) = O(t^{1/4})$ (better bounds can be obtained). Also, taking the logarithm of the Euler product of $\zeta(s)$ we find

$$-\log\zeta(\frac{3}{2} + it) = \sum_{p \text{ prime}} \log(1 - p^{-3/2-it}) = O(1),$$

which proves (i). Differentiating,

$$\frac{\zeta'}{\zeta}(\frac{3}{2}+it) = -\sum_p \frac{p^{-3/2-it}\log p}{1-p^{-3/2-it}} = O(1)$$

also. So

$$\frac{\zeta^{*\prime}}{\zeta^*}(\frac{3}{2}+it) = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{3}{4}+\frac{it}{2}) + \frac{\zeta'}{\zeta}(\frac{3}{2}+it) = O(\log t),$$

proving (ii). □

Terms which have poles from c when $\delta = 0$

Observe that

$$\lambda_{1_1} - \lambda_{1_2} = 1 + i\delta,$$

$$\lambda_{1_1} - \lambda_{1_3} = 3it + 3\epsilon + \frac{1}{2} + \frac{i\delta}{2},$$

and

$$\lambda_{1_2} - \lambda_{1_3} = 3it + 3\epsilon - \frac{1}{2} - \frac{i\delta}{2},$$

so these poles only occur for the permutations which swap 1 and 2, namely (12), (321), and (13). First, the diagonal terms are

$$\frac{e^{(2\epsilon-1,2\epsilon+1,-4\epsilon)(c,0,-c)}}{(6\epsilon+1)(-2)}c(1+i\delta)c(1-i\delta)$$

$$+\frac{e^{(2\epsilon-1,-4\epsilon,2\epsilon+1)(c,0,-c)}}{(-6\epsilon-1)(6\epsilon-1)}|c(3it+3\epsilon+\frac{1}{2}+\frac{i\delta}{2})c(1+i\delta)|^2$$

$$+\frac{e^{(-4\epsilon,2\epsilon-1,2\epsilon+1)(c,0,-c)}}{(-2)(-6\epsilon+1)}|c(3it+3\epsilon-\frac{1}{2}-\frac{i\delta}{2})c(3it+3\epsilon+\frac{1}{2}+\frac{i\delta}{2})c(1+i\delta)|^2.$$

Set $\epsilon = 0$ to put the Eisenstein series on the line $\Re\lambda_1 = \frac{1}{2}\alpha$:

$$\frac{e^{-c}}{-2}c(1+i\delta)c(1-i\delta) + e^{-2c}c(3it+\frac{1}{2}+\frac{i\delta}{2})c(-3it+\frac{1}{2}-\frac{i\delta}{2})c(1+i\delta)c(1-i\delta)$$

$$+\frac{e^{-c}}{-2}c(3it-\frac{1+i\delta}{2})c(3it+\frac{1+i\delta}{2})c(1-i\delta)\times$$
$$c(-3it-\frac{1-i\delta}{2})c(-3it+\frac{1-i\delta}{2})c(1+i\delta).$$

Its double residue is

$$\ll 1 + |c(\frac{1}{2}+3it)|^2 \ll 1$$

by lemma 5.9.4. (Recall that the residue of $c(s)$ at $s = 1$ is a fixed constant.)

The non-diagonal terms (for t large – we are only interested in asymptotics) reduce to bounding the products of the c's, which up to constants are:

For $s_1 = (12), s_2 = (321)$:

$$c(1+i\delta)c(-3it+\frac{1}{2}-\frac{i\delta}{2})c(1-i\delta),$$

for $s_1 = (12), s_2 = (13)$:

$$c(1+i\delta)c(-3it+\frac{1}{2}-\frac{i\delta}{2})c(-3it-\frac{1}{2}+\frac{i\delta}{2})c(1-i\delta),$$

and for $s_1 = (13), s_2 = (321)$

$$c(1+i\delta)c(3it-\frac{1}{2}-\frac{i\delta}{2})c(3it+\frac{1}{2}+\frac{i\delta}{2})c(-3it+\frac{1}{2}-\frac{i\delta}{2})c(1-i\delta).$$

After taking the double residue these are also $O(1)$, noting that

$$|c(\frac{1}{2}+it)c(-\frac{1}{2}+it)| = \left|\frac{c(\frac{1}{2}+it)}{c(\frac{1}{2}-it)}\right| = 1.$$

Thus the terms which have poles from c when $\delta = 0$ contribute $O(1)$ to the inner product.

