Lecture Notes 11

Duals of Banach Spaces

Now we study $B(V,W)$ = bdd linear maps $V \to W$. (maps bdd sets to bdd sets)

& its topologies

Norm If V, W normed spaces

$$\|\Lambda\| := \sup_{\|v\|\le 1} \|\Lambda v\| = \sup_{\|v\|=1} \|\Lambda v\|$$

$$= \sup_{v\neq 0} \frac{\|\Lambda v\|}{\|v\|}$$

exists, ∞ since $\|\cdot\|$ is cts for $\Lambda \in B(V,W)$ & image is bdd.

Proposition $B(V,W)$ is a normed space under this norm, in fact a Banach space (complete normed space) if W is a Banach space.

Pf To show a normed space we need to check $\forall \alpha \in \mathbb{F}$

① $\|\alpha\Lambda\| = |\alpha| \cdot \|\Lambda\|$ (Obvious)

& ② $\|\Lambda_1 + \Lambda_2\| \le \|\Lambda_1\| + \|\Lambda_2\|$

$\left(\text{pf: If } \|v\|=1,\ \|(\Lambda_1+\Lambda_2)v\| \le \|\Lambda_1 v\| + \|\Lambda_2 v\| \le \|\Lambda_1\| + \|\Lambda_2\|\right)$,

& ③ $\|\Lambda\| = 0 \Leftrightarrow \Lambda = 0$ (obvious).

For the second part, we assume now W is a Banach space (complete).
If $\{\Lambda_n\}$ is a Cauchy sequence of operators in the $\mathcal{B}(V,W)$ norm.

$$\|\Lambda_n v - \Lambda_m v\| \leq \|\Lambda_n - \Lambda_m\| \cdot \|v\|$$

$$\to 0 \quad \text{for } \|v\| \leq 1 \text{ as } n,m \to \infty.$$

$\Rightarrow \{\Lambda_n v\}$ Cauchy, converges to limit called $f(v)$.

Now, by properties of limits & TVS.

$$f(v+w) = f(v) + f(w)$$
$$f(\alpha v) = \alpha f(v),$$

Hence f linear
Let $\varepsilon > 0$, & n,m large enough so that $\|\Lambda_n - \Lambda_m\| \leq \varepsilon$
Now $\|fv - \Lambda_n v\|$

$$\leq \|fv - \Lambda_m v\| + \|\Lambda_m v - \Lambda_n v\|$$

let $m \to \infty$

$$\|fv - \Lambda_n v\| \leq \|\Lambda_m - \Lambda_n\| \|v\|$$
$$\leq \varepsilon \|v\|$$

$\Rightarrow \|f - \Lambda_n\| \leq \varepsilon.$

$\Rightarrow \Lambda_n \to f$ in operator norm. $\square$

Now, we specialize to when $W = k$.

(Bad, sometimes) Notation: Let $v \in V$, $\lambda \in V^*$
normed space.

denote $\lambda(v)$ by

$$\langle v, \lambda \rangle.$$

! Careful Not an inner product, just a pairing $V \times V^* \to k$.

But it satisfies Cauchy-Schwartz: $|\langle v, \lambda \rangle| \leq \|v\| \cdot \|\lambda\|$
w/ following def'n

Given $\lambda \in V^*$, $\|\lambda\| = \sup_{\|v\|=1} |\langle v, \lambda \rangle|$.

This Norm Makes V^* into a Banach space — strictly stronger topology, than weak* if space is ∞-dim'l.
by previous prop.
(we will see this now)

Theorem Norm topology on V^* is stronger than weak* top. They coincide if $\dim V < \infty$, but Norm top is strictly stronger if $\dim V = \infty$.

Pf Let $B^* =$ closed unit ball in V^* in norm topology. Then we claim

$$\|v\| = \sup_{\substack{\lambda \in V^* \\ \lambda \in B^*}} |\langle v, \lambda \rangle|$$

(i.e., usual norm on V coincides with

operator norm on V, viewed as a subspace of $(V^*)^*$

Pf of claim: we showed before, that for all $v \in V$ $\exists \lambda \in B^*$ w/ $\langle v, \lambda \rangle \leq \|v\|$

But for all $\mu \in B^*$

$$|\langle v, \mu \rangle| \leq \|v\| \cdot \|\mu\| \leq \|v\|,$$

so indeed, $\|v\| = \sup_{\substack{\cancel{\lambda \in V^*} \\ \lambda \in B^*}} |\langle v, \lambda \rangle|$

Now, the map $\lambda \mapsto \langle v, \lambda \rangle$, for any fixed v, is a bdd linear fnl on V^* w/ norm $\|v\|$, $\Downarrow$ cts,

Thus weak* top is weaker than norm topology, the weakest in which these are all cts

If $\dim V = \infty$, suppose weak* top coincides w/ norm topology, then weak* top is locally bounded. A set is weak(*)-ly bdd $\Leftrightarrow$ all lin fnls on A are bdd.

