Lecture Notes 16

The localization Theorems

Let $\Omega \subseteq \mathbb{R}^n$ be open
& $\omega \subseteq \Omega$,
Then we say two distributions
$\Lambda_1, \Lambda_2 \in \mathcal{D}'(\Omega)$ agree on ω
if
$$\Lambda_1(\varphi) = \Lambda_2(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\omega),$$
(i.e., $\operatorname{supp} \varphi \subseteq \omega$),

The distributions coming from integrating against smooth functions agree locally (on ω)

$\Updownarrow$

the smooth f'ns agree on ω.

The distributions coming from integrating against $L^1_{loc}(\Omega)$ f'ns agree locally

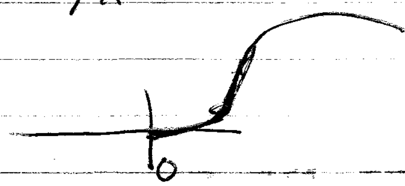
$\Updownarrow$

the L^1_{loc} functions agree locally

We will spend some time discussing local & global relations.

For this we first need to study the important topic of constructing C_c^∞ functions.

Consider $f(x) = e^{-1/x}$, $x > 0$



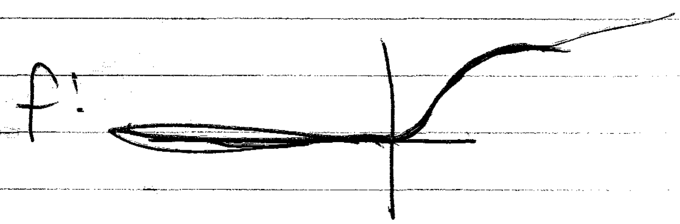
It is easy to see, by the chain rule & induction

$$\left(\frac{d}{dx}\right)^n e^{-1/x} = p_n\left(\frac{1}{x}\right) e^{-1/x}$$

p_n a polynomial.

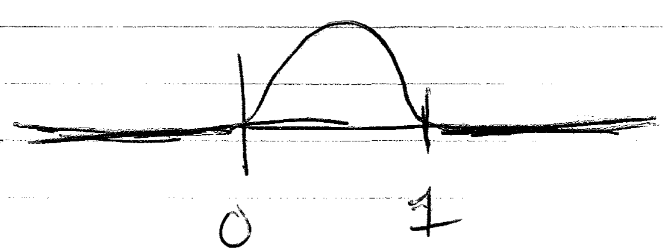
So $f^{(n)}(x) \to 0$ as $x \to 0$, for all $n \geq 0$.

Thus if we extend f to $\mathbb{R}$ by setting $f(x) = 0$ for $x \leq 0$, f is smooth.



Now Let

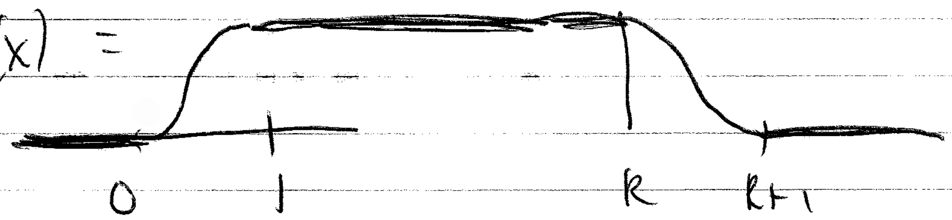
$$g(x) = f(x) \cdot f(1-x) \in C_c^\infty(\mathbb{R})$$



Consider $h(x) := \int_{-\infty}^{x} (g(t) - g(t-R))\,dt$

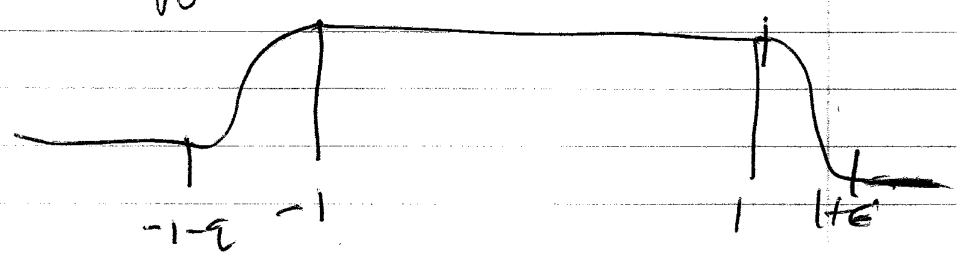
$R >> 1$

Then

$h(x) =$ 

$$h(x) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x \ge R+1 \\ 1 & \text{if } 1 \le x \le R \\ \text{inbetween} & \text{otherwise} \end{cases}$$

