Lecture Notes 7

Various Results about Continuous Linear Maps.

Theorem Let V be a Frechet space (= locally convex w/ complete invariant metric), W be a TVS, and $\Lambda_n : V \to W$ be linear, continuous maps, $n \geq 1$. Suppose that

$$\Lambda x := \lim_{n\to\infty} \Lambda_n x$$

exists for each $x \in V$. Then Λ is also a cts linear map from $V \to W$.

Proof Λ is obviously linear, as follows from basic properties of limits.

Recall Baire's Theorem: a complete metric space is of second category in itself.

Banach-Steinhaus: if $B = \{$all v such that $\{\Lambda_n v\}$ is bounded$\}$, & B second category, then $\{\Lambda_n\}$ is equicontinuous.

By assumption, $B = V$, which is indeed second category.

So $\{\Lambda_n\}$ is equicontinuous, meaning that for each open nbd U of 0 in W, there exists an open neighborhood S of 0 in V such that

$$\Lambda_n(S) \subseteq \mathcal{U},$$

which implies

$$\Lambda(S) \subseteq \overline{\mathcal{U}}.$$

We saw early on that in any local base, each open nhd contains the closure of some other member in the base. Thus if T is an arbitrary open subset $\subseteq W$ containing 0, $T \supseteq \overline{\mathcal{U}}$ for some open nhd $\mathcal{U}$ of 0, & $\Lambda(S) \subseteq T$, proving continuity @ 0 (which implies global continuity). $\square$

~~Rotated Example Suppose V & W are complete normed TVS's (Banach Spaces).~~

~~We have seen that boundedness & continuity are very related.~~

~~In a Banach space these are both measured by the norm.~~

~~Bounded Linear Operators on a Banach Space have a norm:~~

$$\|\Lambda\| = \sup_{v \neq 0} \frac{\|\Lambda v\|}{\|v\|}$$

~~which is finite for bounded operators.~~

Open Mapping Theorem

Assume $\Lambda : V \to W$ is a continuous linear map from a Frechet space V to a TVS W. Assume also that $\Lambda(V)$ is of second category in W. Then, if Λ is one-to-one, Λ is a homeomorphism from V to W. In ~~particular~~ general, it ~~has~~ is an open mapping (sends open sets to open sets).

Proof. We will show it is an open mapping, because it follows the image is a subspace. But the only open subspace is W, because ~~the~~ an open nhd of 0 is absorbing.

Let $\mathcal{U}$ be an open nhd of 0. By translation, it suffices to show $\Lambda(\mathcal{U})$ is open, ~~i.e. conta~~ in particular contains an open nhd of 0.

Let d be the invariant metric of the Frechet space V. Since $\mathcal{U}$ is open, $\mathcal{U} \supseteq B_0(r)$ for some $r > 0$, & of course also $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$

$$\mathcal{U}_k = B_0(2^{-k} r).$$

Now, $\mathcal{U}_n \supseteq V_{n+1} - V_{n+1}$, & thus

$$\overline{\Lambda(\mathcal{U}_n)} \supseteq \overline{\Lambda(V_{n+1}) - \Lambda(V_{n+1})}$$

Thus $\overline{\Lambda(U_n)} \supseteq \overline{\Lambda(U_{n+1})} - \overline{\Lambda(U_{n+1})}$

(Check! $\overline{A} + \overline{B} \subseteq \overline{A+B}$
since if ~~U~~ U is any open nbd of $a+b \in \overline{A}+\overline{B}$, $U \supseteq U_1 + U_2$
open nbds of a, b,
s.t. $U_1 \cap A$ & $U_2 \cap B$ are nonempty)

Now $\Lambda(W) = \bigcup_{k \geq 1} k\Lambda(U_2)$

assumed to be 2nd category

Baire says some $k\Lambda(U_2)$ must be of 2nd category ~~since~~

$\Updownarrow$

$\Lambda(U_2)$ is 2nd category

$\Updownarrow$

$\overline{\Lambda(U_2^{n+1})}$ contains an open subset S.

