## Distributions as Linear Functionals

Let $\Omega \subseteq \mathbb{R}^n$ be an open set

<u>Recall</u> $f : \Omega \to \mathbb{C}$ is <u>locally $L^1$</u> (aka locally integrable) if

$$\int_K |f| < \infty \quad \text{for all compact } K \subseteq \Omega$$

$L^1_{loc}(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ which are locally } L^1 \}$

If $f \in L^1_{loc}$ & $\varphi$ has cpt supp,

$$\int_\Omega f\varphi \quad \text{is well defined.}$$

Now, this is a linear f'nl on $\mathcal{D}(\Omega)$ (below), & our goal is to massively extend to other linear f'nls, yet still use the calculus formalism of integration to represent them.

(e.g. integration by parts, "differentiating" nondifferentiable functions — how do you make sense of $\left(\frac{d}{dx}\right)^2 |x|$ ?)

$\mathcal{D}(\Omega)$

$= \{ f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp} f \text{ is a cpt subset of } \Omega \}$,

$=$ test function space

$$= \bigcup_{\substack{K \text{ compact} \\ K \subseteq \Omega}} \mathcal{D}_K, \quad \mathcal{D}_K = \{ f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp} f \subseteq K \},$$

Recall the Frechet topology on $\mathcal{D}_K$, $K \subseteq \Omega$ cpt.

It is actually given by the (semi)-norms $\|f\|_N =$

$$= \max\{|D^\alpha f| \text{ on } \Omega \mid |\alpha| \leq N\}$$

(earlier we defined seminorms ~~by~~ on $\mathcal{D}(\Omega)$ taking the maximum over cpt sets $K_N$, where $K_1 \subseteq K_2 \subseteq \cdots$ & $\bigcup K_n = \Omega$.
But this is the same if we restrict to functions w/ support in some fixed $K$)

Note $\|f\|_N$ is actually a norm on the vector space $\mathcal{D}(\Omega)$.

Let us ~~a~~ define a topology now on $\mathcal{D}(\Omega)$ by specifiying a local base

$$\mathcal{B} = \left\{ \begin{array}{l} \text{balanced convex subsets } W \subseteq \mathcal{D}(\Omega) \\ \text{s.t. } W \cap \mathcal{D}_K \text{ is open in } \mathcal{D}_K \\ \text{for all cpt } K \subseteq \Omega \end{array} \right\}$$

We need to spend some time getting used to this topology, in particular showing it makes $\mathcal{D}(\Omega)$ into a TVS.

Example

Let's note actually that the $\mathcal{D}(\Omega)$-translates of $\mathcal{B}$ are already a topology. In otherwords, you don't need to take finite intersections. Suppose we have 2 translates

$$f_1 + W_1 \quad \& \quad f_2 + W_2$$

$f$ a function $f$ in their intersection.
We will show $\exists\, W \in \mathcal{B}$ such that

$$f + W \subseteq f_1 + W_1 \ \& \ f_2 + W_2.$$

Now $f, f_1, f_2$ all have cpt supp in $\Omega$,
So $\exists$ a cpt subset $K$ st,

$$f, f_1, f_2 \in \mathcal{D}_K.$$

By def'n of $\mathcal{B}$,

$$\mathcal{D}_K \cap W_1,\ \mathcal{D}_K \cap W_2,\ \mathcal{D}_K \cap W$$

are all open subsets of $\mathcal{D}_K$.

