Lecture Notes 12

Subspaces & Quotients.

Let V be a vector space, for now finite dimensional.

Let V' = dual of V

W = subspace of V

$W^\perp$ = annihilator of W in V'

Then W' is canonically isomorphic to $V'/W^\perp$.

This is because linear functionals on W must extend to linear functionals on V.

2 ~~such~~ f'nls on V agree on W $\Leftrightarrow$ their difference is zero on W $\Leftrightarrow$ their difference lies in $W^\perp$

Likewise: V/W is a vector space & its dual is $\cong$ canonically to $W^\perp$.

So: we see the notion of subspace is dual to the notion of quotient.

We will extend this result to some TVS, but first we need to develop some background, including:

- quotient TVS's
- extending linear functionals.

Now we will go back to some material earlier in Rudin that we skipped over. (§1.40)

Consider projection map $\pi : V \twoheadrightarrow V/W$

quotient space

$\pi(x) = x + W$ equiv class.

π is a linear map between the vector spaces V & V/W, w/ kernel W.

· We now assume V is a TVS. If π is cts, W is closed. Thus we also assume W is a closed subspace. We next define a topology on V/W & prove π is closed.

<u>Quotient topology</u> open sets U in V/W are those for which $\pi^{-1}(U)$ is open in V.

(weakest topology which makes π cts)

<u>Proposition</u> 1) V/W is a TVS

2) π is cts, open, linear

3) If $\mathcal{B}$ is a local base for V's topology, $\pi(\mathcal{B})$ is a local base for V/W's topology

4) If V locally convex, so is V/W

——— locally bdd ———

——— metrizable ———

——— normable ———

——— Frechet ———

——— Banach ———

<u>Proof</u> 1)

First we show this is a topology.
If U_1 & U_2 are open ($\Leftrightarrow$ $\pi^{-1}(U_1)$, $\pi^{-1}(U_2)$ open)
then $\pi^{-1}(U_1 \cap U_2) = \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ is open
$\Leftrightarrow$ $U_1 \cap U_2$ is open.
If U_α are open, similarly
$\pi^{-1}(\bigcup U_\alpha) = \bigcup \pi^{-1}(U_\alpha)$ is open.

So {open sets} are closed under finite intersections & arbitrary unions.
Obviously $\emptyset$, V/W are open, as
$\pi^{-1}(\emptyset) = \emptyset$
$\pi^{-1}(V/W) = V$.
So <u>this is a topology.</u>

As a detour before showing V/W is a TVS, we note that if $S \in V$ is open
$$\pi^{-1}(\pi(S)) = S + W = \bigcup_{w \in W} (w+S) \text{ is open in } V.$$
Thus $\pi(S)$ is open $\therefore$ <u>π is an open mapping</u>

Now, back to TVS axioms.

- If <u>U is a nhd of 0</u> in V/W
  <u>$\pi^{-1}(U)$</u> — V

$\exists$ open nhd S of 0 in V such that $S+S \in \pi^{-1}(U)$

So $\pi(S)+\pi(S) \subseteq \pi(\pi^{-1}(U)) = U$

$\pi(S)$ open since π is an open map,

$\Rightarrow$ addition is continuous.

~~[Recall: scalar mult cts~~ if $\forall x\in V, \alpha\in k$, U nhd of αx, $\exists r>0$ & open nhd S of x s.t. $\beta S \subseteq U$ for all $|\alpha-\beta|$

- If U is an open nhd of 0 in V/W
  $x \in V/W$
  $\alpha \in k$,

then...

$S = \pi^{-1}(U)$ open in V

pick $v \in \pi^{-1}(x) \in V$

scalar mult in V is cts, so $\exists r>0$ & open nhd T of v s.t. for all $|\beta-\alpha|<r$

$$\beta T \subseteq \alpha v + S$$

$$\Rightarrow \beta\pi(T) \subseteq \alpha\pi(v)+\pi(S)$$

$$\beta\pi(T) \subseteq \alpha x + U$$

$\pi(T)$ open since π is an open map

so $\exists r>0$ & open nhd $\pi(T)$ of x s.t. for all $|\beta-\alpha|<r$

$$\beta\pi(T) \subseteq \alpha x + U.$$

Thus scalar mult is cts.

