On the Existence and Temperedness of Cusp Forms for $SL_3(\mathbb{Z})$

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Abstract

We develop a partial trace formula which circumvents some technical difficulties in computing the Selberg trace formula for the quotient $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$. As applications, we establish the Weyl asymptotic law for the discrete Laplace spectrum and prove that almost all of its cusp forms are tempered at infinity. The technique shows there are non-lifted cusp forms on $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$ as well as non-self-dual ones. A self-contained description of our proof for $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is included to convey the main new ideas. Heavy use is made of truncation and the Maass-Selberg relations.

1 Introduction

In the 1950s A. Selberg ([Sel1]) developed his trace formula to prove the existence of non-holomorphic, everywhere-unramified, cuspidal "Maass" forms. These are real-valued functions on the upper-half plane $\mathbb{H} = \{x + iy \mid y > 0\}$ which are invariant under the action of $SL_2(\mathbb{Z})$ by fractional linear transformations. Unlike the holomorphic cusp forms, which can all be explicitly

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described, no Maass form for $SL_2(\mathbb{Z})$ has ever been constructed and they are believed to be intrinsically transcendental.

The non-constant Laplace eigenfunctions in $L^2(SL_2(\mathbb{Z})\backslash\mathbb{H})$ are all Maass forms. Since $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is noncompact, their existence is no triviality; they very likely do not exist on the generic finite-volume quotient of $\mathbb{H}$ (see [Sarnak]). However, Selberg showed that for any congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, the discrete spectrum is as large as one can expect – namely, it obeys the same asymptotics as the spectrum of a compact surface of the same size:

Theorem 1.1. *(Selberg) Let* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ *be the discrete eigenvalues (with multiplicity) of the non-Euclidean laplacian* Δ *on* $\Gamma\backslash\mathbb{H}$*. If* $\Gamma \subset SL_2(\mathbb{Z})$ *is a congruence subgroup, then*

$$N(T) = \#\{\lambda_j \leq T\} \sim \frac{area(\Gamma\backslash\mathbb{H})}{4\pi}T \tag{1.1}$$

as $T \to \infty$*.*

Similar asymptotics hold for any compact manifold by theorems of H. Weyl and others; we will refer to such an asymptotic for $N(T)$ as the *Weyl law* for the space in question. For noncompact manifolds one can count the discrete and continuous spectra together, but it is extremely difficult to decouple the two. The main contribution of this paper is a technical novelty for separating them and proving the Weyl law for certain noncompact quotients.

One can generalize Maass forms to groups other than $SL_2(\mathbb{R})$ in a variety of ways. We are most interested in the linear group and so will focus our attention there, to $G = SL_3(\mathbb{R})$ in particular. Let $K = SO_3(\mathbb{R})$ be a maximal compact subgroup of G, $\mathcal{H} = G/K, \Gamma = SL_3(\mathbb{Z})$, and $X = \Gamma\backslash\mathcal{H}$. The ring $\mathcal{D}$ of (G-)invariant differential operators on the symmetric space $\mathcal{H}$ is a polynomial ring in two generators; it is explicitly described in [Bump], p. 32 for example. We will concentrate first on a particular element of $\mathcal{D}$, the laplacian Δ, which is normalized so that its continuous spectrum on $L^2(\mathcal{H})$ is the interval $[1,\infty)$. Here again, the non-constant discrete eigenfunctions of Δ in $L^2(X)$ are cusp forms (see Section 3 for definitions), which are the appropriate generalizations of Maass forms and our basic objects of study.

By symmetry considerations, it is not difficult to prove the existence of *odd* cusp forms on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and $SL_3(\mathbb{Z})\backslash\mathcal{H}$. These comprise only half of the expected spectrum; the deeper issue is the existence of *even* cusp forms. The only other known cusp forms on $SL_3(\mathbb{Z})\backslash\mathcal{H}$ are Gelbart-Jacquet lifts ([GelJac]) of the forms on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ that Selberg discovered (many of which have been numerically identified–see [Hejhal]). However, no version of the trace formula has been used for the higher rank $SL_3(\mathbb{Z})\backslash\mathcal{H}$ to prove the Weyl law for cusp forms, *ala* Selberg's Theorem 1.1.[1] For example, Arthur's trace formula ([Art2]) computes the traces of certain integral operators over the discrete spectrum, but in it the discrete spectrum is paired with parabolic orbital integrals. Though it appears difficult, it would be very interesting to separate the two and give estimates on the size of the spectrum – especially because Arthur's formula has been developed for general quotients.

We will shortcut the trace formula to prove the Weyl law for $SL_3(\mathbb{Z})\backslash\mathcal{H}$ as well as some qualitative results about the spectrum.

Theorem 1.2. *Let* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ *denote the eigenvalues of* Δ *on* $SL_3(\mathbb{Z})\backslash\mathcal{H}$ *with multiplicity. Then*

$$N(T) := \#\{\lambda_j \leq T\} \sim \frac{vol(SL_3(\mathbb{Z})\backslash\mathcal{H})}{\Gamma(7/2)} \left(\frac{T}{4\pi}\right)^{5/2} \tag{1.2}$$

as $T \to \infty$.

These are the same asymptotics that the spectra of closed five-dimensional manifolds obey. This result confirms a conjecture of Sarnak ([Sarnak]), who asserted that the Weyl law holds for the cuspidal spectrum of the laplacian on any congruence quotient of $SL_n(\mathbb{R})/SO_n(\mathbb{R})$. Theorem 1.2 is actually a special case of the more-general Theorem 5.3, a kind of equidistribution theorem which counts the two-dimensional joint spectrum of $\mathcal{D}$ lying in various sets. This generalization is used to establish many of the other findings below, so first we shall briefly describe the joint spectrum before stating the applications.

[1] In [StaWal] a proof of Theorem 1.2 is claimed. This proof appears to be incorrect because it is based on taking a specifically-chosen sequence of functions in the trace formula of [Wallace]. Unfortunately that trace formula is incomplete, because in its estimation of the continuous spectrum, it ignores the poles of the intertwining operators which arise from contour shifts. This seems to be a complicated problem to fix, though doing so would lead to much more specific information about the discrete spectrum (e.g. perhaps an error term).

The ring of invariant differential operators $\mathcal{D}$ on $\mathcal{H}$ is commutative and we may take an orthonormal set of common eigenfunctions

$$\frac{1}{\sqrt{\text{area}(SL_3(\mathbb{Z})\backslash\mathcal{H})}} = \phi_0, \phi_1, \phi_2, \ldots \tag{1.3}$$

such that

$$\Delta\phi_j = \lambda_j \phi_j.$$

Each ϕ_j in turn induces a homomorphism

$$\lambda : \mathcal{D} \to \mathbb{C}, \quad \text{where } D\phi = \lambda(D)\phi, \ D \in \mathcal{D}.$$

This map is best described by "principal series" or "Langlands" parameters

$$\{\ell_1, \ell_2, \ell_3\}, \quad \ell_1 + \ell_2 + \ell_3 = 0$$

(see Section 4 for more details). It is possible to compute $\lambda(D)$ in terms of these spectral parameters, e.g.

$$\lambda(\Delta) = 1 - \frac{\ell_1^2 + \ell_2^2 + \ell_3^2}{2}. \tag{1.4}$$

The real parts of the ℓ_i are bounded, for example by a trivial bound from representation theory:

$$|\text{Re } \ell_i| < \frac{1}{2} \quad ([\text{JacSha}]). \tag{1.5}$$

Hence the Weyl law essentially counts the number of spectral parameters (ℓ_1, ℓ_2, ℓ_3) lying in a large ball. In Theorem 5.3 we compute the asymptotics of how many points of the joint spectrum lie in certain sets of various shapes and sizes. Along with some representation theory, we can count how many cusp forms obey given properties. The statement of Theorem 5.3 is technical so we will only describe its corollaries here.

A cusp form is called *tempered* if each Re $\ell_i = 0$. The archimedean Ramanujan-Selberg conjecture asserts that all cusp forms on $SL_3(\mathbb{Z})\backslash\mathcal{H}$ are tempered. The only nontrivial improvement of (1.5) on individual forms is the result of [LRS].

Theorem 1.3. *Almost all of the cusp forms on $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$ obey the archimedean Ramanujan-Selberg conjectures in the sense that*

$$\lim_{T\to\infty} \frac{\#\{\lambda_j \le T \mid \Delta\phi_j = \lambda_j\phi_j, \phi_j \text{ tempered}\}}{\#\{\lambda_j \le T\}} = 1.$$

For $SL_2(\mathbb{Z})\backslash\mathbb{H}$ the nontempered cusp forms are those with Laplace eigenvalue less than 1/4, and the Weyl law shows there are only a finite number of these. In higher rank, though, there may be nontempered forms with large Laplace eigenvalues. The archimedean Ramanujan-Selberg conjecture is an analog of Selberg's famous "1/4 conjecture," as it automatically implies that every cuspidal Laplace eigenvalue satisfies

$$\lambda^{cusp}(SL_3(\mathbb{Z})\backslash\mathcal{H}) \ge \lambda_1(\mathcal{H}) = 1.$$

In fact, this has already been proved in [Miller], but without establishing temperedness itself. The technique here is different and further-reaching in that it works whenever the Weyl law can be proved by our method.