~~Since $\overline{\Lambda(U_2)}$ is furthermore balanced~~

$\Updownarrow$

$\exists\, w_0 \in W$ & $\varepsilon > 0$ such that all w in W with $d(w, w_0) < \varepsilon$ are limits of points in $\Lambda(U_2^{n+1})$.

So $\overline{\Lambda(U_n)} \supseteq$ ~~$B(w_0, \varepsilon)$~~ $B_{w_0}(\varepsilon) - B_{w_0}(\varepsilon) \supseteq B_0(\varepsilon)$.

Now $B_0(\varepsilon) \subseteq \overline{\Lambda(U_n)}$, where ε depends on n.

Let $y_n \in \overline{\Lambda(U_n)}$ be arbitrary

Since $\overline{\Lambda(U_{n+1})}$ contains an open nbhd of zero, by continuity one has

$$y_n - \Lambda(U_{n+1}) \cap \Lambda(U_n) \neq \emptyset.$$

$$\Downarrow$$

$$\exists\, x_n \in U_n \text{ s.t. } \Lambda(x_n) \in y_n - \overline{\Lambda(U_{n+1})}$$

This works for any y_n.

Let y_1 be arbitrary &

Let $y_{n+1} = y_n - \Lambda(x_n)$ & repeat.....

$$\Rightarrow d(x_n, 0) < 2^{-n} r$$

so $x_1 + \cdots + x_n$ is a Cauchy sequence which converges to $x \in V$.

$$d(x, 0) < r.$$

$$\Lambda(x_1 + \cdots + x_n) = y_1 - y_{n+1}$$

$$\to y_1 \text{ as } n \to \infty$$

$$\Lambda(x) = y_1 \quad (\text{cty of } \Lambda)$$

so $y_1 \in \Lambda(U)$.

Thus $\underset{\underset{B_0(\varepsilon)}{\cup\!\!/}}{\overline{\Lambda(U_1)}} \subseteq \Lambda(U)$. So map is open $\square$

Other consequences of the Open Mapping Theorem (also known as open mapping theorem by some):

<u>Theorem</u> If $\Lambda: V \to W$ is a cts linear mapping of Frechet spaces & is onto, then it is open (& hence a homeomorphism), if it is one-to-one.

<u>Proof</u> Now in the (regular) O.M.T., $\Lambda(V)$ is of Baire 2nd category in itself (since W is Frechet).

<u>Theorem</u> If V, W are furthermore Banach Spaces, & map is one-to-one, cts & onto, then $\exists\, c_1, c_2 > 0$ s.t.

$$c_1 \|v\| \le \|\Lambda v\| \le c_2 \|v\|,$$

<u>Pf</u> If Λ is cts, then $\Lambda^{-1}([0,1))$ is open $\Rightarrow \exists\, \varepsilon > 0$ s.t. $\|\Lambda v\| < 1$ for all $\|v\| < \varepsilon$

$\Downarrow$

upper bound w/ $c_2 = \varepsilon^{-1}$.

Lower bound follows from same cty statement for Λ^{-1}.

~~Theo~~
<u>Example</u> A Frechet space cannot have a strictly stronger Frechet topology than it already has (since the identity map is 1-1, onto, cts. . .), hence a homeomorphism.

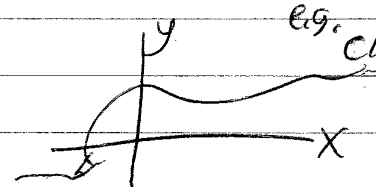
Closed Graph Theorem

a map $f: X \to Y$ of topological spaces has a "graph"

$$\Gamma_f = \{(x, f(x)) \mid x \in X\}$$

in $X \times Y$ under the product topology.

e.g. closed on $\mathbb{R}^2$



<u>Lemma</u> If X, Y top spaces, Y Hausdorff, $f: X \to Y$ cts, then the graph Γ_f is closed.

