Lecture Notes 3

Metric TVS

topology comes from a metric.

Main Thm Metrizable $\Leftrightarrow$ has a countable local base.
w/ balanced open balls
@ origin
& invariant

Additionally, if V is locally convex, we can arrange that all open balls are convex.

The hard part, of course, is how to construct the metric.

Some comments first:

proven at the end of the second class

Recall we showed before that any nhd of 0 contains a balanced nhd of 0. Likewise, all convex nhds contain a balanced convex nhd of 0.

If U is an open nhd of U, then $\exists\, U'$, balanced, $\subseteq U$, s.t. $U' + U' \subseteq U$, U' open

If locally convex, we may assume U' is convex, at the expense of making U' smaller.

Now assume $\{S_n \mid n \geq 1\}$ is a local base. We showed before that there exists a balanced local base, & in a locally convex TVS, a convex, balanced local base.

Let $U_1 = S_1$.

Let $U_2 = S_2 \cap U_1'$ — where U' is the set given on the previous page.
$U_3 = S_3 \cap U_2'$
$\vdots$
$U_{n+1} = S_{n+1} \cap U_n'$
$\vdots$

Then each U_n is open, balanced (...and convex if locally convex...) & since $U_n \subseteq S_n$ & $\{S_n\}$ is a local base, $\{U_n\}$ is also a local base.

Furthermore $U_{n+1} + U_{n+1} \subseteq U_n' + U_n' \subseteq U_n$.

This proves claim (1) on p. 18 of Rudin. I felt more needed to be said.

Now we describe the construction of the invariant metric. Recall that $d(v_1, v_2) = d(0, v_1 - v_2)$, So first we need to describe the function f s.t. $d(V_1, V_2) = f(V_1 - V_2)$.

Consider rational numbers of the form

$$(\#) \quad r = \sum_{n=1}^{N} c_n 2^{-n} \quad \text{, where } N \text{ is arbitrary.}$$

Such r are between $0 \le r < 1$

Let $A(r) = V$ if $r \ge 1$

& $A(r) = c_1 V_1 + c_2 V_2 + \cdots + c_N V_N$ if r has the above form.

Define $f(x) = \inf\{r \mid x \in A(r)\}$.
of that form.

I.e., if $x \notin \bigcup_{N \ge 1} (V_1 + \cdots + V_N)$, $f(x) = 1$.

This means the metric $d \le 1$, but remember, we only really care about <u>nearby</u> points for defining the topology.

(We should note the infimum exists: It does for any subset of $\mathbb{R}_{\ge 0}$.)

Note also for $\mathbb{R}^n$, this describes the metric, at least for points less than distance 1 apart, & if $V_n = 2^{-n}$(unit ball).

<u>Proof of Thm</u> We need to show 3 properties to prove d is a metric:

Property 1) $d(v,v) = f(0) = 0$.

Obvious, since $0 \in$ each V_n.

Property 2) $d(v,w) = f(v-w)$ If $v \neq 0$, $v \notin A(2^{-n}) = V_n$ for some n (Hausdorff...), so $f(v) = 2^{-n} \neq 0$.

$\quad\quad\quad\quad\quad\; d(w,v) = f(w-v)$,

i.e. $f(v) = f(-v)$, obvious because each V_n is balanced.

Property 3) Δ-inequality $f(v+w) \leq f(v) + f(w)$
[of course, the hard one...]

We claim $A(r) + A(s) \subseteq A(r+s)$ for r,s of the form $(*)$.

This implies for $r < t$
$$A(r) \subseteq A(r) + A(t-r) \subseteq A(t).$$

If $f(x)+f(y) \geq 1$, the Δ-inequality is obvious. Otherwise, let $\varepsilon > 0$. Then $\exists\, r,s$ of the form $(*)$ s.t.

$$\begin{aligned} f(x) &< r \quad x \in A(r) \\ f(y) &< s \quad y \in A(s) \end{aligned} \implies x+y \in A(r+s)$$
$$r+s < f(x)+f(y)+\varepsilon,$$

so $f(x+y) \leq r+s < f(x)+f(y)+\varepsilon$. Take $\varepsilon \to 0$, get
$$f(x+y) \leq f(x)+f(y).$$

Now we prove the claim $A(r)+A(s) \subseteq A(r+s)$,
Remember, we still need to prove the open balls are balanced, a local base, (& convex ...).

We prove $A(r)+A(s) \subseteq A(r+s)$ by induction on $N \geq 1$
When $N=1$, $r, s = 0$ or $\frac{1}{2}$

$A(0) = \{0\}$

$A(\frac{1}{2}) = U_1$

$(r,s) = (0,0)$: $\{0\}+\{0\}=\{0\}$
$(\frac{1}{2},0)$: $\{0\}+U_1 = U_1$
or $(0,\frac{1}{2})$: $U_1+\{0\} = U_1$ ✓

Assume for $N-1$. Let $r+s<1$,

$$r = \sum_{j=1}^{N-1} c_j(r) 2^{-n} + c_N(r)2^{-N} = r' + c_N(r)2^{-N}$$

$$s = \sum_{j=1}^{N-1} c_j(s) 2^{-n} + c_N(s)2^{-N} = s' + c_N(s)2^{-N}$$

$$A(r) = A(r') + c_N(r)U_N$$

$$A(s) = A(s') + c_N(s)U_N$$

$$A(r)+A(s) \subseteq A(r')+A(s') + c_N(r)U_N + c_N(s)U_N$$

$$\subseteq A(r'+s') + c_N(r)U_N + c_N(s)U_N$$

If $c_N(r) = c_N(s) = 0$, $r = r'$, $s = s'$, done.