Corollary 1.4. *If*

$$N^{cusp}(T) := \#\{\lambda_j \le T \mid \Delta\phi_j = \lambda_j\phi_j \text{ for a nonzero cusp form } \phi_j\},$$

then

$$N^{cusp}(T) \sim N(T) \sim \frac{vol(SL_3(\mathbb{Z})\backslash\mathcal{H})}{\Gamma(7/2)} \left(\frac{T}{4\pi}\right)^{5/2}$$

as $T \to \infty$.

Proof: The only other eigenfunctions are residues of Eisenstein series, which are not tempered, and Theorem 1.3 shows these have measure zero.

Another and more-precise reason uses the classification of the discrete spectrum established by Mœglin and Waldspurger ([MœWal], confirming the conjecture of [Jacquet]). A consequence is that the only non-cuspidal discrete eigenfunctions of Δ on $SL_p(\mathbb{Z})\backslash SL_p(\mathbb{R})/SO_p(\mathbb{R})$ are constants when p is prime. Thus for $T \ge 0$, $N(T) = N^{cusp}(T) + 1$, as alluded to earlier. $\square$

Corollary 1.5. *There are "native" cusp forms on $SL_3(\mathbb{Z})\backslash\mathcal{H}$ which are not Gelbart-Jacquet lifts of Maass forms from $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Moreover, these native forms comprise 100% of the spectrum in the sense that*

$$\lim_{T\to\infty} \frac{\#\{\lambda_j \le T \mid \Delta\phi_j = \lambda_j\phi_j, \phi_j \text{ a Gelbart-Jacquet lift}\}}{\#\{\lambda_j \le T\}} = 0.$$

A cusp form is called *self-dual* if $\phi(g) = \phi((g^t)^{-1})$ for all $g \in G$. Alternatively, its joint spectral parameters are

$$\{\ell_1, \ell_2, \ell_3\} = \{\mu, 0, -\mu\} \text{ for some } \mu \in \mathbb{C}.$$

Theorem 1.6. *There exists a non-self-dual cusp form on $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$, and in fact these are also of full measure:*

$$\lim_{T\to\infty} \frac{\#\{\lambda_j \le T \mid \Delta\phi_j = \lambda_j\phi_j, \phi_j \text{ self-dual}\}}{\#\{\lambda_j \le T\}} = 0.$$

All Gelbart-Jacquet lifts are self-dual and it is conjectured that the converse is true.

In the next section we present a self-contained account of the main ideas of this paper, but specialized towards giving a new proof of Selberg's Theorem 1.1 for $\Gamma = SL_2(\mathbb{Z})$. The main improvement is a circumvention of the usual derivation of the trace formula. Usually one integrates the automorphic kernel only over a truncated fundamental domain, in order to prevent divergence from the parabolic orbital integrals and the Eisenstein series. After matching the growth rates of these two terms, a trace formula is obtained in the limit as the truncation parameter moves to infinity through the cusp. Here we merely truncate, and by positivity considerations arrive at an inequality giving a *lower* bound on the size of the spectrum. This gives the Weyl lower bound, but with the wrong constant – it involves the area of the truncated fundamental domain instead. By pushing the truncation parameter to infinity only at the end, we recover the correct lower bound; the upper bound has already been established by [Donnelly].

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2 The Weyl Law for $SL_2(\mathbb{Z})\backslash\mathbb{H}$

This is a self-contained section illustrating our method on $SL_2(\mathbb{Z})\backslash\mathbb{H}$. We will reprove the Weyl Law (Theorem 1.1) using the same "partial trace" technique we use in higher rank for Theorem 1.2. We stress that the reason the partial trace is successful has little to do with the complications of higher rank *per se*, but rather because it serves as a substitute for the trace formula in counting the spectrum. As of yet, it appears difficult to give a direct proof of the Weyl law in higher rank using a trace formula, though this is how and why Selberg proved it for $SL_2(\mathbb{Z})\backslash\mathbb{H}$ to begin with. Our technique is also applicable to congruence covers of $SL_2(\mathbb{Z})\backslash\mathbb{H}$, but we will focus on the full-level situation here in order to demonstrate how the main difficulty – non-compactness – is addressed. Basic references for this section are [Sel1], [Sel2], [Hejhal], and [Terras].

2.1 Definitions

The hyperbolic plane can be modeled by the complex upper-half plane

$$\mathbb{H} = \{u + iv \mid v > 0\}$$

with area element

$$dA = \frac{du\,dv}{v^2},$$

line element

$$d\sigma^2 = \frac{du^2 + dv^2}{v^2},$$

and laplacian

$$\Delta = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

The group $G = SL_2(\mathbb{R})$ acts by isometries on $\mathbb{H}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

The stabilizer of any point in $\mathbb{H}$ is a maximal compact subgroup isomorphic to $K = SO_2(\mathbb{R}) = \mathrm{Stab}(i)$, and so $\mathbb{H}$ is isomorphic to the symmetric space G/K.

Let $\Gamma = SL_2(\mathbb{Z}), \bar{\Gamma} = PSL_2(\mathbb{Z})$ and $X = \Gamma\backslash\mathbb{H} = \bar{\Gamma}\backslash\mathbb{H}$. The space X is noncompact but has finite volume, $\frac{\pi}{3}$. Thus the laplacian Δ has a continuous spectrum on $L^2(X)$ furnished by Eisenstein series

$$E_s(z) = \frac{1}{2} \sum_{\left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}|n\in\mathbb{Z}\right\}\backslash\Gamma} Im(\gamma z)^s.$$

(This infinite series is only defined for $Re(s) > 1$, where it converges absolutely, but it has a meromorphic continuation to $s \in \mathbb{C}$.) The continuous spectrum comes from

$$E_{\frac{1}{2}+it}(z) \quad , \quad t \in \mathbb{R}.$$

Also, since Δ annihilates constant functions, it has a discrete spectrum. Our aim here is to count it. Let

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the discrete spectrum of Δ on $L^2(X)$ and

$$\sqrt{\frac{3}{\pi}} = \phi_0, \phi_1, \phi_2, \ldots$$

be an orthonormal set of eigenfunctions

$$\phi_j \in L^2(X), \quad \Delta\phi_j = \lambda_j\phi_j.$$

We will prove

Theorem 2.1. *(=Theorem 1.1)(Selberg) As $T \to \infty$*

$$N(T) = \#\{\lambda_j \le T\} \sim \frac{T}{12}.$$

2.2 Convolution Operators

Every $g \in C_c^\infty(K\backslash\mathbb{H}) = C_c^\infty(K\backslash G/K)$ acts on $f \in L^2(G/K)$ by convolution:

$$(L_g f)(x) = (f * g)(x) = \int_{G/K} f(y)g(y^{-1}x)dy. \tag{2.1}$$

Bi-K-invariant functions are functions of the distance of a point in $\mathbb{H}$ to i. Their convolution operators form a commutative algebra which also commutes with the Laplace operator. These operators share common eigenfunctions as well, so Laplace eigenfunctions on $\mathbb{H} = G/K$ play a special role. The simplest examples are furnished by power functions

$$\Delta v^s = s(1-s)v^s \tag{2.2}$$

and by the bi-K-invariant "spherical" functions

$$\tilde{\phi}_s(z) = \int_K \text{Im } (kz)^s dk, \qquad \int_K dk = 1,$$

$$\Delta\tilde{\phi}_s(z) = s(1-s)\tilde{\phi}_s. \tag{2.3}$$

The integral operators L_g also act on $f \in L^2(\Gamma\backslash G/K)$:

$$\begin{aligned} (L_g f)(x) = & \int_{G/K} f(y)g(y^{-1}x)dy \\ = & \sum_{\gamma\in\bar{\Gamma}} \int_{\Gamma\backslash G/K} f(\gamma^{-1}y)g(y^{-1}\gamma x)dy \\ = & \int_{\Gamma\backslash G/K} f(y)K(x,y)dy, \end{aligned} \tag{2.4}$$

where

$$K(x,y) = \sum_{\gamma\in\bar{\Gamma}} g(x^{-1}\gamma y) \tag{2.5}$$

is called the *automorphic kernel*. Again, the L_g commute with each other and with Δ, so we may assume that the ϕ_j are taken to be an orthonormal set of eigenfunctions of all $L_g, g \in C_c^\infty(K\backslash G/K)$ as well:

$$(L_g\phi)(x) = \int_{G/K} \phi(y)g(y^{-1}x)dy = \hat{g}(\phi)\cdot\phi(x). \tag{2.6}$$

Selberg showed that

$$\int_K \phi_j(kg)dk = \phi_j(I)\tilde{\phi}_{s_j}(g), \qquad \Delta\phi_j = s_j(1-s_j)\phi_j$$

and hence by averaging (2.6) over K he found a formula for $\hat{g}(\phi)$:

Proposition 2.2. *(Selberg's Uniqueness Principle) The L_g-eigenvalue $\hat{g}(\phi_j)$ of any ϕ_j as above depends only on its Laplace eigenvalue. Namely, if*

$$\Delta\phi_j = (\frac{1}{4} - \nu_j^2)\phi_j, \tag{2.7}$$

then

$$L_g\phi_j = \hat{g}(\nu_j)\phi_j,$$

where the Selberg transform

$$\hat{g}(\nu) = \int_{\mathbb{H}} g(u+iv)v^{1/2+\nu}\frac{dudv}{v^2}. \tag{2.8}$$

Remark 2.3. Formula (2.8) is in fact (2.6) applied to the eigenfunction $v^{1/2+\nu}$, and evaluated at the identity.