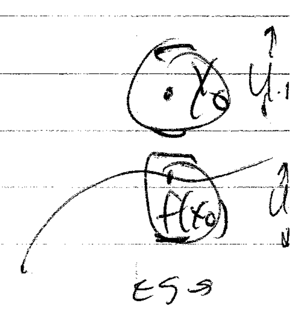
<u>Pf</u> Its complement Ω is open, since given $(x_0, y_0) \in \Omega$

$$\Downarrow$$

$$y_0 \neq f(x_0)$$

we find open nbhds U_1, U_2 in Y s.t.

$y_0 \in U_1$, $f(x_0) \in U_2$, $U_1 \cap U_2 = \emptyset$



By cty, $\exists$ open nhd $S \subseteq X$ of x_0 w/

$$f(S) \subseteq U_2.$$

Now $S \times U_1$ lies in Ω

because given $x \in S$

$y \in U_1$

$f(x) \subseteq U_2$

so $\neq y$ for any $y \in U_1$. $\square$

Example $f(x) = x$ maps $X \to X$.
Closed Graph $\Gamma_f \iff$ Hausdorff.

Closed Graph Theorem

If V, W ~~Frechet Spaces~~ have complete invariant metric (es Frec
$\Lambda : V \to W$ is linear w/ closed graph Γ_Λ

$\implies \Lambda$ is cts.

Remark Closed Graph $\iff$ if $x_n \to x$
$\Lambda x_n \to y$
then $y = \Lambda x$

in metric space.

~~Pf of remark: $\Leftarrow$ let $x, y \in \Gamma_f$, $x_n \to x$~~

Proof First, using complete invariant metrics d_X & d_Y on X & Y,

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

Is a complete inv. metric on $X \times Y$, a TVS

Λ linear $\Rightarrow$ Γ_Λ a ~~close~~ closed subspace

$\Rightarrow$ complete (lies ins ide complete TVS)

$\Rightarrow$ $d_{X \times Y}|_{\Gamma_\Lambda}$ is a complete invariant metric on $X \times Y$.

<u>Use Projections</u> Π_X, Π_Y onto factors ~~are cts & 1-1~~

$\Pi_X : \Gamma_\Lambda \to X$

is 1-1
onto
cts.
$\Rightarrow$ open map
Π_X^{-1} cts.

so

$\Lambda = \Pi_Y \circ \Pi_X^{-1}$ is cts. $\square$

<u>Bilinear Maps</u>

$$B : V_1 \times V_2 \longrightarrow W \quad \cancel{\text{TVS}}$$

linear in each variable

Eg. Quadratic forms,

~~If~~ cts in product topology, cts separately in each variable

Thm If $B: V_1 \times V_2 \to W$ is bilinear & V_2, W are TVS, V_1 has a complete invariant metric, then if B is separately cts in each variable,

$$\begin{matrix} x_n \to x \text{ in } V_1 \\ y_n \to y \text{ in } V_2 \end{matrix} \implies B(x_n, y_n) \to B(x,y) \text{ in } W.$$

~~Furthermore,~~

Pf Let $U \subseteq W$ be a nhd of 0

$S \in U$ — 0 s.t. $S + S \subseteq U$.

Then for each x $\{B(x, y_n) \mid n \geq 1\}$ is a bdd subset of W (by continuity)

So the ~~map~~ family of maps $b_n(x) = B(x, y_n)$, $n \geq 1$ is equicts by Banach-Steinhaus

$\Downarrow$

$\exists$ open nhd T in V_1 of 0 s.t. $b_n(T) \subseteq S$.

Now
$$\begin{aligned} &B(x_n, y_n) - B(x,y) \\ &= B(x_n - x, y_n) + B(x, y_n - y) \\ &= b_n(x_n - x) + B(x, y_n - y) \end{aligned}$$

as $n \to \infty$ $b_n(x_n - x) \in S$

$B(x, y_n - y) \to 0$ by cty so $\in S$

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Thus $B(x_n, y_n) - B(x, y) \in S$ for n large

& any open nbhd S of 0.

So $B(x_n, y_n) \to B(x, y)$. $\square$