Terms which have Singularities in their Denominators

These are terms where either

$$(s_1\lambda_1 + s_2\lambda_2)(\alpha_1^\vee) \text{ or } (s_1\lambda_1 + s_2\lambda_2)(\alpha_2^\vee)$$

vanish for $\delta = 0$. For t large this is only possible when the entries of

$$\lambda_1(\epsilon = 0) = (it + \frac{1}{2}, it - \frac{1}{2}, -2it) \text{ and } \lambda_2(\epsilon = 0) = (-it + \frac{1}{2}, -it - \frac{1}{2}, 2it)$$

match to vanish, which forces $s_1 = s_2(12)$.

We will add the six terms together and first take the $\delta \to 0$ limit to get the double residue. Afterwards, by taking the $\epsilon \to 0$ limit we will get the actual contribution to the continuous spectrum.

Recall

$$\lambda_1 = (it + \epsilon + \frac{1}{2} + \frac{i\delta}{2}, it + \epsilon - \frac{1}{2} - \frac{i\delta}{2}, -2it - 2\epsilon)$$

and

$$\lambda_2 = (-it + \epsilon + \frac{1}{2} - \frac{i\delta}{2}, -it + \epsilon - \frac{1}{2} + \frac{i\delta}{2}, 2it - 2\epsilon).$$

The six terms are

$$e\times(12):\frac{e^{(2\epsilon+i\delta,2\epsilon-i\delta,-4\epsilon)(c,0,-c)}}{(6\epsilon-i\delta)(2i\delta)}c(1-i\delta)$$

$$(12)\times e:\frac{e^{(2\epsilon-i\delta,2\epsilon+i\delta,-4\epsilon)(c,0,-c)}}{(6\epsilon+i\delta)(-2i\delta)}c(1+i\delta)$$

$$(23)\times(321):\frac{e^{(2\epsilon+i\delta,-4\epsilon,2\epsilon-i\delta)(c,0,-c)}}{(-6\epsilon+i\delta)(6\epsilon+i\delta)}c(3it+3\epsilon-\frac{1}{2}-\frac{i\delta}{2})c(-3it+3\epsilon+\frac{1}{2}-\frac{i\delta}{2})c(1-i\delta)$$

$$(321)\times(23):\frac{e^{(2\epsilon-i\delta,-4\epsilon,2\epsilon+i\delta)(c,0,-c)}}{(-6\epsilon-i\delta)(6\epsilon-i\delta)}c(3it+3\epsilon+\frac{1}{2}+\frac{i\delta}{2})c(1+i\delta)c(-3it+3\epsilon-\frac{1}{2}+\frac{i\delta}{2})$$

$$(13)\times(123):\frac{e^{(-4\epsilon,2\epsilon-i\delta,2\epsilon+i\delta)(c,0,-c)}}{(-2i\delta)(-6\epsilon+i\delta)}c(3it+3\epsilon+\frac{1}{2}+\frac{i\delta}{2})c(3it+3\epsilon-\frac{1}{2}-\frac{i\delta}{2})$$

$$\times c(-3it+3\epsilon-\frac{1}{2}+\frac{i\delta}{2})c(1+i\delta)c(-3it+3\epsilon+\frac{1}{2}-\frac{i\delta}{2})$$

$$(123)\times(13):\frac{e^{(-4\epsilon,2\epsilon+i\delta,2\epsilon-i\delta)(c,0,-c)}}{(2i\delta)(-6\epsilon-i\delta)}c(3it+3\epsilon-\frac{1}{2}-\frac{i\delta}{2})c(3it+3\epsilon+\frac{1}{2}+\frac{i\delta}{2})$$

$$\times c(-3it+3\epsilon-\frac{1}{2}+\frac{i\delta}{2})c(-3it+3\epsilon+\frac{1}{2}-\frac{i\delta}{2})c(1-i\delta).$$