The kernel $K(x,y)$ thus has two expansions:

Proposition 2.4. *(Spectral Expansion) For $g \in C_c^\infty(K\backslash G/K)$,*

$$K(x,y) = \sum_{\gamma\in\bar{\Gamma}} g(y^{-1}\gamma x) =$$

$$\sum_{\substack{j\geq 0 \\ \textit{disc. } L^2 \textit{ spec.}}} \hat{g}(\nu_j)\phi_j(x)\phi_j(y) + \frac{1}{4\pi}\int_{\mathbb{R}} \hat{g}(it)E_{1/2+it}(x)\overline{E_{1/2+it}(y)}dt. \tag{2.9}$$

We will now recall some analytic properties of the Selberg transform that will be needed in constructing our choices of g later. First is the inversion formula

$$g(x) = \int_{\mathbb{R}} \hat{g}(it)\tilde{\phi}_{1/2-it}(x)\frac{t\tanh\pi t}{4\pi}dt. \tag{2.10}$$

Secondly, we need to clarify the relationship between the Selberg transform and the Mellin transform $\mathcal{M}$. If

$$\bar{g}(v) = \int_{\mathbb{R}} g(u + iv)du \tag{2.11}$$

denotes the "Harish transform," then (2.8) expresses

$$\hat{g}(\nu) = \mathcal{M}\left(\frac{\bar{g}(x)}{\sqrt{x}}\right)(\nu). \tag{2.12}$$

Through the Mellin inversion formula, formula (2.10) essentially amounts to a statement about the Harish transform. A more-precise fact was proven by Ehrenpreis and Mautner (see also [Gangolli]).

Theorem 2.5. *([EhrMau]). The Harish transform $g \mapsto \bar{g}$ is a bijection between*

$$C_\sigma^\infty(K\backslash\mathbb{H}) = \{g \in C_c^\infty(K\backslash\mathbb{H}) \mid \mathit{supp}(g) \subset K \cdot i(\sigma^{-1}, \sigma)\}$$

and

$$C_\sigma^\infty(0,\infty) = \{h \in C_c^\infty(0,\infty) \mid h(v) = h(v^{-1}), \mathit{supp}(h) \subset (\sigma^{-1}, \sigma)\}.$$

2.3 Construction of Functions

Recall that there exist smooth, non-negative functions on $\mathbb{R}$ with arbitrarily-small support whose Fourier transforms are also non-negative. (The easiest way to make one is to convolve a non-negative, compactly supported function with itself and rescale.) Using (2.12) and Theorem 2.5, we may fix some $g \in C_\sigma^\infty(K\backslash\mathbb{H})$ such that $g \geq 0$, $\int_{\mathbb{R}} \hat{g}(it)dt = 1$, and $\hat{g}(it) \geq 0$ for $t \in \mathbb{R}$. We would like to scale $\hat{g}$ so that it resembles the characteristic function of a large interval. However, in this non-euclidean setting it is difficult to control g and $\hat{g}$ simultaneously. Instead, we will convolve $\hat{g}(it)$ with $\chi_{[-T,T]}$:

$$\hat{g}_T(it) = \int_{-T}^{T} \hat{g}(it + ir)dr \ \geq 0, \tag{2.13}$$

which is the Mellin transform of

$$\frac{\bar{g}(x)}{\sqrt{x}} \int_{-T}^{T} x^{ir} dr \ \in C_\sigma^\infty(0,\infty).$$

Hence Theorem 2.5 also guarantees the existence of some function $g_T \in C_\sigma^\infty(K\backslash\mathbb{H})$ whose Selberg transform

$$\widehat{g_T} = \hat{g}_T.$$

We will use the g_T's in the automorphic kernel. Our construction allows us to conclude two important analytic properties:

Proposition 2.6.

$$\max_{x\in\mathbb{H}} |g_T(x)| = g_T(i).$$

Proof: This follows from the inversion formula (2.10), the positivity of $\hat{g}_T$ (2.13), and the inequality

$$|\tilde{\phi}_{1/2+it}(x)| \leq 1 = \tilde{\phi}_{1/2+it}(i)$$

(see [DKV], (2.13)). □

Proposition 2.7. *For any $m \geq 0$,*

$$\hat{g}_T(it) = \begin{cases} 1 + O_m\left((|T| - |t| + 1)^{-m}\right) & t < T, \\ O_m\left((|t| - |T| + 1)^{-m}\right) & t \geq T. \end{cases}$$

Proof: Since g is smooth, $\hat{g}(it) = O\left((1+|t|)^{-m}\right)$ for any $m \geq 0$. Hence

$$\int_T^\infty \hat{g}(it)dt = O\left((1+T)^{-m}\right), \quad T > 0$$

and the proposition follows because of our normalization $\int_{\mathbb{R}} \hat{g}(it)dt = 1$. □

2.4 Partial Trace

Let $C \geq 1$ be a fixed truncation parameter. We will adjust it only in the final step of our analysis. Denote the usual fundamental domain for X as

$$\mathcal{F} = \{z \in \mathbb{H} \mid |z + \bar{z}| \leq 1 \leq |z|\}$$

and its truncation as

$$\mathcal{F}_C = \{z \in \mathcal{F} \mid Im(z) \le C\}.$$

Of course, $\mathcal{F}_C$ is compact and has area $\frac{\pi}{3} - \frac{1}{C}$; as $C \to \infty$, it exhausts $\mathcal{F}$. If X were compact, we would take the trace of the integral operator L_g by integrating the two expressions for $K(x,x)$ in (2.9) over $\mathcal{F}$. However, these integrals diverge and we instead take a partial trace over $\mathcal{F}_C$:

$$\int_{\mathcal{F}} K(x,x)dx = \sum_{\gamma \in \bar{\Gamma}} \int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx$$

$$= \sum_{j \ge 0} \hat{g}(\nu_j) \int_{\mathcal{F}_C} |\phi_j(x)|^2 dx + \frac{1}{4\pi} \int_{\mathbb{R}} \hat{g}(it) \int_{\mathcal{F}_C} |E_{1/2+it}(x)|^2 dx dt. \tag{2.14}$$

Up until now we have essentially followed Selberg's derivation of the trace formula. His next step is the understand the divergence of each side of (2.14) as a function of C. He is able to cancel it from each side and derive a trace formula for $\sum \hat{g}(\nu_j)$ by letting $C \to \infty$. As a final step, he uses a family of functions such as the g_T to deduce the spectral asymptotics.

This cancellation can be found in great generality (e.g. [Art2]), but the resulting formula is quite complicated. We will now deviate from this path by keeping *C fixed* for now, and performing an analysis of (2.14) with the g_T. By noting

$$\int_{\mathcal{F}_C} |\phi_j(x)|^2 dx \le \int_{\mathcal{F}} |\phi_j(x)|^2 dx = 1, \tag{2.15}$$

(2.14) becomes an inequality, giving a lower bound on $\sum \hat{g}(\nu_j)$ (which, as we will make precise later, roughly equals $N(T^2)$).

Since only a finite number of Γ-translates of $\mathcal{F}_C$ neighbor it, we may take σ to be small enough so that

$$g(x^{-1}\gamma x) \ne 0, \quad x \in \mathcal{F}_C$$

only for the finitely-many $\gamma \in \bar{\Gamma}$ which have a fixed-point on the boundary of $\mathcal{F}_C$.

Lemma 2.8. *For σ small,*

$$\left|\sum_{\substack{\gamma\in\bar{\Gamma}\\ \gamma\neq I}}\int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx\right| = O\left(\sigma^2 \max g\right).$$

Proof: By the above remark, it suffices to show

$$\int_{\mathbb{H}} |g(x^{-1}\gamma x)|dx = O\left(\sigma^2 \max g\right)$$

for γ a rotation. Since rotations sweep distant points a proportional amount, only x of distance $O(\sigma)$ from the fixed point can have $g(x^{-1}\gamma x) \neq 0$. □

Applying (2.14), (2.15), and Proposition 2.6 we have that

$$\int_{\mathcal{F}_C} g_T(x^{-1}Ix) \leq \begin{array}{c} \sum_{j\geq 0}\hat{g}_T(\nu_j) \\ +\frac{1}{4\pi}\int_{\mathbb{R}}\hat{g}_T(it)\int_{\mathcal{F}_C}|E_{1/2+it}(x)|^2 dxdt \\ +O(\sigma^2 g_T(I)). \end{array} \tag{2.16}$$

The next subsection is devoted to analyzing the Eisenstein series term. In Lemma 2.10 we show that it is $O(T\log T)$. Thus for some $c_0 > 0$

$$\begin{aligned}\left(\text{area}(\mathcal{F}_C) - c_0\sigma^2\right) g_T(I) &= \left(\text{area}(\mathcal{F}_C) - c_0\sigma^2\right)\frac{1}{4\pi}\int_{\mathbb{R}}\hat{g}_T(it)t\tanh(\pi t)dt \\ &\leq \sum_{j\geq 0}\hat{g}_T(\nu_j) + O(T\log T).\end{aligned}$$

Proof of Theorem 2.1:
We need to show

$$\lim_{T\to\infty}\frac{N(T)}{T} = \frac{1}{12}.$$

The general theorem of [Donnelly], for example, shows that

$$\limsup_{T\to\infty}\frac{N(T)}{T} \leq \frac{1}{12}. \tag{2.17}$$

By Proposition 2.7

$$\begin{aligned}\int_{\mathbb{R}} \hat{g}_T(it)t\tanh(\pi t)dt &= \int_{-T}^{T} t\tanh(\pi t)dt + O(T) \\ &= T^2 + o(T^2). \end{aligned} \tag{2.18}$$