By above $f - f_1 \in W_1 \cap \mathcal{D}_K$ & $f - f_2 \in W_2 \cap \mathcal{D}_K$ so $\exists N$ large so that $f-f_1,\ f-f_2 \in S$,

where

$$\underbrace{S_0 = \{\varphi \in \mathcal{D}_K \mid \|\varphi\|_N < \varepsilon\}}_{\text{this is a basic open nbd of } 0 \text{ in } \mathcal{D}_K} \quad \text{for some } N \& \varepsilon,$$

Now $f - f_1 \in S$ means

$$r(f - f_1) \in S \quad \text{for some}$$

$r$ slightly $> 1$

(since $r\|f - f_1\|$ is still $< \varepsilon$, if $r$ is slightly $> 1$)

So $f-f_1 \in (1-\delta)S$ for some small $\delta>0$.
$f-f_2$

$S \subseteq W_1 \cap D_K$
$W_2 \cap D_K$ & each balanced, so
of $W_1, W_2$

thus $(1-\delta)S \subseteq (1-\delta)W_1 \cap D_K$
& $(1-\delta)W_2 \cap D_K$,

Since $W_1, W_2$ convex

$$f-f_1+\delta W_1 \subseteq (1-\delta)W_1+\delta W_1 \subseteq W_1$$
$$\text{or } f-f_2+\delta W_2 \subseteq (1-\delta)W_2+\delta W_2 \subseteq W_2.$$

So let $W=(\delta W_1)\cap(\delta W_2)$, & then

$$f \in f_1+W$$
$$f \in f_2+W$$

This is a good exercise in the definitions.

<u>Theorem</u> $\mathcal{D}(\Omega)$ is a locally convex TVS with this topology.

<u>Pf</u> First let $f \in \mathcal{D}(\Omega)$ be arbitrary, & $g \in \mathcal{D}(\Omega)$, $g \neq f$,
let $W=\{\varphi \in \mathcal{D}(\Omega) \mid \|\varphi\|_0 < \|f-g\|_0\}$
just sup norm.

So $W$ is balanced & its intersection w/ all $\mathcal{D}_K$ is of course open. P.5/14

Also $f \notin g + W$.
So complement of $\{f\}$ is open
$\Uparrow$
any singleton $\{f\}$ is closed.

Cty of addition Given an open nbd $W$ of $0$ $W \in \mathcal{B}$,
$\frac{1}{2}W \in \mathcal{B}$
& $\frac{1}{2}W + \frac{1}{2}W \subseteq W$ by convexity,
so addition is cts (Sufficed to produce $U = \frac{1}{2}W$ in $\mathcal{B}$).

Cty of scalar mult Let $\alpha \in k$, $f \in \mathcal{D}(\Omega)$.
Let $W \in \mathcal{B}$.
$\exists$ some small $\delta > 0$ so that $\delta f \in \frac{1}{2}W$.

Now, if $|\beta - \alpha| < \delta$ & $\varphi - f \in \frac{1}{2(|\alpha|+\delta)} W$,

then

$$\beta\varphi - \alpha f = \beta(\varphi - f) + (\beta - \alpha)f$$
$$\in \tfrac{1}{2}W + \tfrac{1}{2}W \subseteq W.$$

So scalar mult is cts. $\square$.

**Lemma** Let $V$ be an open subset of $\mathcal{D}(\Omega)$, let $K \subseteq \Omega$ be cpt.
Then $\mathcal{D}_K \cap V$ is an open subset of $\mathcal{D}_K$.

**Pf** Let $f \in \mathcal{D}_K \cap V$.
$V$ is open, so $\exists W \in \mathcal{B}$ such that
$f + W \subseteq V$.
Then $f + \underbrace{W \cap \mathcal{D}_K}_{\text{open in } \mathcal{D}_K} \subseteq \mathcal{D}_K \cap V$

$f$ is arbitrary, so Thus $\mathcal{D}_K \cap V$ is open. $\square$

**Corollary** Every open, convex balanced subset of $\mathcal{D}(\Omega)$ lies in $\mathcal{B}$.

(immediate from def'n of $\mathcal{B}$ on p.12).

**Propositon** The subspace topology of any $\mathcal{D}_K \subseteq \mathcal{D}(\Omega)$, $K \subseteq \Omega$ cpt, coincides w/ $\mathcal{D}_K$'s usual topology

Thus $\mathcal{D}_K$ embeds into $\mathcal{D}(\Omega)$ as a subspace.