Now, by translating, we can have a smooth f'n of cpt support on $\mathbb{R}$

ψ 

$\psi(x) = \psi(-x)$

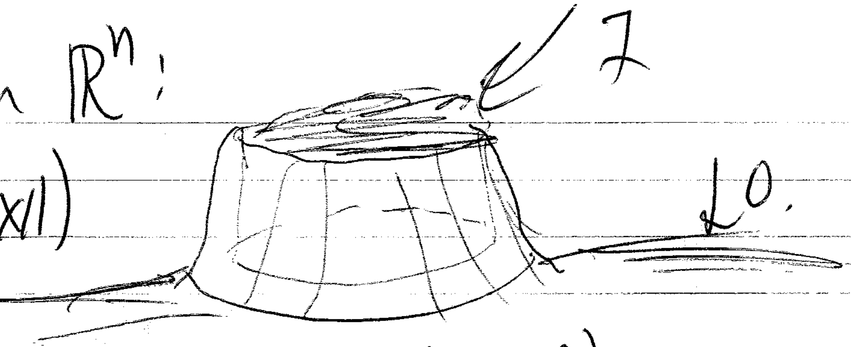
$\psi \equiv 1$ on $[-1, 1]$

$\equiv 0$ on $(-\infty, -1-\varepsilon)$

$\mathbb{R}$ $(1+\varepsilon, \infty)$.

Finally on $\mathbb{R}^n$:

$\psi(\|x\|)$



$\in C_c^\infty(\mathbb{R}^n)$

(Herodian)

This is a good toolkit, which of course you can scale & translate.

To study distributions locally, we use the smooth cutoff fns to kill the distributions behaviour outside of a ball. They need to be smooth because distributions cannot be integrated against harsh cutoff functions

Partition of Unity Theorem

Let $\mathcal{C} = \{U_\alpha\}$ be a collection of open sets in $\mathbb{R}^n$ whose union is denoted Ω.

Then $\exists$ a sequence of functions

$$\psi_1, \psi_2, \ldots \in \mathcal{D}(\Omega),$$

$\psi_j \geq 0$,

such that 1) $\operatorname{supp} \psi_j \subseteq$ some U_α

2) $\sum_{j=1}^{\infty} \psi_j \equiv 1$ on Ω ("partition of unity")

& 3) For each cpt $K \subseteq \Omega$ $\exists m \in \mathbb{N}$
& an open set $W \supseteq K$
s.t. $\psi_1 + \cdots + \psi_m \equiv 1$ on W.

This is called a "locally finite" (3)
partition of unity, "subordinate
to the open cover $\mathcal{C}$ of Ω"

Each $x \in \Omega$ has an open nbd W
which intersects the supports
of only finitely many ψ_j.

Pf $\mathbb{R}^n$ has a countable dense set, $\mathbb{Q}^n$.
Let $S = \mathbb{Q}^n \cap \Omega$.

$\{B_k\}$ - countable collection of all
balls around pts in S w/
rational radii,
& which are subsets of
some $U_\alpha \in \mathcal{C}$.