Again using (2.17),

$$\sum_{j\geq 0} \hat{g}_T(\nu_j) \leq N((T+\sqrt{T})^2) + O(1).$$

This proves

$$\liminf_{T\to\infty} \frac{N(T)}{T} \geq \frac{\frac{\pi}{3} - \frac{1}{C} - c_0\sigma^2}{4\pi}.$$

Taking $\sigma \to 0$ and $C \to \infty$, we conclude the Weyl law. □

2.5 The Eisenstein Series Bounds

Recall that the Eisenstein series $E_s(z)$ has the constant term

$$c(v,s) = \int_0^1 E_s(u+iv)du = v^s + \phi(s)v^{1-s},$$

where

$$Z(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s),$$

and

$$\phi(s) = \frac{Z(2s-1)}{Z(2s)} = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}.$$

Define the *truncated Eisenstein series* for $u+iv \in \mathcal{F}$ as

$$\Lambda^C E_s(u+iv) = \begin{cases} E_s(u+iv) - c(v,s), & v > C \\ E_s(u+iv), & v \leq C. \end{cases} \tag{2.19}$$

The truncated $\Lambda^C E_s(z)$ can be extended from $\mathcal{F}$ to $\Gamma\backslash\mathbb{H}$; it decays rapidly as $v \to \infty$, and so is in $L^2(\Gamma\backslash\mathbb{H})$. Thus for any $s_1, s_2 \in \mathbb{C}$ which are not poles of the Eisenstein series $E_s(z)$,

$$\int_{\Gamma\backslash\mathbb{H}} \Lambda^C E_{s_1}(z)\Lambda^C E_{s_2}(z)\frac{dudv}{v^2} < \infty.$$

In fact the same is true if we only truncate $E_{s_1}(z)$:

Lemma 2.9. *With s_1 and s_2 as above,*

$$\int_{\Gamma\backslash\mathbb{H}} \Lambda^C E_{s_1}(z)\Lambda^C E_{s_2}(z)\frac{dudv}{v^2} = \int_{\Gamma\backslash\mathbb{H}} \Lambda^C E_{s_1}(z)E_{s_2}(z)\frac{dudv}{v^2}.$$

Proof: Unfolding, their difference

$$\int_{\Gamma\backslash\mathbb{H}} \Lambda^C E_{s_1}(z)\left(E_{s_2}(z)-\Lambda^C E_{s_2}(z)\right)\frac{dudv}{v^2} = \int_C^\infty \frac{dv}{v^2}c(y,s_2)\int_0^1 \left(E_{s_1}(z)-c(v,s_1)\right)du$$

$$= \int_C^\infty \frac{dv}{v^2}c(v,s_2)\left(c(v,s_1)-c(v,s_1)\right) = 0.$$

$\square$

It will simplify notation to introduce $\delta_C(z)$, the characteristic function of $\{z \in \mathbb{C} \mid \mathrm{Im}\ z > C\}$. Then for $z \in \mathcal{F}$

$$\Lambda^C E_s(z) = E_s(z) - \delta_C(z)c(v,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left[\mathrm{Im}\,(\gamma z)^{s_1} - \delta_C(\gamma z)c(\mathrm{Im}\,(\gamma z), s_1)\right].$$

Because the rightmost expression is automorphic, it agrees with $\Lambda^C E_s(z)$ for all $z \in \mathbb{H}$; at most one of the terms indicated by the δ_C functions is actually subtracted. We can thus unfold this series representation of $\Lambda^C E_{s_1}(z)$ in the integral:

$$\int_{\Gamma\backslash\mathbb{H}} \Lambda^C E_{s_1}(z)E_{s_2}(z)\frac{dudv}{v^2} = \int_{\Gamma_\infty\backslash\mathbb{H}} E_{s_2}(z)\left(v^{s_1}-\delta_C(z)c(v,s_1)\right)\frac{dudv}{v^2}$$

$$= \int_0^\infty \frac{dv}{v^2}\left(v^{s_1}-\delta_C(z)c(v,s_1)\right)\int_0^1 du E_{s_2}(z)$$

$$= \int_0^\infty \left(v^{s_2}+\phi(s_2)v^{1-s_2}\right)\left(v^{s_1}-\delta_C(v)\left(v^{s_1}+\phi(s_1)v^{1-s_1}\right)\right)\frac{dv}{v^2}$$

$$= \int_0^C v^{s_2+s_1-2}dv + \phi(s_2)\int_0^C v^{s_1-s_2-1}dv$$

$$-\phi(s_1)\int_C^\infty v^{s_2-s_1+1}dv - \phi(s_1)\phi(s_2)\int_C^\infty v^{-s_1-s_2}dv$$

$$= \frac{C^{s_2+s_1-1}}{s_2+s_1-1} + \frac{\phi(s_2)C^{s_1-s_2}}{s_1-s_2} + \frac{\phi(s_1)C^{s_2-s_1}}{s_2-s_1} + \phi(s_1)\phi(s_2)\frac{C^{1-s_2-s_1}}{1-s_2-s_1}, \tag{2.20}$$

provided Re $s_1 > 1 +$ Re s_2. Each side has a meromorphic continuation in s_1, s_2, and together are the *Maass-Selberg relations.*

Because the Eisenstein series and hence their truncations are holomorphic for $Re(s_1) = Re(s_2) = \frac{1}{2}$, the Maass-Selberg relations must have a removable singularity there. The limiting value is

$$\int_{\Gamma\backslash\mathbb{H}} |\Lambda^C E_{1/2+it}(z)|^2 \frac{dudv}{v^2} = 2\log C - \frac{\phi'}{\phi}(\frac{1}{2}+it) + \frac{C^{2it}\phi(\frac{1}{2}-it) - C^{-2it}\phi(\frac{1}{2}+it)}{2it}. \tag{2.21}$$

The functional equations

$$Z(s) = Z(1-s)\ ,\ Z(\bar{s}) = \overline{Z(s)}$$

show that

$$|\phi(\frac{1}{2}+it)| = \left|\frac{Z(it)}{Z(it+1)}\right| = 1. \tag{2.22}$$

Also,

$$\frac{\phi'}{\phi}(\frac{1}{2}+it) = 2\left(\frac{Z'}{Z}(2it) + \frac{Z'}{Z}(-2it)\right) \tag{2.23}$$

and

$$\frac{Z'}{Z}(s) = \sum_{\rho|Z(\rho)=0} \frac{1}{s-\rho} - \frac{1}{s} - \frac{1}{s-1}, \tag{2.24}$$

a sum over the poles and critical zeroes of $\zeta(s)$.

Lemma 2.10.

$$\int_{\mathbb{R}} \hat{g}_T(it) \int_{\mathcal{F}_C} |E_{1/2+it}(x)|^2 dxdt = O(T\log T).$$

Proof: By the definition of the truncation,

$$\int_{\mathcal{F}_C} |E_{1/2+it}(x)|^2 dx \leq \int_{\mathcal{F}} |\Lambda^C E_{1/2+it}(x)|^2 dx,$$

which is calculated in (2.21). In view of (2.22) and (2.23), it suffices to show

$$\text{Re} \int_{\mathbb{R}} \hat{g}_T(it) \frac{Z'}{Z}(2it) dt = O(T \log T).$$

This follows from (2.24) and the zero counting estimate (see Proposition 7.1 or [Titchmarsh])

$$\text{Re} \int_{i(H-1)}^{i(H+1)} \sum_{\rho | Z(\rho)=0} \frac{ds}{s-\rho} = O(\#\{\rho \mid H-1 \leq \text{Im } (\rho) \leq H+1\}) = O(\log H).$$

$\square$

3 Coordinates on $SL_3(\mathbb{R})$ and $\mathcal{H} = SL_3(\mathbb{R})/SO_3(\mathbb{R})$

The Iwasawa decomposition $G = SL_3(\mathbb{R}) = NAK$ states that every element $g \in G$ can be uniquely factored as $g = nak$, where

$$n \in N_0 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

$$a \in A_0 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in G \ , \ a_i > 0 \right\},$$

and

$$k \in K = SO_3(\mathbb{R}).$$

The minimal standard parabolic is

$$P_0 = N_0 A_0 = A_0 N_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}.$$

There are two associate maximal standard parabolics

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} \quad , \quad P_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\}.$$

Each can be decomposed uniquely as $P_i = M_i' N_i A_i = N_i M_i' A_i$, where

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

$$M_1' = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \in G \right\},$$

$$A_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} \in G \ , \ a > 0 \right\},$$

$$N_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

$$M_2' = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\},$$

and

$$A_2 = \left\{ \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in G \ , \ a > 0 \right\}.$$

The *period* of an automorphic function $\psi \in L^2(\Gamma \backslash G/K)$ along the parabolic P is defined as the integral

$$\psi_P(g) = \int_{\Gamma \cap N \backslash N} \psi(ng) dn. \tag{3.1}$$

If all periods along each of the parabolics P_0, P_1, and P_2 vanish, then ψ is called a *cusp* form.