Finally points are closed since they are images of points, & π is an open map.

This proves 1) & 2).

3): Let $\mathcal{B}$ be a local base, & U an open subset of V/W.

Then some $B \in \mathcal{B}$ is contained in $\pi^{-1}(U)$

$$\pi(B) \subseteq \pi(\pi^{-1}(U)) = U$$

remember this equality is automatic for onto maps

Thus $\pi(\mathcal{B})$ is a local base.

4) Laundry list:

linear maps preserve convexity so "local convexity" part is OK. ✓

π cts $\Rightarrow$ bdd $\Rightarrow$ bdd sets are mapped to bdd sets

so "locally bdd" part is OK. ✓

Metrizability w/ invariant metric $\Leftrightarrow$ countable local base.

so "metrizable" part is OK ✓

Normable $\Leftrightarrow$ $0 \in$ a convex bdd nbhd

so inherited by linear map ✓

Now: Frechet (locally convex complete w/ invariant metric)

Here's how to make an invariant metric on V/W:

$$\rho(\pi(x), \pi(y)) := \inf(d(x-y, z) \mid z \in W)$$

Check: obviously well defined & invariant. & a metric.

Open Ball in ρ is image of open ball in V.

So ρ is a compatible metric.

In a normed space, this is the quotient norm

$$\|\pi(x)\| = \inf\{\|x-z\| \mid z \in W\}.$$

So, to wrap up, we need to show
ρ is complete ~~if d is~~ [result for norm is a special case]

Let u_n be a Cauchy sequence in V/W under ρ. Then, passing to a subsequence,

$\rho(u_n, u_{n+1}) < 2^{-n}$

By def'n of ρ as inf, $\exists\, x_n \in V$ s.t.

$d(x_n, x_{n+1}) < 2^{-n}.$

$\Rightarrow x_n \to x \in V.$

π cts $\Rightarrow \pi(x_n) = u_n \to \pi(x) =: u.$

But this means original sequence $\to u$!

$\therefore$ ~~So~~ V/W is complete under ρ. $\square$

Applications

Prop Let $V = $ TVS

N, F subspaces. N closed, $\dim F < \infty$. Then $N+F$ is closed.

Slick proof $\pi: V \to V/N$ is cts
~~give it~~ quotient topology

$\pi(F) \subseteq V/N$ is finite dimensional
Hence closed.

Inverse image $\pi^{-1}(\pi(F)) = N+F$
is closed since
π is a cts map. $\square$

Example Let V = vector space
p = seminorm

$N = \{v \mid p(v)=0\} \subseteq V$, subspace

$\pi: V \to V/N$ quotient map.

Quotient seminorm $\tilde{p}(\pi(x)) = p(x)$
is actually a norm!

Eg. $\mathcal{L}^r$ = all msrble f'ns
on $[0,1]$ s.t. $\|f\|_r < \infty$
$p = \|\cdot\|_r$
not a norm.
But $\tilde{p} = \|\cdot\|$ on $L^r = \mathcal{L}^r/(\text{a.e. zero f'ns})$
is
a Banach space.

Now, back to Chapter 4.

Thm Let V = Banach Space

$M \subseteq V$ closed subspace.

Then Hahn-Banach extends any $\lambda \in M^*$ to a fnl. Λ on V^*.

The map
$$\sigma : \lambda \mapsto \Lambda + M^\perp$$
is an isometric isomorphism of M onto $V^*/M^\perp$

Pf.