V_k = same ball as B_k
but w/ half the radius.

$$\bigcup B_k = \bigcup V_k = \Omega.$$

Now, use the function construction 2 pages ago: $\exists\, \varphi_k \in \mathcal{D}(\Omega)$, $0 \le \varphi_k \le 1$

$$\varphi_k \equiv 1 \quad \text{on } U_k$$

$$\operatorname{supp} \varphi_k \subseteq B_k,$$

Define $\psi_1 = \varphi_1$

$$\psi_2 = (1-\varphi_1)\varphi_2$$

$$\psi_3 = (1-\varphi_1)(1-\varphi_2)\varphi_3$$

$$\vdots$$

$$\psi_k = (1-\varphi_1)\cdots(1-\varphi_{k-1})\varphi_k$$

Then each $\psi_i \ge 0$, & vanish off B_i

so 1) holds, as each B_j lies in some U_α. ✓

Now $\psi_1 + \psi_2 = 1 - (1-\varphi_1)(1-\varphi_2)$

$$\psi_1 + \psi_2 + \psi_3 = 1 - (1-\varphi_1)(1-\varphi_2) + (1-\varphi_1)(1-\varphi_2)\varphi_3$$

$$= 1 - (1-\varphi_1)(1-\varphi_2)(1-\varphi_3)$$

etc.

$$\psi_1 + \cdots + \psi_k = 1 - (1-\varphi_1)\cdots(1-\varphi_k)$$ (use induction if you get stuck)

~~Also~~ thus $\psi_1(x) + \cdots + \psi_k(x)$

$= 0$ if all $\varphi_i(x) = 0$

~~1~~ if ~~all~~ some $\varphi_i(x) = 1$.

We see for $x \in \Omega$, $x \in$ some V_ℓ

so

as long as $k \geq \ell$,

so $\sum_{j=1} \psi_j \equiv 1$ on Ω, hence (2) ✓

If K is cpt as in 3), $\exists$ a finite cover $\{V_1, \ldots, V_m\}$

so w/ $W = V_1 \cup \cdots \cup V_m$

$\psi_1 + \cdots + \psi_m \equiv 1$ on W. $\square$

Gluing Distributions together w/ partitions of unity

Thm Let $\mathcal{C} = \{U_\alpha\}$ be an open cover of some open subset $\Omega \subseteq \mathbb{R}^n$.

If for each U_α $\exists$ a distribution $\Lambda_\alpha \in \mathcal{D}'(U_\alpha)$ s.t.

(compatibility cond'n) $\Lambda_\alpha = \Lambda_\beta$ on $U_\alpha \cap U_\beta$ as long as $U_\alpha \cap U_\beta \neq \emptyset$,

then $\exists!$ $\Lambda \in \mathcal{D}'(\Omega)$ s.t.

$$\Lambda = \Lambda_\alpha \text{ on } U_\alpha$$

for all $U_\alpha \in \mathcal{C}$.

Pf Wolg we may assume

$$\bigcup U_\alpha = \Omega.$$

So, we have the same setting as the last thm.

Let ψ_i be the partition of unity coming from there.

Notation: Let $U_{\alpha_i} \in \mathcal{C}$ be an open set

as in part i) there which contains
supp ψ_i.

Now, given $\varphi \in \mathcal{D}(\Omega)$ we break it up as a sum:

$$\varphi = \sum \psi_i \varphi$$

φ has cpt support, & on cpt sets all φ_i except a finite # are nonzero,

So this is a finite sum.

Define

$$\Lambda\varphi = \sum \Lambda_{\alpha_i}(\psi_i \varphi).$$

(This just reassembles the distributions together).

$\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$ is obviously linear.

We have written $\Lambda = \sum \underbrace{\Lambda_{\alpha_i} * \psi_i}$

multiplication of distribution by smooth fn is still a distribution,

Given a sequence $\varphi_j \to 0$ in $\mathcal{D}(\Omega)$,
$\exists$ a cpt set K containing suppφ_j for all j
(see Lecture Notes 14, p. 9).

So $\varphi_j \to 0$ in $\mathcal{D}(\Omega) \Rightarrow \Lambda_{\alpha_i} \psi_i \varphi_j \to 0$ as $j \to \infty$

$\Downarrow$

finite sum $\Lambda \varphi_j \to 0$ as $j \to 0$

$\Downarrow$

Λ is sequentially cts

$\Updownarrow$

$\boxed{\Lambda \text{ is cts}}$ (for distributions).

Now we show $\Lambda|_{U_\alpha} = \Lambda_\alpha$.