We shall use the following sets of coordinates for the Lie algebras. Let

$$\mathbf{a}_0 = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \mid h_1, h_2, h_3 \in \mathbb{R} \ , \ h_1 + h_2 + h_3 = 0 \right\}$$

$$\simeq \{H = (h_1, h_2, h_3) \in \mathbb{R}^3 \mid h_1 + h_2 + h_3 = 0\}.$$

It has simple roots $\alpha_1, \alpha_2 \in \mathbf{a}_0^*$, which are the following linear functions on $\mathbf{a}_0$:

$$\alpha_1(H) = h_1 - h_2 \ , \ \alpha_2(H) = h_2 - h_3.$$

There is also a third root $\alpha_3 = \alpha_1 + \alpha_2$ which acts by $\alpha_3(H) = h_1 - h_3$. We can similarly coordinatize

$$\mathbf{a}_0^* = \left\{ \lambda = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3 \mid \ell_1 + \ell_2 + \ell_3 = 0 \right\}$$

and its complexification

$$\mathbf{a}_{0\mathbb{C}}^* = \left\{ \lambda = (\ell_1, \ell_2, \ell_3) \in \mathbb{C}^3 \mid \ell_1 + \ell_2 + \ell_3 = 0 \right\}.$$

Since these are subsets of $\mathbb{C}^3$, we will use the standard norm there to define norms in these spaces. The simple roots are bases of $\mathbf{a}_0^*$ and $\mathbf{a}_{0\mathbb{C}}^*$ as vector spaces over $\mathbb{R}$ and $\mathbb{C}$, respectively. Also, they have corresponding co-roots

$$\alpha_1^\vee = (1, -1, 0) \in \mathbf{a}_0 \ , \ \alpha_2^\vee = (0, 1, -1) \in \mathbf{a}_0.$$

The Weyl groups $\Omega(\mathbf{a}_0)$ and $\Omega(\mathbf{a}_0^*)$ are isomorphic to the symmetric group S_3, and act in a compatible way by permuting the standard basis vectors in $\mathbb{R}^3$ or $\mathbb{C}^3$ (see the chart in the appendix for more details). The Cartan subgroups A_1, A_2 of the maximal parabolics have one-dimensional Lie algebras, coordinatized by

$$\mathbf{a_1} = \{H = (h, h, -2h) \mid h \in \mathbb{R}\} \simeq \{h \in \mathbb{R}\}$$

and

$$\mathbf{a_2} = \{H = (2h, -h, -h) \mid h \in \mathbb{R}\} \simeq \{h \in \mathbb{R}\}.$$

These each have one-dimensional dual spaces, $\mathbf{a_1}^*$ and $\mathbf{a_2}^*$ respectively. Also, each dual has a special vector ρ, half the sum of the positive roots:

$$\rho_0 = \frac{\alpha_1 + \alpha_2 + \alpha_3}{2} : (h_1, h_2, h_3) \mapsto h_1 - h_3,$$

$$\rho_1 : (h, h, -2h) \mapsto 3h,$$

and

$$\rho_2 : (2h, -h, -h) \mapsto 3h.$$

Under these coordinates ρ_1 and ρ_2 are naturally identified with α_2 and α_1, respectively.

Each parabolic P_i has the Langlands decomposition

$$P_i = N_i A_i M_i'$$

and logarithm maps $H_i : G \to \mathbf{a_i}$ such that

$$a = e^{H_i(a)} \quad , \quad a \in A_i \tag{3.2}$$

and

$$g \in N_i e^{H_i(g)} M_i' K.$$

These maps are well-defined despite the fact that the decomposition $G = P_i K$ is in general not unique. Similarly for the maximal parabolics, there are maps m_1 and m_2 mapping G onto $M_1'/(K \cap M_1')$ and $M_2'/(K \cap M_2')$, respectively.

Truncation

Let Δ_P denote the set of simple roots which do not vanish identically on P:

$$\Delta_{P_0} = \{\alpha_1, \alpha_2\} \ , \ \Delta_{P_1} = \{\alpha_2\} \ , \ \Delta_{P_2} = \{\alpha_1\}.$$

The group G can also be viewed as a parabolic, with $\Delta_G = \{\}$.

We can now define Langlands' truncation ([Lan1], see [Art1]). For any parabolic $P = P_0, P_1, P_2$, or G, define $\hat{\tau}_P(x)$ to be the characteristic function of

$$\{x = c_1 \alpha_1^\vee + c_2 \alpha_2^\vee \in \mathbf{a}_0 \mid c_i > 0 \ \forall \alpha_i \in \Delta_P\}.$$

Thus, $\hat{\tau}_{P_0}$ is the characteristic function of

$$\{x = c_1 \alpha_1^\vee + c_2 \alpha_2^\vee \in \mathbf{a}_0 \mid c_1, c_2 > 0\},$$

$$\hat{\tau}_{P_1} \text{ of } \{x = c_1 \alpha_1^\vee + c_2 \alpha_2^\vee \in \mathbf{a}_0 \mid c_2 > 0\},$$

$$\hat{\tau}_{P_2} \text{ of } \{x = c_1 \alpha_1^\vee + c_2 \alpha_2^\vee \in \mathbf{a}_0 \mid c_1 > 0\},$$

and

$$\hat{\tau}_G \text{ of } \mathbf{a}_0.$$

Let $C \in \mathbf{a}_0$ be a fixed parameter. The truncation of an automorphic form ψ is a sum over all standard parabolic subgroups

$$(\Lambda^C \psi)(x) := \sum_P (-1)^{\dim A} \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \hat{\tau}_P(H_0(\gamma x) - C) \int_{\Gamma \cap N \backslash N} \psi(n\gamma x)dn$$

$$= \sum_P (-1)^{\dim A} \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \hat{\tau}_P(H_0(\gamma x) - C)\psi_P(\gamma x), \tag{3.3}$$

which itself is clearly an automorphic form. It can be proven that it also decays rapidly in the cusp because of the way its constant terms have been removed (see [Art1]). Note that if ψ is a cusp form to begin with, by definition all its constant terms ψ_P vanish identically in proper parabolics, and thus $\Lambda^C \psi = \psi$.

Spectral Density

We will later need to use the spectral density function $\beta(\lambda)$ on $i\mathbf{a_0}^*$, which is a constant multiple of

$$\left| \frac{\Gamma(\frac{1}{2} + \lambda(\alpha_1^\vee))}{\Gamma(\lambda(\alpha_1^\vee))} \frac{\Gamma(\frac{1}{2} + \lambda(\alpha_2^\vee))}{\Gamma(\lambda(\alpha_2^\vee))} \frac{\Gamma(\frac{1}{2} + \lambda(\alpha_3^\vee))}{\Gamma(\lambda(\alpha_3^\vee))} \right|^2. \tag{3.4}$$

This function will appear below in (4.2); see [DKV] for more information. For λ such that $\lambda(\alpha_1^\vee), \lambda(\alpha_2^\vee)$, and $\lambda(\alpha_3^\vee)$ are all large, Stirling's formula shows that $\beta(\lambda)$ behaves as a constant times $|\lambda(\alpha_1^\vee)\lambda(\alpha_2^\vee)\lambda(\alpha_3^\vee)|$.

The spectral density function is normalized so that

$$\int_{\|\lambda\| \leq T} \beta(\lambda)d\lambda \sim \frac{T^{5/2}}{\Gamma(7/2)(4\pi)^{5/2}}.$$

One can also view this as normalizing the measure $d\lambda$.

4 Convolution operators

Every $g \in C_c^\infty(K\backslash G/K)$ acts by convolution on $L^2(G/K)$:

$$(L_g f)(x) = (f * g)(x) = \int_G f(y)g(y^{-1}x)dy = \int_{G/K} f(y)g(y^{-1}x)dy.$$

The convolution operator L_g also acts on $f \in L^2(\Gamma\backslash G/K)$ by

$$(L_g f)(x) = \int_{G/K} f(y)g(y^{-1}x)dy$$

$$= \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G/K} f(\gamma^{-1}y)g(y^{-1}\gamma x)dy = \int_{\Gamma\backslash G/K} f(y)K(x,y)dy,$$

where

$$K(x,y) = \sum_{\gamma\in\Gamma} g(y^{-1}\gamma x)$$

is the automorphic kernel.

Suppose furthermore that $g(x)$ is real; then L_g is an operator on $L^2(\Gamma\backslash G/K)$ which commutes with the laplacian Δ and all other invariant differential operators. We may thus choose an orthonormal set

$$\frac{1}{\sqrt{vol(\Gamma\backslash G/K)}} = \phi_0, \phi_1, \phi_2, \ldots$$

of common eigenfunctions of L_g and the ring of invariant differential operators $\mathcal{D}$. Continuing, we may resolve any multiplicities that remain by using the Hecke operators, which commute with these other operators.

Proposition 4.1. *(Selberg's Uniqueness Principle)([Sel1]) If ϕ is a common eigenfunction of the ring of invariant differential operators, then*

$$(L_g\phi)(x) = \hat{g}(\phi)\phi(x),$$

where $\hat{g}(\phi)$ only depends on ϕ's eigenvalues under $\mathcal{D}$. In fact, one can always find some $\lambda \in \mathbf{a}_{\mathbb{C}}^*$ such that the function $\phi_\lambda = e^{(\lambda+\rho)(H_0(x))}$ on G/K has the same eigenvalues as ϕ under any $D \in \mathcal{D}$. This provides the formula

$$\hat{g}(\phi) = \hat{g}(\lambda) = (L_g\phi_\lambda)(I) = \int_{G/K} g(x)e^{(\lambda+\rho)(H_0(x))}dx. \tag{4.1}$$

If s is any permutation in the Weyl group $\Omega(\mathbf{a}_0^*)$, then in fact $\phi_\lambda(x) = e^{(\lambda+\rho)(H_0(x))}$ has the same eigenvalues as $\phi_{s\lambda}$. The uniqueness principle thus implies that $\hat{g}$ is invariant under $\Omega(\mathbf{a}_0^*)$. Formula (4.1) shows that the transform $\hat{g}(\lambda)$ is the composition of an average over N_0, and a Fourier transform on A_0. More precisely, if g is a function on G/K, denote

$$\bar{g}(a) = \int_{N_0} g(na)dn.$$

For a function f on A_0, define the Fourier transform as

$$(\mathcal{F}f)(\lambda) = \int_{A_0} f(a)e^{\lambda(H_0(a))}da \quad , \quad \lambda \in i\mathbf{a}^*_{\mathbf{0}\mathbb{C}}.$$

Then $\hat{g} = \mathcal{F}\left(\bar{g}(\cdot)e^{-\rho(H_0(\cdot))}\right)$.