If $c_N(r) = 0$, $c_N(s) = 1$ $\quad r' = r$

$$A(r) + A(s) \subseteq A(r + s') + U_N$$
$$= A(r+s) \quad \text{(no carrying of the last bit in the addition)}.$$

likewise $c_N(r) = 1$, $c_N(s) = 0$.

Finally, if $c_N(r) = c_N(s) = 1$

$$A(r) + A(s) \subseteq A(r' + s') + \underbrace{U_N + U_N}_{\subseteq\, U_{N-1}}$$
$$= A(r' + s') + A(2^{1-N})$$
$$\overset{\text{induction}}{\subseteq} A(r' + s' + 2^{1-N})$$
$$= A(r+s).$$

Finally, why a local base? open balls

$$B_\delta(0) = \bigcup_{r < \delta} A(r)$$

so $\quad B_\delta(0) \subseteq U_N \quad$ for $\delta < 2^{-N}$

Convexity is a little more subtle:

$\{U_n\}$ all convex $\implies$ $A(r)$ are all convex

~~as~~ $tA(r)+(1-t)A(r)$

$$= \sum c_n(r)\left(tU_n+(1-t)U_n\right)$$

$$\subseteq \sum c_n(r)U_n = A(r),$$

Why is $\bigcup_{r<\delta} A(r)$ convex?

We showed $A(r) \subseteq A(t)$ for $r \le t$

so this union is an increasing union of convex sets, hence convex (... the line between any 2 points lies in this after a finite amount of time),

$\square$

New topic: <u>Bounded</u> Linear Operators.

Means: maps bounded sets into bounded sets,

(Not the same as "bounded function," since obviously we know most linear maps are not bounded.)

Thm If $\Lambda : V \to W$ is a linear mapping of TVS's, then

$$\underset{(a)}{\Lambda \text{ cts}} \Longrightarrow \underset{(b)}{\Lambda \text{ bdd}} \Longrightarrow \overset{(c)}{\text{the set } \{\Lambda(v_n)\} \text{ is bdd in } W \text{ for any sequence } v_n \to 0 \text{ in } V.}$$

Moreover, if V is metrizable, then these 3 properties are equivalent, in particular also to the property $\underset{(d)}{v_n \to 0 \Rightarrow \Lambda(v_n) \to 0}$.

Proof Assume (a), & let E be a bounded subset of V.
We must show $\Lambda(E) \subseteq W$ is bdd.
Let U be a nbd of 0 in W.
Λ cts $\Rightarrow \exists$ open set S, containing 0 in V, s.t.
$$\Lambda(S) \subseteq U.$$
E bdd $\Rightarrow$ for large t $E \subseteq tS$
Therefore $\Lambda(E) \subseteq \Lambda(tS) = t\Lambda(S) \subseteq tU$ for large t
$\Lambda(E)$ is bounded, (a) $\Rightarrow$ (b) ✓

Assume (b). (c) follows if we can show

convergent sequences are bdd.

Indeed, Let $\mathcal{U}$ be an open nbd of 0.
$\mathcal{U}$ contains a balanced open nbd $\mathcal{U}'$ of 0.
For large n, $v_n \in \mathcal{U}'$ (def'n of convergence to 0)

As $\bigcup_{k \geq 1} k\mathcal{U}' = V$,
the finitely-many points $v_1, \cdots, v_N$ not in $\mathcal{U}'$ are contained in $k\mathcal{U}'$ for some $k \geq 1$. $\mathcal{U}'$ is balanced, so $\mathcal{U}' \subseteq k\mathcal{U}'$ & $\{v_n\} \subseteq k\mathcal{U}'$

For all $t \geq k$,
$\{v_n\} \subseteq k\mathcal{U}' \underset{\substack{\uparrow \\ \text{balanced}}}{\subseteq} t\mathcal{U}' \subseteq t\mathcal{U}$, so by def'n $\{v_n\}$ is bdd.
So (b) $\Rightarrow$ (c).

To prove the rest, now assume V is metrizable.
We need to show (c) $\Rightarrow$ (d) & (d) $\Rightarrow$ (a).

First we prove (d) $\Rightarrow$ (a). Assume (a) is false,
i.e. $\exists$ an open set $\mathcal{U} \subseteq W$ s.t. $\Lambda^{-1}(\mathcal{U})$ is not open

V is metrizable, so has a countable local base, none of whose members are subsets of $\lambda^{-1}(U)$.
Let $\{S_1, S_2, \cdots\}$ be this local base & let $v_n \in S_n$ be an element not contained in $\lambda^{-1}(U)$.
Then $v_n \to 0$ (by def'n) but $\lambda(v_n)$ never gets inside U, so (d) is false.

Finally, we come to proving $(c) \Rightarrow (d)$

$v_n \to 0$ implies for each $k \geq 1$ $\exists N = N_k$ such that $d(v_n, 0) \leq k^{-2}$ for all $n \geq N_k$.
Let $c_n = k$ if $N_k \leq n < N_{k+1}$
$= 1$ if $n < N_1$
Then $d(c_n v_n, 0) < \frac{1}{k}$
and $c_n v_n \to 0$.

So $\{c_n v_n\}$ is bdd in V
& by assumption of (c), $\{\lambda(c_n v_n)\} = \{c_n \lambda(v_n)\}$ is bdd in W.

Let U be any nhd of 0 in W, & U' a balanced sub-nhd $U' \subseteq U$ containing 0.
Then for large t
$\{c_n \lambda(v_n)\} \subseteq tU'$

For n large, $c_n \geq t$

$c_n \lambda(v_n) \in tU'$

$\Downarrow$

$\lambda(v_n) \in \frac{t}{c_n} U' \subseteq U'$ since U' is balanced,

$\subseteq U$

so $\lambda(v_n) \in U$ for n large.

U arbitrary, so That means $\lambda(v_n) \to 0$. $\square$