Let $\varphi \in \mathcal{D}(U_\alpha)$

$\psi_i \varphi \in \mathcal{D}(U_\alpha \cap U_{\alpha_i})$

By assumption $\Lambda_\alpha(\psi_i \varphi) = \Lambda_{\alpha_i}(\psi_i \varphi)$

Thus

$$\Lambda \varphi = \sum \Lambda_{\alpha_i}(\psi_i \varphi) = \sum \Lambda_\alpha(\psi_i \varphi) \quad \text{sum is finite}$$
$$= \Lambda_\alpha(\sum \psi_i \varphi)$$

$= \Lambda_\alpha(\varphi)$, since $\sum \psi_i \equiv 1$ on $\operatorname{supp}\varphi \subseteq U_\alpha$.

so $\Lambda|_{U_\alpha} \equiv \Lambda_\alpha$.

Finally, uniqueness: the equality $\Lambda|_{U_\alpha} \equiv \Lambda_\alpha$ implies $\Lambda|_{U_{\alpha_i}} \equiv \Lambda_{\alpha_i}$

so since $\varphi = \sum_{\text{finite}} \psi_i \varphi$, $\operatorname{supp}\psi_i\varphi \subseteq U_{\alpha_i}$

$$\Lambda(\varphi) = \sum \Lambda(\psi_i \varphi)$$

$$= \sum \Lambda_{\alpha_i}(\cancel{\varphi}\psi_i \varphi).$$

since $\Lambda(\psi_i\varphi) = \Lambda_{\alpha_i}(\psi_i\varphi)$. $\square$

Support of a distribution

Given $\Lambda \in \mathcal{D}'(\Omega)$,

$\operatorname{supp}\Lambda$ is the smallest closed set s.t. $\Lambda(\varphi) = 0$ for all φ w/ support off of it.

I.e. $\Lambda|_{\Omega - \operatorname{supp}\Lambda} \equiv 0$ & $\operatorname{supp}\Lambda$ is the smallest such closed set.

open

it makes sense to restrict distributions to open sets.

By last thm, we see this is equivalent to saying Λ vanishes on all open subsets disjoint from $\operatorname{supp}\Lambda$.

Thm Let $\Lambda \in \mathcal{D}'(\Omega)$.

obvious
a) If $\varphi \in \mathcal{D}(\Omega)$ & $\operatorname{supp}\varphi \cap \operatorname{supp}\Lambda = \emptyset$, $\Lambda\varphi = 0$

b) $\operatorname{supp}\Lambda = \emptyset \iff \Lambda \equiv 0$

c) $\psi \in C^\infty(\Omega)$, $\psi \equiv 1$ on an open set containing $\operatorname{supp}\Lambda \Rightarrow \psi\Lambda = \Lambda$.

d) ~~S_Λ~~ If $\operatorname{supp}\Lambda \subseteq \Omega$ is cpt then Λ has finite order.
Moreover $\exists\, C < \infty$ & $N \geq 0$ s.t. $|\Lambda\varphi| \leq C\,\|\varphi\|_N$ for all $\varphi \in \mathcal{D}(\Omega)$,
& Λ extends uniquely from $\mathcal{D}'(\Omega)$ to $C^\infty(\Omega)$ dual.

<u>Pf</u> c) Let $\varphi \in \mathcal{D}(\Omega)$. Then $\varphi - \psi\varphi$ is zero on $\operatorname{supp}\Lambda$;
so $\Lambda\varphi = \Lambda\psi\varphi = \psi\Lambda\varphi$ ✓.

d) Let $K = S_\Lambda$. By partitions of unity on a finite subcover of K, $\exists\, \psi \in C_c^\infty(\Omega)$ s.t. $\psi \equiv 1$ on K.

Let $K' = \operatorname{supp}\psi$

Recall earlier we showed using Leibniz differentiation formula that for any N

$$\|\psi\varphi\|_N \le C(\psi)\cdot\|\varphi\|_N$$

for any $\varphi \in \mathcal{D}(\Omega)$. ($C(\psi)$ depends only on ψ).