We will make use of the following theorem in constructing our choices of functions g. Before stating it, let us introduce the notation

$$A(\sigma) = \{a \in A_0 \mid \|H_0(a)\| \leq \sigma\}.$$

Theorem 4.2. *([Gangolli]) The map $g \mapsto \bar{g}$ provides a bijection between the sets*

$$\{g \in C^\infty(K\backslash G/K) \mid supp\ g \subset KA(\sigma)K\}$$

and

$$\{h \in C^\infty(A) \mid supp\ h \subset A(\sigma)\ and\ h(sa) = h(a)\ for\ all\ s \in \Omega(\mathbf{a_0})\}.$$

Construction of functions

Let $\sigma > 0$ be fixed for the remainder of this section. It is possible to find a non-negative function $g \in C^\infty(K\backslash G/K)$ which is supported in $KA(\sigma)K$, and whose transform $\hat{g}$ is non-negative on the joint spectrum. Having such a function without the positivity requirement on $\hat{g}$ is straightforward; one then rescales and convolves it with itself to achieve positivity (see lemma 6.2 of [DKV] for example). We shall again normalize g so that

$$\iint_{i\mathbf{a}_\mathbf{0}^*} \hat{g}(\lambda)d\lambda = 1.$$

Let us state a property of g which is implied by the positivity of $\hat{g}$ on $i\mathbf{a_0}^*$:

Proposition 4.3. *If $g \in C_C^\infty(K\backslash G/K)$ is such that $\hat{g}(\lambda) \geq 0$ for all $\lambda \in i\mathbf{a_0}^*$, then*

$$\max_{x \in G} |g(x)| = g(I).$$

Remark 4.4. As the proof will demonstrate, the analogous fact is true for the usual Euclidean Fourier transform.

Proof of Proposition 4.3: The proof uses the inversion formula

$$g(x) = \int_{i\mathbf{a_0}^*} \hat{g}(\lambda)\overline{\tilde{\phi}_\lambda(x)}\beta(\lambda)d\lambda, \tag{4.2}$$

where

$$|\tilde{\phi}_\lambda(x)| \leq \tilde{\phi}_\lambda(I) = 1$$

is a spherical function (e.g. (2.3)) and $\beta(\lambda)$ is the spectral density. Trivially

$$|g(x)| \leq \int_{i\mathbf{a_0}^*} \hat{g}(\lambda)\beta(\lambda)d\lambda = g(I).$$

□

Now if Σ is a measurable, bounded subset of $i\mathbf{a_0}^*$ which is invariant under the Weyl group, define

$$\hat{g}_\Sigma(\lambda) = \hat{g} * \chi_\Sigma(\lambda) = \int_\Sigma \hat{g}(\lambda - \mu)d\mu. \tag{4.3}$$

The function $\hat{g}_\Sigma(\lambda)$ is roughly concentrated on Σ, especially for large Σ; we shall use it to estimate the spectrum in Σ. Also $\hat{g}_\Sigma(\lambda)$ decays rapidly, since g_Σ is a smooth function of compact support. If Σ is open and its boundary has a finite Hausdorff length, then

$$\hat{g}_{t\Sigma}(\lambda) \ll (1 + dist(\lambda, t\Sigma))^{-m} \tag{4.4}$$

(cf. p. 85 of [DKV]). Of course $\hat{g}_\Sigma(\lambda)$ is the Fourier transform of

$$\bar{g}(a) \int_\Sigma e^{(-\mu-\rho)(H_0(a))}d\mu,$$

which is a smooth function on A_0 whose support is contained in $A(\sigma)$. It is furthermore invariant under the action of the Weyl group $\Omega(\mathbf{a_0})$ because Σ is. Thus by Theorem 4.2 there exists a smooth, bi-K-invariant function g_Σ supported in $KA(\sigma)K$ such that

$$\overline{g_\Sigma}(a) = \bar{g}(a) \int_\Sigma e^{-\mu(H_0(a))} d\mu \tag{4.5}$$

and

$$\widehat{g_\Sigma} = \hat{g}_\Sigma.$$

In summary, while not changing the support of our function g, we can still smear its transform $\hat{g}$ over Σ. Of course g_Σ becomes more concentrated near the identity as Σ gets larger; in the classical Euclidean case this is analogous to multiplying a function by Fejer's kernel. In (5.5) we will use a result analogous to Proposition 2.7 comparing $\hat{g}_\Sigma$ to χ_Σ.

5 The Partial Trace

In addition to the discrete spectrum there is also a continuous spectrum, furnished by Eisenstein series. Because it is complicated we will just refer to its terms in the spectral expansion as "$Eis_g(x,y)$" until we need to be more explicit:

$$K(x,y) = \sum_{j\geq 0} \hat{g}(\lambda_j)\phi_j(x)\phi_j(y) + Eis_g(x,y) = \sum_{\gamma\in\Gamma} g(x^{-1}\gamma y). \tag{5.1}$$

Let $\mathcal{F}$ be a fundamental domain for $\Gamma\backslash G/K$ and $\mathcal{F}_C$ be a compact subset of $\mathcal{F}$. Then

$$\begin{aligned}\int_{\mathcal{F}_C} K(x,x)dx = & \textstyle\sum_{\gamma\in\Gamma} \int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx \\ = & \textstyle\sum_{j\geq 0} \hat{g}(\lambda_j) \int_{\mathcal{F}_C} \phi_j(x)^2 dx + \int_{\mathcal{F}_C} Eis_g(x,x)dx \end{aligned} \tag{5.2}$$

Now,

$$\int_{\mathcal{F}_C} \phi_j(x)^2 dx \leq \int_{\mathcal{F}} \phi_j(x)^2 dx = 1$$

so

$$\sum_{\gamma\in\Gamma}\int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx \leq \sum_{j\geq 0}\hat{g}(\lambda_j) + \int_{\mathcal{F}_C} Eis_g(x,x)dx. \tag{5.3}$$

Now we will analyze the integrals

$$\int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx$$

more systematically. Note that we *do not* group these into conjugacy classes as is done in the trace formula.

Proposition 5.1. *For σ sufficiently small and supp $g \subset A(\sigma)$, the integral*

$$\int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx = 0$$

for all but finitely many γ – those which have fixed points in the closure of $\mathcal{F}_C$.

Proposition 5.2. *If $\gamma \neq I$ has a fixed-point in the closure of $\mathcal{F}_C$ and supp $g \subset A(\sigma)$, then*

$$\int_{\mathcal{F}_C} g(x^{-1}\gamma x)dx \ll |\max g| vol(A(\sigma)).$$

Although the implied constant above depends on $\mathcal{F}_C$, it may be taken to be independent of $\gamma \in SL_3(\mathbb{Z})$ since only finitely many elements produce a non-zero integral.

Clearly

$$\int_{\mathcal{F}_C} g(x^{-1}Ix)dx = vol(\mathcal{F}_C)\cdot g(I).$$

Now take $g = g_\Sigma$ as in (4.5). Using Propositions 4.3 and 5.2, the inequality (5.3) becomes

$$vol(\mathcal{F}_C)g_\Sigma(I) + O\left(g_\Sigma(I)vol(A(\sigma))\right) \leq \sum_{j=0}^{\infty}\hat{g}_\Sigma(\lambda_j) + \int_{\mathcal{F}_C} Eis_{g_\Sigma}(x,x)dx. \tag{5.4}$$

The proof of Theorem 8.5 in [DKV] shows that both

$$\left|\sum_{j=0}^{\infty} \hat{g}_{\Sigma}(\lambda_j) - \#\{\mathrm{Im}\ \lambda_j \in \Sigma\}\right| \ll \int_{B\Sigma} \beta(\lambda) d\lambda$$

and

$$\left|g_{\Sigma}(I) - \int_{\Sigma} \beta(\lambda) d\lambda\right| \ll \int_{B\Sigma} \beta(\lambda) d\lambda,$$

$$B\Sigma = \{\lambda \in i\mathbf{a}_{\mathbf{0}}^* \mid \mathrm{dist}(\lambda, \partial\Sigma) \leq 1\}, \tag{5.5}$$

assuming Σ is a bounded, open, and Weyl-group invariant subset of $i\mathbf{a}_{\mathbf{0}}^*$. Now to control the error term we shall assume Σ's boundary is piece-wise smooth (cf. Lemma 8.7 of [DKV]). Then (3.4) and (5.5) show that for large t

$$g_{t\Sigma}(I) \sim \int_{t\Sigma} \beta(\lambda) d\lambda,$$

and by using (5.4),

$$\left(\int_{t\Sigma} \beta(\lambda) d\lambda\right) (vol(\mathcal{F}_C) + O(vol(A(\sigma))))\, (1 + o(1))$$

$$\leq \#\{\mathrm{Im}\ \lambda_j \in t\Sigma\} + \int_{\mathcal{F}_C} Eis_{g_{t\Sigma}}(x, x) dx.$$