σ well defined, since the difference of 2 extns is trivial on M, hence in $M^\perp$

Obviously σ is linear

Let $\Lambda \in V^*$, & $\lambda := \Lambda|_M$. Then $\sigma(\lambda) = \Lambda + M^\perp$ so σ is onto.

Now we prove an estimate.

Given $\lambda \in M^*$ & Λ extending it

$$\|\lambda\| \le \|\Lambda\|$$

since each is a sup of $|\Lambda v|$, but the RHS over more v.

Quotient norm $\|\Lambda + M^\perp\| = \inf \|\Lambda'\|$, Λ' an extn of λ,

$\|\sigma\lambda\|$.

Thus

$$\|\lambda\| \le \|\sigma\lambda\| \le \|\Lambda\|$$

for any ext'n Λ.

Old Hahn-Banach: $\exists$ an extn Λ of λ w/ same norm

so $\|\sigma\lambda\| = \|\lambda\|$,

σ therefore is 1-1 & isometric.

Another Similar Thm Let V = Banach Space

M = closed subspace.

$\pi : V \to V/M$ quotient map.

$W = V/M$.

Given $\Lambda \in W^*$, define $\tau : \Lambda \mapsto \Lambda \circ \pi$

$: W^* \to V^*$.

Then τ is an isometric embedding of W^* onto $M^\perp$

Pf If $v \in V$

$\Lambda \in W^*$

$\pi v \in W$

$\therefore v \mapsto (\Lambda \circ \pi)(v)$ is a cts linear map vanishing on M.

$\Rightarrow \tau\Lambda \in M^\perp$.

τ obviously linear

Now we prove it is onto,

Let $\mu \in M^\perp$, $N = \ker \mu \supseteq M$.

$\exists$ linear functional Λ on W p.10/11
such that $\Lambda \circ \pi = \mu$
(factors through kernel).
we don't immediately know $\Lambda \in W^*$ (is cts),
though.

Here $\Lambda = \pi(N)$ (by def'n of quotient)
closed in W,
since $N = \ker \mu$ & μ is cts.
Old thm: for nontrivial linear fnl, closed ker $\Rightarrow$ cts,
— trivial ——— also cts,
So Λ is cts, $\in W^*$.
$\tau\Lambda = \mu$, so τ is onto.

Now we prove $\|\tau\Lambda\| = \|\Lambda\|$,
which is the isometry statement
(& again implies τ is one-to-one).
Let $\Lambda \in W^*$.
Let $w \in W$ w/ $\|w\| = 1$, & $r > 1$.
By def'n of $W = V/M$'s quotient norm as an mp
$\exists x \in V$ s.t.
$\pi x = w$
$\|x\| < r$.

Thus
$$|\langle w, \Lambda\rangle| = |\Lambda w| = |\Lambda \pi x|$$
$$= |(\tau\Lambda)(x)|$$
$$\leq \|\tau\Lambda\| \cdot \|x\|$$
True for all $\leq r\|\tau\Lambda\|$
$\|w\| = 1 \implies \|\Lambda\| \leq r\|\tau\Lambda\|$, for all $r > 1$
$\therefore \|\Lambda\| \leq \|\tau\Lambda\|$.

Also, $\|\pi v\| \le \|v\|$ for all $v \in V$
(def'n of inf.)

So
$$\|(\tau \lambda)(v)\| = \|\lambda \pi v\| \le \|\lambda\| \cdot \|\pi v\|$$
$$\le \|\lambda\| \, \|v\|$$
for all $v \in V$.

Thus
$$\|\tau \lambda\| \le \|\lambda\|$$
$$\& \ \|\lambda\| = \|\tau \lambda\|. \quad \square.$$

<u>Remarks</u> We have shown what we had ~~set out to~~, namely that

① if V = Banach space
$W \subseteq V$ closed subspace
W^* canonically isometric to $V^*/W^\perp$

② $(V/W)^*$ canonically isometric to $W^\perp$

So quotient & subspace are dual notions.