Λ cts $\Rightarrow \exists\, C>0, N\ge 0$ s.t.

$$|\Lambda(\varphi)| \le C\|\varphi\|_N \text{ for all } \varphi \in \mathcal{D}_K$$

Thus for all $\varphi \in \mathcal{D}(\Omega)$

$$\psi\varphi \in \mathcal{D}_{K'}$$

$$|\psi\Lambda(\varphi)| = |\Lambda(\psi\varphi)| \le C\|\psi\varphi\|_N$$
$$\le C\cdot C(\psi)\cdot\|\varphi\|_N$$

as claimed.

Finally, the extension:

Define $\mu: C^\infty(\Omega) \to \mathbb{C}$ by

$$\mu(f) = \Lambda(\psi f) \quad , \ f \in C^\infty(\Omega).$$

extends obviously, since $\psi \equiv 1$ on supp Λ.

cty: $f_i \to 0$ in $C^\infty(\Omega)$ means all derivatives $\to 0$ on compacta.
Leibniz: $\psi f_i \to 0$ in $\mathcal{D}_{K'}$ uniformly...
Λ cts on $\mathcal{D}_{K'} \Rightarrow \Lambda \psi f_i \to 0$. (above estimate) ✓

Uniqueness: Let K_0 be a cpt subset of Ω - $f \in C^\infty(\Omega)$.
By partitions of unity $\exists \varphi \in \mathcal{D}(\Omega)$ s.t. $\varphi \equiv f$ on K_0.
Thus $\mathcal{D}(\Omega)$ is dense in $C^\infty(\Omega)$. (uniform convergence on cpta!)
So Λ has at most one cts extn

Part (f), sufficiency p. 154

Thm (Distributions supported at a point).

$\boxed{\Lambda \in \mathcal{D}'(\Omega) \ \& \text{ supp } \Lambda = \{p\} \ \& \text{ order } N \text{ (finite by part d)}}$

$\boxed{\Lambda = \text{lin comb of } D^\alpha \delta_p \text{ over } |\alpha| \le N.}$

Pf

$\Rightarrow$ obvious.

$\Leftarrow$ We may assume, to make calcs easier, $p=0$. p.15

Let $\varphi \in \mathcal{D}(\Omega)$ s.t. $(D^\alpha \varphi)(0) = 0$ for all $|\alpha| \le N$.

Assume for now this implies $\Lambda\varphi = 0$.

If so,

Λ vanishes on intersection of null spaces of $D^\alpha \delta_0$, $|\alpha| \le N$

Earlier lemma: a linear fnl is in the span of lin fnls $\lambda_1, \dots, \lambda_k$ $\iff$ it vanishes on ker λ_j for each j

(Uses alg algebra)

So then Λ is a lin comb of the $|\alpha| \le N$.

Now ~~sh~~ we show $\Lambda\varphi = 0$.

Let $\varepsilon > 0$.

Then by cty of the derivatives $\exists$ cpt ball K around 0 s.t.

$$|(D^\alpha \varphi)(x)| < \varepsilon \quad \forall x \in K, \ |\alpha| \le N.$$

Claim

Claim

$(*)$ $|D^\alpha \varphi(x)| <$ $< \varepsilon \cdot (|x| \cdot n)^{N-|\alpha|}$ p(16)

for all $x \in K$, $|\alpha| \le N$.

True for $|\alpha| = N$. we prove by downwards induction. If true for $|\alpha| = i$ & $|\beta| = i-1$, we look at

Gradient

$$|\nabla(D^\beta \varphi)| = |(D_{x_1} D^\beta \varphi, D_{x_2} D^\beta \varphi, \dots, D_{x_n} D^\beta \varphi)|$$

each entry $= D^\alpha \varphi$, for some α w/ $|\alpha| = i$.

so each entry $< \varepsilon \cdot (|x| n)^{N-|\alpha|}$ $= \varepsilon (|x| n)^{N-i}$.

$$|\nabla D^\beta \varphi| \le n^{N-i+1} \cdot \varepsilon |x|^{N-i}$$
$$= n^{N-|\beta|} \varepsilon |x|^{N-i}$$

