An Easier Way to Show $\zeta(3) \notin \mathbf{Q}$

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In 1978 R. Apery proved that

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is an irrational number. His proof was shortened by F. Beukers, who translated it into equivalent statements about integrals of Legendre polynomials. Nevertheless, there is some "magic" in Beukers' proof involving a complicated change of variables. In this note we will use a different integral than Beukers which gives the same approximation as Beukers' (and Apery's), but which is much easier to maniuplate.

1 Some Formulas

For the sake of reference, let us state some key integral formulas

$$\int_0^1 \int_0^1 \frac{s^a t^a}{1-st} ds dt = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2} \tag{1}$$

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$$\int_0^1 \int_0^1 \frac{s^a t^a \ln(st)}{1-st} ds dt = -2 \sum_{n=1}^{\infty} \frac{1}{(n+a)^3} \tag{2}$$

$$\int_0^1 \int_0^1 \frac{s^a t^a \ln(t)}{1-st} ds dt = -\sum_{n=1}^{\infty} \frac{1}{(n+a)^3}. \tag{3}$$

If $a \neq b$,

$$\int_0^1 \int_0^1 \frac{s^a t^b}{1-st} ds dt = \frac{1}{b-a} \sum_{n=1}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+b} \right) \tag{4}$$

$$\int_0^1 \int_0^1 \frac{s^a t^b \ln(s)}{1-st} ds dt = \frac{1}{a-b} \sum_{n=1}^{\infty} \frac{1}{(n+a)^2} + \frac{1}{(a-b)^2} \sum_{n=1}^{\infty} \left(\frac{1}{n+a} - \frac{1}{n+b} \right) \tag{5}$$

$$\int_0^1 \int_0^1 \frac{s^a t^b \ln(t)}{1-st} ds dt = \frac{1}{b-a} \sum_{n=1}^{\infty} \frac{1}{(n+b)^2} + \frac{1}{(a-b)^2} \sum_{n=1}^{\infty} \left(\frac{1}{n+b} - \frac{1}{n+a} \right) \tag{6}$$

$$\int_0^1 \int_0^1 \frac{s^a t^b \ln(st)}{1-st} ds dt = \frac{1}{a-b} \sum_{n=1}^{\infty} \left(\frac{1}{(n+a)^2} - \frac{1}{(n+b)^2} \right). \tag{7}$$

Formulas (1) and (4) are proven directly by expanding

$$\frac{1}{1-st} = \sum_{n=0}^{\infty} s^n t^n$$

and integrating. The remaining formulas follow by differentiating (1) and (4) with respect to a and b.

If $p(s,t) \in \mathbf{Z}[s,t]$ is a polynomial of degree n with integral coefficients, then it follows from (2) and (7) that

$$\int_0^1 \int_0^1 \frac{p(s,t) \log(st)}{1-st} ds dt = \frac{a_n + b_n \zeta(3)}{d_n^3}, \tag{8}$$

where $a_n, b_n, d_n \in \mathbf{Z}$ and d_n is the least common multiple of the integers $1, 2, \ldots, n$. It is not hard to see (using the Prime Number Theorem) that for any fixed $\epsilon > 0$, $d_n \leq e^{(1+\epsilon)n}$ for large n.

Legendre Polynomials

Consider $P_n(s) = \frac{1}{n!}\frac{d^n}{ds^n}(s - s^2)^n$, a polynomial with integral coefficients. We will use the integrals

$$\int_0^1 \int_0^1 \frac{P_n(s)P_n(t)\log(st)}{1-st} dsdt = \int_0^1 \int_0^1 \int_0^1 \frac{P_n(s)P_n(t)}{1-(1-st)u} dsdtdu. \tag{9}$$

Since

$$\int_0^1 \frac{1}{1-(1-x)t} dx = -\frac{\log(x)}{1-x}. \tag{10}$$

and $P_n(1-s) = (-1)^n P_n(s)$, this equals

$$(-1)^n \int_0^1 \int_0^1 \int_0^1 \frac{P_n(s)P_n(t)}{1-(1-(1-s)t)u} dsdtdu.$$

Lemma 1.1 *For* $s, t \in (0,1)$ *fixed,*

$$\int_0^1 \frac{1}{1-(1-(1-s)t)u} du = \int_0^1 \frac{1}{(1-(1-u)s)(1-(1-t)u)} du. \tag{11}$$

Proof: Partial fractions give that

$$\frac{1}{(1-(1-u)s)(1-(1-t)u)} = \frac{1}{1-(1-s)t}\left(\frac{s}{1-(1-u)s} - \frac{1-t}{1-(1-t)u}\right), \tag{12}$$

so

$$\int_0^1 \frac{1}{(1-(1-u)s)(1-(1-t)u)} du \tag{13}$$

$$= \frac{1}{1-(1-s)t}\left(-s\frac{\log(1-s)}{s} + (1-t)\frac{\log(t)}{t-1}\right) = -\frac{\log(t(1-s))}{1-(1-s)t}. \tag{14}$$

Using (10) with $x = 1-(1-s)t$, we see that the two integrals are equal.

$\square$

Beukers used the integral (9) and, by using integration by parts and changing variables, deduced from it that $\zeta(3)$ is irrational. We will do some easy integrations-by-parts, but avoid his change of variables.

The lemma implies that

$$\begin{aligned}
&(-1)^n \int_0^1\int_0^1\int_0^1 \frac{P_n(s)P_n(t)}{1-(1-(1-s)t)u} dsdtdu \\
&= (-1)^n \int_0^1\int_0^1\int_0^1 \frac{P_n(s)P_n(t)}{(1-(1-u)s)(1-(1-t)u)} dsdtdu \qquad (15)\\
&= \int_0^1\int_0^1\int_0^1 \frac{P_n(s)P_n(t)}{(1-(1-u)s)(1-tu)} dsdtdu.
\end{aligned}$$

We already know that this is of the form $\frac{a_n+b_n\zeta(3)}{d_n^3}$. We will show it is very small, in fact $o\left(\frac{1}{d_n^3}\right)$, which shows that $\zeta(3) \notin \mathbf{Q}$.

By integrating by parts in the variables s and t, each n times, we get that

$$\int_0^1\int_0^1\int_0^1 \frac{P_n(s)P_n(t)}{(1-(1-u)s)(1-tu)} dsdtdu \qquad (16)$$

$$= \int_0^1\int_0^1\int_0^1 \frac{(s-s^2)^n(t-t^2)^n(u-u^2)^n}{((1-(1-u)s)(1-tu))^{n+1}} dsdtdu. \qquad (17)$$

The function

$$f(s,t,u) = \frac{s(1-s)t(1-t)u(1-u)}{(1-(1-u)s)(1-tu)} \qquad (18)$$

vanishes on the boundary of $[0,1]\times[0,1]\times[0,1]$ and has its maximum at

$$(s,t,u) = (2-\sqrt{2}, \sqrt{2}-1, \frac{1}{2}),$$

where

$$f(2-\sqrt{2}, \sqrt{2}-1, \frac{1}{2}) = 17-12\sqrt{2} \approx .029437. \qquad (19)$$

Thus

$$\frac{a_n+b_n\zeta(3)}{d_n^3} = O((.029437)^n). \qquad (20)$$

However, $d_n^3 = O(e^{3.01n})$, and since $e^{3.01}(17-12\sqrt{2}) \approx .597205$, we conclude that

$$\left|\frac{a_n+b_n\zeta(3)}{d_n^3}\right| = o\left(\frac{1}{d_n^3}\right), \qquad (21)$$

proving indeed that $\zeta(3) \neq \mathbf{Q}$.