Its double residue at $\delta=0$ (up to a constant) is

$$\frac{e^{6\epsilon c}}{\epsilon}-\frac{e^{-6\epsilon c}}{\epsilon}c(3it+3\epsilon-\frac{1}{2})c(3it+3\epsilon+\frac{1}{2})c(-3it+3\epsilon-\frac{1}{2})c(-3it+3\epsilon+\frac{1}{2})$$

(the (321) − (23) terms only have single poles). This can be rewritten as

$$\frac{1}{\epsilon}\left(e^{6\epsilon c} - e^{-6\epsilon c}\frac{\zeta^*(-\frac{1}{2}+3it+3\epsilon)}{\zeta^*(\frac{3}{2}+3it+3\epsilon)}\frac{\zeta^*(-\frac{1}{2}-3it+3\epsilon)}{\zeta^*(\frac{3}{2}-3it+3\epsilon)}\right)$$

$$= \frac{1}{\epsilon}\left(e^{6\epsilon c} - e^{-6\epsilon c}\frac{\zeta^*(\frac{3}{2}-3it-3\epsilon)}{\zeta^*(\frac{3}{2}-3it+3\epsilon)}\frac{\zeta^*(\frac{3}{2}+3it-3\epsilon)}{\zeta^*(\frac{3}{2}+3it+3\epsilon)}\right),$$

which by lemma 5.9.4 is $O(\log T)$.

THE END

Bibliography

[Arthur 1980] James Arthur, *A trace gormula for reductive groups. II. Applications of a truncation operator*, Compositio Math. **40** (1980), no. 1, 87–121.

[Arthur 1984] James Arthur, *The trace formula for noncompact quotient* in *Proceedings of the International Congress of Mathematicians*, vol. II, pp. 849–859 (1984).

[Borel 1974] Armand Borel *Cohomology of Arithmetic Groups*, Proc. Int. Congr. of Mathematicians, Vancouver 1974, **1** (1975), 435-442.

[Borel 1997] Armand Borel *Automorphic Forms on $SL(2, \mathbf{R})$*, Cambridge Tracts in Mathematics, **130**, Cambridge University Press, 1997.

[Bump] Daniel Bump *Automorphic Forms and Representations*, Graduate Studies in Advanced Mathematics **55**, Cambridge University Press, 1997.

[Cartier] Pierre Cartier *Computers in number theory*, in *Computers in Number Theory*, Proceedings of the Science Research Council Atlas Symposium No. 2, A. O. L. Atkin and B. J. Birch., editors, Academic Press, New York, 1971.

[Donnelly] H. Donnelly, *On the cuspidal spectrum for finite volume symmetric spaces*, J. Diff. Geom. **17** (1982), 239-253.

[Fermigier] S. Fermigier, *Annulation de la cohomologie cuspidale de sous-groupes de congruence de* $GL_n(\mathbf{Z})$ (French) [Vanishing of the cuspidal cohomology of congruence subgroups of $GL_n(\mathbf{Z})$], Math. Ann. **306** (1996), no. 2, 247–256.

[Gelbart] Stephen Gelbart, *Automorphic Forms on Adele Groups*, Princeton University Press, 1975.

[GelJac] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. 4^e, Serie 11 (1978), **4**, 471–542.

[HarishChandra] Harish-Chandra, *Automorphic forms on semisimple lie groups*, L.N.M. **62**, Springer, 1968.

[Humphreys] James Humphreys, *Arithmetic groups*, Lecture Notes in Mathematics **789**, Springer-Verlag, Berlin, 1980.

[Jacquet] Hervé Jacquet, *On the residual spectrum of* $GL(n)$, in *Lie group representations, II*, 185–208, Lecture Notes in Math., **1041**, Springer, 1984.