We will see at the end of Section 7 that

$$\int_{\mathcal{F}_C} Eis_{g_{t\Sigma}}(x, x) dx = o\left(\int_{t\Sigma} \beta(\lambda) d\lambda\right). \tag{5.6}$$

Proof of Theorem 1.2: Taking $t \to \infty$

$$\liminf_{t\to\infty} \frac{\#\{\mathrm{Im}\ \lambda_j \in t\Sigma\}}{\int_{t\Sigma} \beta(\lambda) d\lambda} \geq vol(\mathcal{F}_C) + O(vol(A(\sigma))).$$

We had insisted that *supp* $g \subset A(\sigma)$, so taking $\sigma \to 0$ and exhausting $\mathcal{F}$ through compact sets $\mathcal{F}_C$ we conclude that

$$vol(\mathcal{F}) \leq \liminf_{t\to\infty} \frac{\#\{\mathrm{Im}\ \lambda_j \in t\Sigma\}}{\int_{t\Sigma} \beta(\lambda) d\lambda}. \tag{5.7}$$

If

$$\Sigma = \{\lambda = (\ell_1, \ell_2, \ell_3) \in i\mathbf{a}_\mathbf{0}^* \mid \|\lambda\|^2 \leq 1\},$$

then this indicates

$$vol(\mathcal{F}) \leq \liminf_{T\to\infty} \frac{N(T)}{\left(\frac{T}{4\pi}\right)^{5/2} \frac{1}{\Gamma(7/2)}}$$

because the Laplace eigenvalue of $e^{(\lambda+\rho)(H(g))}$ is $1 - \frac{\ell_1^2+\ell_2^2+\ell_3^2}{2}$.[2] (Note that $|\text{Re } \ell_j| < \frac{1}{2}$ by unitary – [JacSha].)

The upper bound

$$\limsup_{T\to\infty} \frac{N(T)}{\left(\frac{T}{4\pi}\right)^{5/2} \frac{1}{\Gamma(7/2)}} \leq vol(\mathcal{F}) \tag{5.8}$$

due to [Donnelly] shows in fact that

$$N(T) \sim \left(\frac{T}{4\pi}\right)^{5/2} \frac{vol(\mathcal{F})}{\Gamma(7/2)}. \tag{5.9}$$

□

Theorem 5.3. *(Spectral Equidistribution) If Σ is a bounded, open, Weyl-group invariant subset of $i\mathbf{a}_\mathbf{0}^*$ with a piece-wise smooth boundary, then*

$$\lim_{t\to\infty} \frac{\#\{Im\ \lambda_j \in t\Sigma\}}{\int_{t\Sigma} \beta(\lambda) d\lambda} = vol(\mathcal{F}). \tag{5.10}$$

Proof: The lower bound is in (5.7). To prove the upper bound, we will use this lower bound along with the asymptotics of the Weyl law for counting in balls. Let r be the length of the longest vector in Σ (which is bounded by assumption). By defining Σ_c as the complement of Σ inside the r-ball B_r, we have that

$$\#\{\text{Im } \lambda_j \in t\Sigma\} + \#\{\text{Im } \lambda_j \in t\Sigma_c\} = \#\{\text{Im } \lambda_j \in t(\Sigma \cup \Sigma_c)\}.$$

[2]A check of this normalization is provided by the compact case, where there are no Eisenstein series and the Weyl law is already known.

Given any $\epsilon > 0$ we can find t large enough so that

$$\#\{\mathrm{Im}\ \lambda_j \in t\Sigma\} \leq (1+\epsilon)\left(\int_{t(\Sigma\cup\Sigma_c)} \beta(\lambda)d\lambda - \int_{t\Sigma_c} \beta(\lambda)d\lambda\right), \tag{5.11}$$

which implies the upper bound when $\epsilon \to 0$. □

Proofs of Theorems 1.6 and 1.3: If the cusp form ϕ_j is not tempered, then its spectral parameter $\lambda = (\ell_1, \ell_2, \ell_3)$ is not a purely-imaginary vector. By the classification of the unitary dual, we have equality of the sets

$$\{\ell_1, \ell_2, \ell_3\} = \{-\bar{\ell}_1, -\bar{\ell}_2, -\bar{\ell}_3\}.$$

We know that $|Re(\ell_i)| < \frac{1}{2}$ by unitary ([JacSha]), so the vectors λ all lie near the hyperplanes defined by

$$\lambda(\alpha_1^\vee) = 0\ ,\ \lambda(\alpha_2^\vee) = 0\ ,\ \lambda(\alpha_1^\vee + \alpha_2^\vee) = 0.$$

Similarly, if ϕ is self dual then

$$\{\ell_1, \ell_2, \ell_3\} = \{\mu, 0, -\mu\} \text{ for some } \mu.$$

This again constrains λ to lie along a hyperplane.

However, by taking the shape Σ in Theorem 5.3 to be further and further away from any fixed hyperplane, we conclude that no hyperplane has a positive percentage of the spectrum near it. Thus each of the exceptional sets we are considering is of measure zero compared to the rest of the spectrum. □

A related argument was used in [DKV] for cocompact subgroups Γ.

Proof of Theorem 1.5: All Gelbart-Jacquet lifts are self-dual forms on SL_3.

Another proof would be simply by counting: the lift quadruples the Laplace eigenvalue of a cusp form on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, and the number of these with $SL_3(\mathbb{R})$-Laplace eigenvalue $\leq T$ is $O(T)$ by Selberg's theorem 1.1. Though forms ψ on congruence covers of $SL_2(\mathbb{Z})\backslash\mathbb{H}$ may also lift to $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$, such ψ are actually twists of forms on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, so one need only count the lifts from $SL_2(\mathbb{Z})\backslash\mathbb{H}$ itself, and not from other congruence covers. □

6 Eisenstein Series

These periodized functions on $\Gamma\backslash G/K$ are constructed using the Langlands decompositions of G's parabolics. Recall that any element $p \in P = NAM'$ factors uniquely into a product $p = nam$ of elements from their respective subgroups. Writing the diagonal matrices $a = e^{H_P(p)}$, we get a map which extends to G:

$$g \mapsto H_P(g)\ ,\ g \in N e^{H_P(g)} M' K.$$

For the parabolics P_1 and P_2, where $M' \simeq GL_2(\mathbb{R})$, there are corresponding maps $m : G \to M'/(K \cap M')$ (see (3.2)).

If $\lambda \in \mathbf{a}^*_{0\mathbb{C}}$ and $g \in G$, the minimal parabolic Eisenstein series is defined as

$$E(P_0, g, \lambda) = E(P_0, g, \lambda, 1) = \sum_{\Gamma \cap P_0 \backslash \Gamma} e^{(\lambda+\rho_0)(H_0(\gamma g))}. \tag{6.1}$$

This sum only converges when $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ have large real parts, but it has a meromorphic continuation to all of $\mathbf{a}^*_{0\mathbb{C}}$.

Since

$$\Gamma \cap M_1' \backslash M_1'/(K \cap M_1') \simeq GL_2(\mathbb{Z})\backslash \mathbb{H}\,,$$

the discrete eigenfunctions on the former are just the even[3] discrete eigenfunctions for $SL_2(\mathbb{Z})$. Take such a cusp form and $\lambda \in \mathbf{a}^*_{1\mathbb{C}}$, and define the maximal parabolic Eisenstein series

$$E(P_1, g, \lambda, \phi) = \sum_{\Gamma \cap P_1 \backslash \Gamma} e^{(\lambda+\rho_1)(H_1(\gamma g))} \phi(m_1(\gamma g)). \tag{6.2}$$

There is a similar Eisenstein series $E(P_2, g, \lambda_1, \phi)$ for the other maximal parabolic P_2, related through a functional equation. Each series again is initially only defined for certain values of λ but extends via a meromorphic continuation to $\mathbf{a}^*_{\mathbb{C}}$ ([Lan1],[Lan3]).

We are now in a position to define what "$Eis_g(x, y)$" is. The spectral expansion of the automorphic kernel is

$$K(x,y) = \sum_{\substack{\phi_j \text{ an } L^2 \text{ discrete eigenfunction} \\ \text{on } SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})}} \hat{g}(\lambda_j)\phi_j(x)\phi_j(y) + Eis_g(x,y),$$

[3] i.e. $f(x+iy) = f(-x+iy)$.

and up to a normalizing constant for the measure, $Eis_g(x,y)$ is ([Art2])

$$\frac{1}{3(2\pi i)^2}\iint_{i\mathbf{a}_0^*}\hat{g}(\lambda)E(P_0,x,\lambda,1)\overline{E(P_0,y,\lambda,1)}d\lambda +$$

$$\frac{1}{2\pi i}\sum_{\substack{\phi_j \text{ an even } L^2 \text{ discrete eigenfunction} \\ \text{on } SL_2(\mathbb{Z})\backslash\mathbb{H}\ ,\ \Delta\phi_j=(\frac{1}{4}-\nu_j^2)\phi_j}}\int_{i\mathbf{a}_1^*}\hat{g}(\lambda+(\nu_j,-\nu_j,0))E(P_1,x,\lambda,\phi_j)\overline{E(P_1,y,\lambda,\phi_j)}d \tag{6.3}$$

We also may assume that each ϕ_j on $\mathbb{H}$ is a Hecke eigenform. There is a beautiful formula for the inner-products of truncated Eisenstein series, due to Langlands. It generalizes the Maass-Selberg formula (2.20) for $SL_2(\mathbb{R})$. See [Art1] for details.