**Pf** The lemma shows one direction. We must show if $U \subseteq \mathcal{D}_K$ is open in the usual topology, $U = \mathcal{D}_K \cap V$ for some open subset $V$ in $\mathcal{D}(\Omega)$.

Given $\varphi \in U$, open, $\exists\ N > 0$ & $\varepsilon > 0$ s.t.

$$\{ f \in \mathcal{D}_K \mid \|f - \varphi\|_N < \varepsilon \} \subseteq U.$$

So let $W_\varphi = \{ f \in \mathcal{D}_K \mid \|f\|_N < \varepsilon \}$

& $W_\varphi \in \mathcal{B}$, of course, since you only need one seminorm to define an open set.

$\varphi + W_\varphi \subseteq U$.

Also $\mathcal{D}_K \cap [\varphi + W_\varphi] = \varphi + [\mathcal{D}_K \cap W_\varphi]$
$\subseteq U$

Let $V = \bigcup\limits_{\varphi \in U} \underbrace{[\varphi + W_\varphi]}_{\text{open set}}$.

Then indeed $V$ is open in $\mathcal{D}(\Omega)$
& $U = \mathcal{D}_K \cap V$. $\square$

<u>Prop</u> The bounded subsets of $\mathcal{D}(\Omega)$ are precisely those which are contained in $\mathcal{D}_K$ for some cpt $K \subset \Omega$
& on which each $\|\cdot\|_N$ is bdd.

<u>Pf</u> First suppose $E \subseteq \mathcal{D}_K$ for some cpt $K \subseteq \Omega$
& $\|f\|_N \leq$ some $M_N < \infty$ for each $f \in E$.

Let $U$ be an open nbd in $\mathcal{D}_K$,
which we may assume is Balanced.
Then $E \subseteq tU$ for some large $t$,
because $U$ contains a basic subnbd defined by a finite # of seminorms.
Hence $E$ is bdd as a subset of $\mathcal{D}_K$.
By last Prop, bdd in $\mathcal{D}(\Omega)$
(bddness is inherited in larger spaces).

Conversely, now, suppose $E \subseteq \mathcal{D}(\Omega)$ is bounded, & that $E \not\subseteq \mathcal{D}_K$ for any cpt $K \subseteq \Omega$.

Thus $\bigcup_{f \in E} \operatorname{supp}(f)$ is not cpt.

There thus then exist a sequence $x_1, x_2, \dots$ in that Union which has no limit point. For each such $x_m$, $\exists f_m \in E$ s.t. $f_m(x_m) \neq 0$, of course.

Let $W = \{ \varphi \in \mathcal{D}(\Omega) \mid |\varphi(x_m)| < \frac{1}{m} |\varphi_m(x_m)| \}$ for all $m$.

since all $\varphi$ here have cpt supp, this is a finite cond'n for each $\varphi$.

Also, given $K$ cpt $\subseteq \Omega$, only a finite # of points $x_1, -, x_m, -$ lie in $K$.

so $W \cap \mathcal{D}_K = \{ \varphi \in \mathcal{D}_K \mid |\varphi(x_j)| < \delta_j \}$

for a finite # of points $x_j$, & $\delta_j > 0$

This is an open subset of $\mathcal{D}_K$, since if you perturb $\varphi$ by a fn $f$ w/ $\|f\|_0$ small, $\varphi + f \in W \cap \mathcal{D}_K$ also.

$W$ is also balanced & convex.

so $W \in \mathcal{B}$.

$\varphi_m \notin mW$, though

so $E \not\subseteq mW$ for any $m \geq 1$.

So $E$ is actually not bounded, contradiction. UPSHOT $E$ is $\subseteq \mathcal{D}_K$ for some cpt $K \subseteq \Omega$

Thus $E$ is also bdd in $\mathcal{D}_K$, since the subspace topology of $\mathcal{D}_K \subseteq \mathcal{D}(\Omega)$ coincides with its usual topology (prop, p.6)

Let $\|\cdot\|_N$ be one of the seminorms. Its open ball is open in $\mathcal{D}_K$

$E \subseteq \mathcal{D}_K$ bdd $\Rightarrow$ $E \subseteq t\cdot$(open ball) for some $t$

$\Rightarrow \|\ \|_N \leq t$ on $E$.