Let U = open nbd of 0 in weak* top.

$\cancel{\text{of}} \supseteq$ intersection of $\{\lambda \mid |\langle v, \lambda \rangle| < \varepsilon\}$ for several v's.

$\supseteq$ ker of such v's in V^*

So any open nbd U contains a subspace of ∞ dimension (Ker cannot have finite dim, since it has finite codim, & space is ∞-dim'l),

Let $\lambda \in N$ be nontrivial & v a vector for which $\langle v, \lambda \rangle \neq 0$, thus v is not bdd on N.
$\Rightarrow$ v is not bdd on U.
$\Rightarrow$ U is not weak(*)ly bdd, a contradiction. $\square$

Proposition In V^*, closed unit ball B^* in norm topology is weak* cpt.

Proof $\lambda \in B^* \iff |\langle v, \lambda \rangle| \leq 1$ for all $x \in$ open unit ball B in V.

Banach-Alaoglu! $\{\lambda \in V^* \mid |\lambda v| \leq 1 \text{ for all } v \in B\} = B^*$ is weak*-cpt
works for any open nbd of 0.
$\square$

Proposition Let V, W be normed spaces & $\Lambda \in B(V,W)$. Then

$$\|\Lambda\| = \sup_{\substack{\|v\| \le 1 \\ \|\lambda\| \le 1}} |\langle \Lambda v, \lambda \rangle|$$

p. 8/C

Pf We have, for any ~~xxx~~ $v \in V$

$$\|\Lambda v\| = \sup_{\|\lambda\| \le 1} |\langle \Lambda v, \lambda \rangle|$$

by claim in pf of last Thm

$\|\Lambda\| = \sup_{\|v\| \le 1} \|\Lambda v\|$ so prop follows. $\square$.

Dual of Dual Last Thm showed we have an isometric embedding $V \hookrightarrow (V^*)^*$

↑ normed TVs.

V Banach, complete, isometric image is complete. Hence closed,

so we may view V as a closed subspace of $(V^*)^*$.

Characterization of Image those elts of $(V^*)^*$ which are cts in weak* topology.

Reflexive $V = V^{**}$ (more correctly, image is onto) $\iff$ weak* top coincides w/ norm top on V^{**} (induced by norm top on V^*, (itself induced from V.))

Annihilators

Let V = Banach Space
$M \subset V$ subspace
$N \subseteq V^*$ subspace

Annihilators: $M^{\perp} = \{ \lambda \in V^* \mid \langle v, \lambda \rangle = 0 \text{ for all } v \in M \}$

$${}^{\perp}N = \{ v \in V \mid \langle v, \lambda \rangle = 0 \text{ for all } \lambda \in N \}$$

we put the $\perp$ superscript on the left or right to distinguish between subspaces of V^* & V, respectively.

$M^{\perp}$ is a weak*-closed subspace, since it is the intersection of closed subspaces: inverse images of $\{0\}$ under the cts maps $\lambda \mapsto \langle v, \lambda \rangle$.

${}^{\perp}N$ is a closed subspace, since each λ is cts.

<u>Thm</u> In this setup, ${}^{\perp}(M^{\perp})$ = closure of M in V under its (norm) topology (also in weak topology, too since V is locally convex, & then weak & strong closures of convex sets coincide)

$({}^{\perp}N)^{\perp}$ = weak*-closure of N in V^*

<u>Proof</u>

First, $v \in M \Rightarrow v \in {}^{\perp}(M^{\perp})$ obviously
$\lambda \in N \Rightarrow \lambda \in ({}^{\perp}N)^{\perp}$ - - -

so $M \subseteq {}^{\perp}(M^{\perp})$ $\Rightarrow$ $\overline{M} \subseteq {}^{\perp}(M^{\perp})$
$N \subseteq ({}^{\perp}N)^{\perp}$ $\Rightarrow$ $\overline{N} \subseteq ({}^{\perp}N)^{\perp}$,

we need to show the reverse inclusions.

If $v \notin \overline{M}$, then since V is locally convex
$\overline{M}$ closed subspace, $\{v\}$ cpt.
in particular convex balanced

$\Downarrow$ version of Hahn-Banach

$\exists \lambda \in V^*$ s.t. $\lambda \in M^{\perp}$, $\lambda v \neq 0$.

That means $v \notin {}^{\perp}(M^{\perp})$. So $\overline{M} = {}^{\perp}(M^{\perp})$.

Now assume $\Lambda \notin \overline{N}^{w}$ = weak* closure of N (not norm closure).

Again, V^* locally convex so exists cts linear fnl (i.e. $v \in V$) s.t. $\Lambda v \neq 0$, but $v \in {}^{\perp}N$, so $\overline{N} = ({}^{\perp}N)^{\perp}$. $\square$

(dual logic)

Corollary Let M be a normed-closed subspace of V. Then M = annihilator of its annihilator. Likewise, N = weak*-closed subspace of V^* $\Rightarrow$ N = annihilator of its annihilator.