[Langlands 1966] Robert Langlands, *Eisenstein series*, in *Albegraic groups and discontinuous subgroups*, A. Borel and G. Mostow, ed., Proc. Symp. Pure Math. **IX**, 1966, 235–252.

[Langlands 1971] Robert Langlands, *Euler products*, Yale Mathematical Monographs **1**, Yale University Press, 1971.

[Langlands 1976] Robert Langlands, *On the functional equations satisfied by Eisenstein series*, L.N.M. **544**, Springer, 1976.

[Langlands 1989] Robert Langlands, *Eisenstein series, the trace formula, and the modern theory of automorphic forms*, in *Number theory, trace formulas, and *discrete groups*, K. E. Aubert, E. Bombieri, and Dorian Goldfeld, ed. Academic Press, 1989, pp. 125-155.

[Maass] HANS MAASS *Siegel's modular forms and Dirichlet series*, Lecture Notes in Mathematics, **216**, Springer, 1971.

[Mestre] J.-F. Mestre, *Formules Explicites et Minorations de Conductors de Varietes Algebraiques*, Compositio Mathematica **58** (1986), pp. 209-232.

[Miller] S. Miller, *Spectral and Cohomological Applications of the Rankin-Selberg Method*, Internat. Math. Res. Notices, 1996, No. 1, 15-26.

[MœWal] C. MŒGLIN and J.-L. WALDSPURGER, *Spectral decomposition and Eisenstein series*, Cambridge Univ. Press, 1995.

[Odlyzko] ANDREW ODLYZKO, *Bounds for discriminants and related estimates for class numbers, regulators and zeroes of zeta functions: a survey of recent results*, Séminaire de Théorie des Nombres, Bordeaux **2** (1990), 119-141.

[Roelcke] W. Roelcke *Über die Wellengleichungen bei Grenzkreisgruppen erster Art*, Sitzungsberichte Heidelberg Akad. Wiss. (1955) NR. 4 109 pp..

[Rud-Sar] Z. Rudnick and P. Sarnak *Zeroes of Principal L-Functions and Random Matrix Theory*, Duke Jour. of Math, **81** no. 2 (1996), 269-322.

[Sarnak 1984] PETER SARNAK, *On cusp forms*, in *The Selberg trace formula and related topics*, Dennis Hejhal, Peter Sarnak, and Audrey Terras, ed., Contemporary Math. **53**, Amer. Math. Soc., 1984, 393-407.

[Sarnak 1995] PETER SARNAK, *Spectra of Hyperbolic Surfaces*, preprint, 1995.

[Selberg 1956] Atle Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.

[Selberg 1965] Atle Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math., Amer. Math. Soc., Vol. VIII (1965), pp. 1-15.

[Shahidi] Freydoon Shahidi, *On nonvanishing of L-functions*, Bull. Amer. Math. Soc. **2** (N.S.), No. 3 (1980), pp. 462-464.

[Speh 1981] Birgit Speh, *The unitary dual of* $GL(3, \mathbf{R})$ *and* $GL(4, \mathbf{R})$, Math. Ann. **258** (1981), 113-133.

[Speh 1983] Birgit Speh, *Unitary representations of* $GL(n, \mathbf{R})$ *with non-trivial* (g, K)*-cohomology*, Invent. Math. **71**, 443-465 (1983).

[Stark] Harold Stark, *Some effective cases of the Brauer-Siegel theorem*, Invent. math. **23** (1974), 135-152.

[Terras] A. Terras *Harmonic Analysis on Symmetric Spaces and Applications, Volume II*, Springer-Verlag, New York, 1988.

[Titchmarsh] E.C.Titchmarsh, *The Theory of the Riemann Zeta-function*, second edition revised by D.R. Heath-Brown, Clarendon Press, Oxford, 1986.

[Vakhutinski] I. Vakhutinski, *Unitary representations of* $GL(3, \mathbf{R})$, Math. USSR-Sb. **75** (1968).