$|\nabla|$ gradient is fastest increase so going from 0 to x
$D^\beta \varphi(0) = 0$ $D^\beta \varphi(x) = ?$,

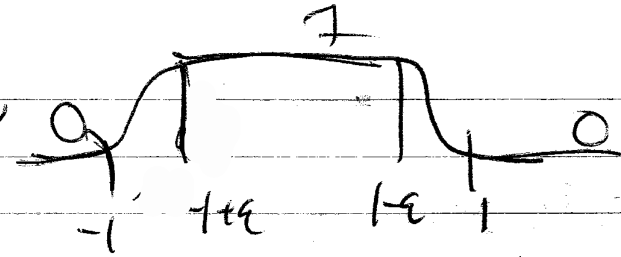
we bound

$D^\beta \varphi(x)$ by $|x| \cdot \max\limits_{|y| \le x} |\nabla D^\beta \varphi(y)|$ gradient

$< \varepsilon n^{N-|\beta|} |x|^{N-|\beta|}$ ✓

So $(*)$ holds for all $|\alpha| \le N$.

Take ψ



& let $\psi_r(x) = \psi\left(\frac{x}{r}\right)$ scaling,

For r small enough, $\operatorname{supp}\psi_r \subseteq K$,

Liebniz $\quad D^\alpha(\psi_r\varphi)(x) = \sum_{\beta\le\alpha} C_{\alpha\beta}\,(D^{\alpha-\beta}\psi)\left(\frac{x}{r}\right) r^{|\beta|-|\alpha|} * (D^\beta\varphi)(x)$

$(*)$: $\Rightarrow \left|D^\alpha(\psi_r\varphi)(x)\right| \le c\,\varepsilon \sum_{\beta\le\alpha} r^{|\beta|-|\alpha|}\,|x|^{N-|\beta|}\,r^{N-|\beta|}$

n is fixed, power of r is ≤ 0

$|x|$ may assumed to be <1 so

$|D^\alpha(\psi_r\varphi)(x)| < c\,\varepsilon$ ← constant indep of ε.

$\Rightarrow \|\psi_r\varphi\|_N < c\,\varepsilon.$

Λ has order $N \Rightarrow \exists\, C_1$ s.t. $\quad |\Lambda f| \le C_1\|f\|_N$

$\forall f \in \mathcal{D}_K$

$\psi_r\Lambda = \Lambda$, since $\operatorname{supp}\Lambda = \{0\}$,

$|\Lambda\varphi| = |\Lambda(\psi_r\varphi)| \le C_1\|\psi_r\varphi\|_N \le C_1\varepsilon.$

so $\Lambda\varphi = 0$. $\square$.

Now we come to the important topic of when a distribution can be expressed as $D^\alpha f$, f cts.

Thm (One of the nicest of the course!)

Let $\Lambda \in D'(\Omega)$ & K a cpt subset of Ω. Then $\exists f \in C(\Omega)$ s.t.

$$\Lambda \equiv D^\alpha f \quad \text{on } K$$

for some α.

(Distributions are derivatives on cpt sets!)

Note multiplying by cutoff ψ s.t. $\psi\Lambda = \Lambda$ we can ensure $f \in C_c$

Pf We may assume, to simplify things, that $K \subseteq [0,1]^n = Q$ (wolg).

Mean Value Thm: for any $\psi \in D_Q$:

$$|\psi| \leq \max_{\substack{1 \leq i \leq n \\ x \in Q}} |(D_{x_i}\psi)(x)|$$

$$|D^\alpha \psi| << \|\nabla D^\alpha \psi\| \text{ sup over } Q.$$

close to what was used in last proof

Let $T = D_{x_1} D_{x_2} \cdots D_{x_n}$.

(FTC) Then $\psi(y) = \int_{Q(y)} (T\psi)(x)\,dx \qquad \psi \in D_Q$

where $Q(y)$ is the subcube w/ $\{0 \leq x_i \leq y_i\}$.

This shows T is injective $T: D_Q \to D_Q$.