So each $\|\ \|_N$ is uniformly bdd on $E$. $\square$.

**Thm** $\mathcal{D}(\Omega)$ is complete.

Moreover

a) it has the Heine-Borel property (closed + bdd $\Rightarrow$ cpt)

b) $\{\varphi_j\} \subseteq \mathcal{D}(\Omega)$ is Cauchy $\Leftrightarrow$ $\{\varphi_j\} \subseteq \mathcal{D}_K$ for some cpt $K \subseteq \Omega$ & $\lim_{i,j\to\infty} \|\varphi_i - \varphi_j\|_N = 0$ $\forall N$.

c) $\varphi_j \to 0$ in $\mathcal{D}(\Omega)$ $\Leftrightarrow$ $\exists$ cpt $K \subseteq \Omega$ s.t. $\operatorname{supp} \varphi_j \subseteq K$ $\forall j$

& $D^\alpha \varphi_j \to 0$ unif on $K$ for each $\alpha$.

**Pf** a): $\mathcal{D}_K$ has the Heine Borel property (proven earlier), so if $E \subseteq \mathcal{D}(\Omega)$ is bdd, it lives in some $\mathcal{D}_K$, where it is also closed & bdd

& hence cpt in $D_K$. But topologies coincide, so (subspace of $D_K$)
E is also cpt in $D(\Omega)$, ✓

(b). Let $\{\varphi_j\} \subseteq D(\Omega)$ be a Cauchy sequence. It is bdd, so lies in some $D_K$ for $K$ cpt $\subseteq \Omega$.
It is Cauchy in the subspace $D_K$, again because topologies coincide.

completeness

Conversely, Cauchy in $D_K \Rightarrow$ Cauchy in $D(\Omega)$ since topologies again coincide.
$D_K$ is Frechet! So $\varphi_j$ has a limit in $D_K$. But convergence there $\Rightarrow$ convergence in $D$ since topologies coincide

(c) $\Rightarrow$: restatement of b), ~~which is complete~~ since $\varphi_j$ is also a Cauchy sequence & $D_K$ is Frechet (complete anyhow).

$\Leftarrow$: The assumption is a convergent sequence in $D_K$.

completeness

So Cauchy in $D_K \Rightarrow$ Cauchy in $D(\Omega)$
$\Rightarrow$ Convergent, since completeness has now been shown.

□.

Now we come to an important statement about Linear functionals.

Prop Let $\Lambda : \mathcal{D}(\Omega) \to V$ be a linear mapping, where $V$ is a locally convex TVS p.11/14

<u>TFAE</u> a) $\Lambda$ is cts

b) $\Lambda$ is bdd

c) $\varphi_j \to 0$ in $\mathcal{D}(\Omega) \Rightarrow \Lambda\varphi_j \to 0$ in $V$

d) $\Lambda|_{\mathcal{D}_K}$ is cts on $\mathcal{D}_K$.

Pf a) $\Rightarrow$ b) was shown before in great generality.
But c) does not follow since $\mathcal{D}(\Omega)$ does not have a metric.
But $\mathcal{D}_K$ does. So, . . .

b) $\Rightarrow$ c): $\varphi_j \to 0$ in some $\mathcal{D}_K$ by the last thm.

so $\Lambda\varphi_j$ is bdd (in general)
$\mathcal{D}_K$ is Frechet, so before we saw this implies $\Lambda\varphi_j \to 0$ actually.

c) $\Rightarrow$ d). If $\varphi_j \to 0$ in $\mathcal{D}_K$, it does in $\mathcal{D}(\Omega)$ (last thm)
So $\Lambda\varphi_j \to 0$ by c)
$\mathcal{D}_K$ metrizable, so this sequential continuity implies cty.

