# Lecture Notes 15



## Working w/ Distributions

Again, $\Omega \subseteq \mathbb{R}^n$ is open.

First of all, let's see how nice functions are also distributions, then we will see how "distributions" transform like "functions" once we set up the measure of integration appropriately.

__Recall__ a linear f'nl $\Lambda$ on $\mathcal{D}(\Omega)$ is a distribution if for each cpt $K \subseteq \Omega$ $\exists$ $N \in \mathbb{Z}_{\geq 0}$ & constant $C < \infty$ s.t.

$$|\Lambda \varphi| \leq C \cdot \|\varphi\|_N$$

for all $\varphi \in \mathcal{D}_K$ (last thm).

$$L^1_{loc}(\Omega) = \left\{ f : \Omega \to \mathbb{C}, \text{ Lebesgue msrble } \& \int_K |f| < \infty \text{ for all cpt } K \subseteq \Omega \right\}$$

__Prop__ each elt $f$ in $L^1_{loc}$ defines a distribution $\Lambda_f \in \mathcal{D}'(\Omega)$ by

$$\Lambda_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{D}(\Omega)$$

Proof First of all, since supp $\varphi$ is cpt, this integral converges.
Now we check this above cond'n.
Let $K \subseteq \Omega$ be cpt.
If $\varphi \in \mathcal{D}_K$, then

$$|\Lambda_f(\varphi)| \le \left[\int_K |f|\right] \cdot \sup_K \varphi$$
$$= \left[\int_K |f|\right] \cdot \|\varphi\|_0$$
$$= C \cdot \|\varphi\|_N$$

w/ $N=0$, $C = \int_K |f|$.

So $\Lambda_f$ is a distribution. $\square$

So locally $L^1$ functions are distributions.

Note the role of the measure.
Really we paired $\varphi$ vs. $f\,dx$
or $\varphi\,dx$ vs. $f$,
which should it be?!

Let $\mu$ be a complex-valued Borel measure on $\Omega$ s.t.
$|\mu|(K) < \infty$ for all cpt $K \subseteq \Omega$

Then $\Lambda_\mu(\varphi) = \int \varphi \, d\mu$ is a distribution

(Same proof!),

Includes previous example.

Our convention is that distributions actually act on $\varphi dx$, not just $\varphi$.

That way, if $\tau \in \mathcal{D}'(\Omega)$, we like to write

$$\int \tau(x)\varphi(x)\,dx = \Lambda\varphi$$

even though this integral doesn't neccesarily make any sense.

But for the distributions given by $L^1_{loc}(\Omega)$, the 2 notations agree.

$\boxed{\text{Distributions are dual to smooth, cptly supported measures}}$

Now, some examples of how distributions act like fns.

Basically, we take an identity valid for $C_c^\infty(\Omega) \subseteq L^1_{loc}(\Omega)$ & extend it.

## Derivation on $\mathcal{D}'(\Omega)$

**Definition** Let $\Lambda \in \mathcal{D}'(\Omega)$, $\alpha$ multindex. Then we define

$D^\alpha \Lambda$ to be the distribution

$$D^\alpha(\Lambda)(\varphi) = \Lambda((-1)^{|\alpha|} D^\alpha \varphi).$$

This holds for $C_c^\infty(\Omega) \subseteq L^1_{loc}(\Omega)$ by integration by parts.

It defines a distribution because if $K \subseteq \Omega$ is cpt &

$$|\Lambda\varphi| \le C \cdot \|\varphi\|_N \quad \forall \varphi \in \mathcal{D}_K,$$

then $|(D^\alpha \Lambda)(\varphi)| \le C \|\varphi\|_{N+|\alpha|}$.

Example of how to do calculus w/ distributions!

**Prop** $D^\alpha D^\beta \Lambda = D^{\alpha+\beta}\Lambda$

**pf** Let $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned}
&(D^\alpha(D^\beta \Lambda))(\varphi) \\
&= (D^\beta \Lambda)((-1)^{|\alpha|} D^\alpha \varphi) \\
&= \Lambda((-1)^{|\alpha|+|\beta|} D^\beta D^\alpha \varphi) \quad \checkmark \text{ mixed partials} \\
&= \Lambda((-1)^{|\alpha+\beta|} D^{\alpha+\beta}\varphi) \\
&= (D^{\alpha+\beta}\Lambda)(\varphi). \quad \Box
\end{aligned}$$

All this makes good sense for $C_c^\infty(\Omega) \subseteq L^1_{loc}(\Omega)$.

But there can be bad examples in $L^1_{loc}(\Omega)$ where a derivative exists, but gives an inconsistent distribution.

$\boxed{Eg}$ Let $\Omega = [0,1] \subseteq \mathbb{R}$
f ~~left~~ continuous fn on $\Omega$ of bdd variation.

Thm (we won't prove)
$\frac{d}{dx} f$ exists a.e. in $L^1_{loc}(\Omega)$.

But $D\Lambda_f \neq \Lambda_{Df}$ is $f dx$ is not absolutely cts.

(see Rudin).

$\mathcal{D}'(\Omega)$ as module for $C^\infty$ functions

Let $\Lambda \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$
We can multiply $\Lambda$ by $f$:

$(\Lambda f)(\varphi) = \Lambda(f\varphi)$,

which agrees w/

$\int (\Lambda f) \cdot \varphi \cdot dx = \int \Lambda \cdot (f\varphi) \cdot dx$.