So $\|\Psi\|_0 \le \int_{Q(y)} |T\Psi|(x)\,dx$

Also $D^\alpha\Psi \in \mathcal{D}_Q$, so

$$\|D^\alpha\Psi\|_0 \le \int_{Q(y)} |TD^\alpha\Psi|(x)\,dx.$$

Λ cts, K cpt $\Rightarrow \exists\, C$ & N s.t.

$$|\Lambda\varphi| \le C\|\varphi\|_N.$$

So $|\Lambda\varphi| \le C\cdot\max_{|\alpha|\le N}\int_{Q(y)} |TD^\alpha\varphi|(x)\,dx$

$TD^\alpha : \mathcal{D}_Q \to \mathcal{D}_Q$ is 1-1 since we can integrate back.

$TD^\alpha : \mathcal{D}_K \to \mathcal{D}_K$ also 1-1.

So $\exists$ linear f'nl $\Lambda_1 : \text{Range}(TD^\alpha) \to \mathbb{C}$ defined by

$$\Lambda_1 TD^\alpha\varphi = \Lambda\varphi.$$

(algebra).

Let $\Psi \in \text{Range}(TD^\alpha)$

$\Psi = TD^\alpha\varphi$

Then $|\Lambda_1\Psi| = |\Lambda\varphi| \le C\cdot\max_{|\alpha|\le N}\int_{Q(y)} |TD^\alpha\varphi|(x)\,dx$

$$\le C \quad \|\Psi\|_{L^1(K)},$$

Thus Λ is a bdd linear f'nl on a subspace of $L^1(K)$

Hahn-Banach - extends to $\in L^1(K)^*$.

$\Updownarrow$

$\exists$ a bdd Borel f'n $g : K \to \mathbb{C}$ s.t.

$$\Lambda\varphi = \Lambda_1 T D^\alpha \varphi = \int_K g(x)\,(T D^\alpha \varphi)(x)\,dx.$$

extend g to $\mathbb{R}^n$ by letting it be zero off K.

Let $f = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} g(x)\,dx_n \cdots dx_1$

$Tf = g$

So f is cts.

$$\Lambda\varphi = \int Tf(x)\,(T D^\alpha \varphi)(x)\,dx$$

integ by parts $= \int g \cdot$ deriv of cts f'n $\square$

Thm Given any $\Lambda \in \mathcal{D}'(\Omega)$ $\exists$ cts f'ns $g_\alpha : \Omega \to \mathbb{C}$ s.t.

each K cpt $\subseteq \Omega$ intersects only finitely many g_α

& $\Lambda = \sum_\alpha D^\alpha g_\alpha$

(So on cpts Λ is a finite sum)

If finite order, then need only finite # of g_α globally

Example $\sum_{n \in \mathbb{Z}} \delta_n^{(|n|)}$ is a distribution, not of finite order.

Pf

Write Ω as a countable union of disjoint cubes $Q_i \subseteq \Omega$.
$Q_i \subseteq$ open set $V_i \subseteq \Omega$
can be done (by padding slightly) so that no cpt subset of K intersects ∞ many V_i.

Make partition of unity ψ_i
$\operatorname{supp} \psi_i \subseteq V_i$
$\psi_i \equiv 1$ on Q_i.
$\sum \psi_i = 1$.

on each $\psi_i \Lambda = \sum_{\text{finite sum}} D^\alpha f_{i,\alpha}$
for some $f_{i,\alpha}$ w/ $\operatorname{supp} \subseteq V_i$.

Let $g_\alpha = \sum_{i=1}^{\infty} f_{i,\alpha}$, finite on cpt ~~sets~~ by ~~fact that~~ cpt sets intersect only finitely many V_i.

$\Rightarrow g_\alpha$ cts & each cpt K meets only finitely many $\operatorname{supp}(g_\alpha)$.

Let $\varphi \in \mathcal{D}(\Omega)$

$$\varphi = \sum \psi_i \varphi$$

$$\Lambda\varphi = \sum \Lambda(\psi_j \varphi) = \sum (\psi_j \Lambda)\varphi$$

so $\Lambda = \sum \psi_j \Lambda$ globally

$$= \sum D^\alpha g_\alpha . \quad \square$$