d) $\Rightarrow$ a). Let $U \subseteq V$ be a balanced convex nbd.
It suffices to show $\Lambda^{-1}(U)$ is open.
It is balanced & convex, since it is an inverse image.

by d) $\Lambda^{-1}(U) \cap D_K$ is open in $D_K$ for each $D_K$,

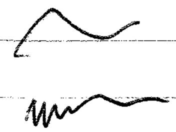
so $\Lambda^{-1}(U)$ lies in $\mathcal{B}$,

$\Box$.

<u>Corollary</u> (Important) each $D^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is cts,

<u>Remark</u> in most topologies, differentiation is <u>not</u> cts

(intuitively: worsens bounds on f'ns) e.g. in $L^2([0,1])$



$\frac{d}{dx} e^{2\pi i n x} = 2\pi i n \; e^{2\pi i n x}$

magnifies norm by $4\pi^2 n^2$!

<u>Pf</u> (Sketch) This works because we use all seminorms, not just the sup norm,

Suffices to show for $|\alpha| = 1$, by composition

Let $E$ be a bounded set, $\|f\|_k < \infty$ for all $f \in E$, $k$ fixed,

Then $\|Df\|_n \le \|f\|_{n+1}$ $\forall n \ge 0$.

so $DE$ is bdd!

$D$ is a bdd operator, $\Leftrightarrow$ to cty by previous Prop

$\Box$.

**Distributions** = $\mathcal{D}(\Omega)^*$

But we write $\mathcal{D}'(\Omega)$ for it, keeping w/ traditions.

Last Prop, when $V = k$ says: (puts ued)

$\Lambda$ a linear fnl on $\mathcal{D}(\Omega)$ is a distribution

$\Updownarrow$

For each cpt $K \subseteq \Omega$ $\exists$ some $N \geq 0$ & const $c < \infty$ s.t.

(*). $|\Lambda \varphi| \leq c \|\varphi\|_N$ for all $\varphi \in \mathcal{D}_K$.

Proof The prop says this is equiv to

$\Lambda|_{\mathcal{D}_K}$ being cts on $\mathcal{D}_K$, for each $K$ cpt $\subseteq \Omega$.

$\Updownarrow$

$N$ s.t.

$|\Lambda \varphi| \leq 1$ for all $\varphi$ s.t. $\|\varphi_N\| < \frac{1}{N}$

$\Updownarrow$

the condition (*) above

**Definition** A distribution $\Lambda$ has finite order $N$ if $\exists$ some $N$ s.t. $\frac{|\Lambda \varphi|}{\|\varphi\|_N}$ is bounded on each $K$

(this can be unbounded as you vary $K$;
the point is just that we need only one
$N$ but maybe various $C$ in (*).)

Otherwise, $u$ has infinite order.

Example for $x \in \Omega$, let $\delta_x$ = Delta fn

$$\delta_x(\varphi) = \varphi(x).$$

A distribution of order zero, of course
($N=0$, $C=1$ works!).

Corollary

$$D_K = \bigcap_{x \notin K} \ker \delta_x$$ is thus a closed subspace of $\mathcal{D}(\Omega)$

Prop. Interior of $D_K$ is empty for each $K \subsetneq \Omega \subseteq \mathbb{R}^n$.

Pf If $\mathcal{U} \subseteq D_K$,

then

$$S = \bigcup_{n \geq 1} n\mathcal{U} \subseteq D_K$$

$\parallel$ $\mathcal{D}(\Omega)$ so $\mathcal{D}(\Omega) \subseteq D_K$, contradiction
since $\Omega - K$ is nonempty. $\square$

Thus $\mathcal{D}(\Omega)$ is first category in itself,
being a finite union of $D_K$'s. It is complete

$\boxed{\text{Baire: } \mathcal{D}(\Omega) \text{ cannot be metrizable!}}$