The reason this is well-defined is that $f\varphi \in \mathcal{D}_K$ if $\varphi \in \mathcal{D}_K$.

Also, for each $N$

$$\|f\varphi\|_N << \|\varphi\|_N$$

since $f$ is smooth (uses Chain rule $D^\alpha fg = \sum_{\beta \le \alpha} c_{\alpha\beta} (D^{\alpha-\beta} f)(D^\beta g)$

So mult by $f$ is a cts operator on $\mathcal{D}(\Omega)$

$\Lambda \mapsto f\Lambda$ is the adjoint action on the duals

Hence $f\Lambda \in \mathcal{D}'(\Omega)$.

(We could see this more directly, since if $|\Lambda\varphi| << \|\varphi\|_N$ for $\varphi \in \mathcal{D}_K$, $|f\Lambda\varphi| << \|\varphi\|_N$ also).

this is the criterion on p.1.

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<u>Proposition</u> (chain rule) If $f \in C^\infty(\Omega)$ & $\Lambda \in \mathcal{D}'(\Omega)$ then $D^\alpha(f\Lambda) = \sum_{\beta \le \alpha} c_{\alpha\beta} (D^{\alpha-\beta} f)(D^\beta \Lambda)$.

(another example of how distributions formally behave like functions).

<u>Pf</u> We check by integration both sides against $\varphi \in \mathcal{D}(K)$ — it suffices to show those integrals agree.

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So

$$\int_\Omega D^\alpha(f\Lambda)\varphi \overset{?}{=} \int_\Omega \sum C_{\alpha\beta}(D^{\alpha-\beta}f)(D^\beta\Lambda)\cdot\varphi$$

Integ by parts

$$(-1)^{|\alpha|}\int_\Omega f\cdot\Lambda\cdot(D^\alpha\varphi) \overset{?}{=} \sum C_{\alpha\beta}(-1)^{|\beta|}\int_\Omega \Lambda\cdot D^\beta(D^{\alpha-\beta}f\cdot\varphi)$$

We claim that

$$\sum_{\beta\leq\alpha} C_{\alpha\beta}(-1)^{|\beta|}D^\beta(D^{\alpha-\beta}f\cdot\varphi) = (-1)^{|\alpha|}f\,D^\alpha\varphi$$

from which the Prop follows.

Now, this is just a calculation about smooth fns one can check. But it has to hold, because when $\Lambda$ is given by localized, smooth $L^1_{loc}(\Omega)$ functions it does!

$\square$

# Topology on Distributions $\mathcal{D}'(\Omega)$

There can be many!

Let's study the weak* topology,
Always locally convex
(open basic balls given by linear fnl inequalities).

weak* converge $\Lambda_i \to \Lambda$ means $\forall \varphi \in \mathcal{D}_K$

$$\Lambda_i \varphi \to \Lambda \varphi.$$

(like in example a while ago about "empirical" vs. "theoretical" probability distributions.

**Thm** Differentiation: $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is sequentially cts under the weak* topology, + (we already knew it is in seminorms)

**Pf**

Let $\Lambda_i \to \Lambda$ (weak*),
$\Lambda_i \in \mathcal{D}'(\Omega)$. Then we claim also $\Lambda \in \mathcal{D}'(\Omega)$
$\mathcal{D}_K$ is Frechet

B-S Thm
If $T_n$ are cts, linear from Frechet to TVS, pointwise limit is cts also

Banach-Steinhaus: $\Lambda|_{\mathcal{D}_K}$ is cts.
so extends to distribution.
(earlier thm).

So $\Lambda \in \mathcal{D}'(\Omega)$.

Now $(D^\alpha \Lambda_i)(\varphi) = (-1)^{|\alpha|} \Lambda_i(D^\alpha \varphi)$

$$\downarrow$$

$(-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$

$= (D^\alpha \Lambda)(\varphi)$. $\square$

Similarly

Thm (left) mult of distributions by smooth f'ns is sequentially cts (again, know it is cts already).

More generally, if $g_n \to g$ in the Frechet top on $C^\infty(\Omega)$ (under seminorms $\sup \left\{ \begin{array}{l} D^\alpha f \\ \text{on } K_N, \\ |\alpha| \le N \end{array} \right\}$)

& $\Lambda_n \to \Lambda$ (weak*)

then $g_n \Lambda_n \to g\Lambda$ weak*.

Pf let $\varphi \in \mathcal{D}_{T_1}$

$$\begin{tikzcd} \int g_m \wedge_n \varphi \arrow[r, "\text{cty of } \wedge_n"'] \arrow[d, "\text{def of weak limit}"'] & \int g \wedge_n \varphi \quad \text{as } m \to \infty \\ \int g_m \wedge \varphi & \end{tikzcd}$$

as $n \to \infty$

We had an earlier thm saying

separate ~~seque~~ sequential cty $\Rightarrow$ joint
if one of the input spaces is Frechet.
($C^\infty(\Omega)$ is Frechet here).

So $\int g_n \wedge_n \varphi \to \int g \wedge \varphi$. $\square$