Theorem 6.1. *Langlands' inner product formula ([Lan1], [Art1])*

$$\int_{\Gamma\backslash G/K}\left[(\Lambda^C E)(P,x,\phi,\lambda_1)\right]\left[(\Lambda^C E)(P,x,\phi,\lambda_2)\right]dx$$

$$= \sum_{P\sim P' \ \text{associate}} vol(\mathbf{a}'/<\alpha^\vee \mid \alpha\in\Delta_{P'}>)\times$$

$$\sum_{s_1,s_2\in\Omega(\mathbf{a},\mathbf{a}')}\frac{e^{(s_1\lambda_1+s_2\lambda_2)(C)}}{\prod_{\alpha\in\Delta_{P'}}(s_1\lambda_1+s_2\lambda_2)(\alpha^\vee)}\left\langle M(s_1,\lambda_1)\phi, M(s_2,\lambda_2)\phi\right\rangle.$$

Here $M(s,\lambda)$ is an intertwining operator, which sends ϕ to a cusp form on the potentially-different parabolic P'. We will discuss it in the instances it arises for us, where it essentially acts as scalar multiplication. The last expression $\langle\psi,\psi'\rangle$ is an inner product over $\Gamma\cap M'\backslash M'$, and the Weyl group $\Omega(\mathbf{a},\mathbf{a}')$ is the set of isomorphisms of $\mathbf{a}$ to $\mathbf{a}'$ coming from restrictions of elements in $\Omega(\mathbf{a}_0)$.

Interlude: SL_2

Write $C=(c,-c), c>0$, and let $\lambda_1=(it+\epsilon,-it-\epsilon), \lambda_2=(-it,it)$ so that

$$\int_{\mathcal{F}}\Lambda^C E(P,g,\lambda_1,\phi)E(P,g,\lambda_2,\phi)dg$$

is a constant times

$$\frac{e^{(\epsilon,-\epsilon)C}}{2\epsilon}+\frac{e^{(2it+\epsilon,-2it-\epsilon)C}}{4it+2\epsilon}R(-2it)+\frac{e^{(-2it-\epsilon,2it+\epsilon)C}}{-4it-2\epsilon}R(2it+2\epsilon)+\frac{e^{(-\epsilon,\epsilon)C}}{-2\epsilon}R(-2it)R(2it+2\epsilon).$$

Here the intertwining operator is

$$R(s)=\sqrt{\pi}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{\zeta(s)}{\zeta(s+1)}=\frac{Z(s)}{Z(s+1)} \tag{6.4}$$

with

$$Z(s)=\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)=Z(1-s).$$

Take $\epsilon\to 0$ so that $\lambda_1\to\bar{\lambda}_2$. Then the last expression approaches

$$\int_{\mathcal{F}}|\Lambda^C E(P,g,\lambda)|^2 dg=(const)\left[4c-\frac{R'}{R}(2it)+\frac{e^{4itc}}{4it}R(-2it)-\frac{e^{-4itc}}{4it}R(2it)\right].$$

In (2.20) we derived this from a direct calculation. It is a key step in Selberg's trace formula for $SL_2(\mathbb{Z})$.

7 Bounding the Eisenstein Contribution

In this final section we will complete the proofs by establishing the estimate in (5.6). From (6.3) it is sufficient to show that both

$$\int_{\mathcal{F}_C}\iint_{i\mathbf{a}_0^*}\hat{g}_{t\Sigma}(\lambda)|E(P_0,x,\lambda,1)|^2 d\lambda dx=o\left(\int_{t\Sigma}\beta(\lambda)d\lambda\right), \tag{7.1}$$

and

$$\int_{\mathcal{F}_C}\int_{i\mathbf{a}_1^*}\sum_{\substack{\phi_j \text{ an even } L^2 \text{ discrete} \\ \text{eigenfunction} \\ \text{on } SL_2(\mathbb{Z})\backslash\mathbb{H} \\ \Delta\phi_j=(\frac{1}{4}-\nu_j^2)\phi_j}}\hat{g}_{t\Sigma}(\lambda+(\nu_j,-\nu_j,0))|E(P_1,x,\lambda,\phi_j)|^2 d\lambda dx=o\left(\int_{t\Sigma}\beta(\lambda)d\lambda\right). \tag{7.2}$$

To do this we shall use the rapid decay of $\hat{g}_{t\Sigma}$ in (4.4). Among other benefits, this allows us to interchange the order of integration. Keep in mind that $\hat{g}_{t\Sigma}$ is roughly the characteristic function of $t\Sigma$.

Some background on L-functions

We will require some information about the density of zeroes of certain L-functions: the Riemann ζ function and the standard L-function of an even cusp form ϕ on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, $L(s,\phi)$. Each is defined as an Euler product[4] over the primes

$$\zeta(s)=\prod_p(1-p^{-s})^{-1} \quad , \quad L(s,\phi)=\prod_p(1-\alpha_p p^{-s})^{-1}(1-\alpha_p^{-1}p^{-s})^{-1}, \quad (7.3)$$

where the α_p satisfy the bound $|\alpha_p|\le p^{5/28}$ ([BDHI]). Each L-function can be completed:

$$Z(s)=\Gamma_{\mathbb{R}}(s)\zeta(s)\ ,\ \Lambda(s,\phi)=\Gamma_{\mathbb{R}}(s+\nu)\Gamma_{\mathbb{R}}(s-\nu)L(s,\phi),$$

where

$$\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$$

and ν is related to ϕ's Laplace eigenvalue by

$$\Delta\phi=(\frac{1}{4}-\nu^2)\phi.$$

With this convention $Z(s)=Z(1-s),\Lambda(s,\phi)=\Lambda(1-s,\phi)$, and each is entire except for the simple poles of $Z(s)$ at $s=0,1$.

The following estimate will be used to bound Eisenstein series integrals later. The analogous statement for $Z(s)$ is classical (e.g. see [Titchmarsh]) and the proof for $\Lambda(s,\phi)$ is essentially identical. However we do not know of a reference in the literature and include it for completeness.

Proposition 7.1. *For $T\in\mathbb{R}$*

$$Re\int_{T-1}^{T+1}\frac{\Lambda'}{\Lambda}(1+it,\phi)dt \ll \log(|T|+|\nu|). \quad (7.4)$$

Proof: Using entirety of $\Lambda(s,\phi)$ and its Mittag-Leffler expansion,

$$\frac{\Lambda'}{\Lambda}(s)=\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+\nu)+\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s-\nu)+\frac{L'}{L}(s,\phi)=\sum_{\{\rho|\Lambda(\rho)=0\}}\frac{1}{s-\rho}$$

[4]Recall that ϕ is assumed to be both a Hecke and Laplace eigenfunction.

(the sum of the zeroes is actually only conditionally convergent, so the term with ρ should always be summed with the term containing the zero at $1-\rho$). From the Euler product,

$$\frac{L'}{L}(s,\phi) = -\sum_p \sum_{n=1}^{\infty} (\alpha_p^n + \alpha_p^{-n}) p^{-ns} \log p,$$

so the bound $|\alpha_p| \le p^{5/28}$ implies that $|\frac{L'}{L}(2+it,\phi)|$ is uniformly bounded in both ν and t. By Stirling's formula,

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s) = -\frac{1}{2}\log\pi + \frac{1}{2}\log s + O(1/|s|)$$

and

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(2+it+\nu) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(2+it-\nu) \le \log(|t|+|\nu|) + O(1).$$

Thus,

$$\sum_\rho \frac{1}{1+|t-\gamma|^2} \ll \sum_\rho \frac{1}{2+it-\rho} \le \log(|t|+|\nu|) + O(1). \tag{7.5}$$

It follows that there are no more than $O(\log(|T|+|\nu|))$ zeroes between with imaginary part between $T-1$ and $T+1$. By writing

$$\mathrm{Re} \int_{T-1}^{T+1} \frac{\Lambda'}{\Lambda}(1+it,\phi)dt = \mathrm{Re} \int_{T-1}^{T+1} \sum_\rho \frac{1}{1+it-\rho} dt$$

$$= \mathrm{Re} \int_{T-1}^{T+1} \sum_{|\rho-iT|\le 2} \frac{1}{1+it-\rho} dt + \mathrm{Re} \int_{T-1}^{T+1} \sum_{|\rho-iT|>2} \frac{1}{1+it-\rho} dt,$$

invoking (7.5), and using the fact that

$$\mathrm{Re} \int_{1+i(T-1)}^{1+i(T+1)} \frac{ds}{s-\rho} = O(1),$$

we bound each of the terms above by $O(\log(|T|+|\nu|))$. □

Minimal Parabolic Eisenstein Series

We have appended a table of the 36 terms in the Langlands inner-product formula (also referred to as the Maass-Selberg relations) for the minimal parabolic.

The calculation is aided by the identity

$$M(s,(\ell_1,\ell_2,\ell_3)) = \prod_{\substack{1\le i<j\le 3\\ s(i)>s(j)}} R(\ell_i-\ell_j). \tag{7.6}$$

This is a special case of a general result of Langlands ([Lan2],pp. 36-47;[Lan4], p. 134;[Art2],p. 854); the function R here is the same as the one used above in (6.4).

Using a limiting procedure as in the interlude we can compute:

Proposition 7.2. *(Diagonal terms) Let $C=(c,0,-c)$, $\lambda_1=(it_1+\epsilon_1,it_2+\epsilon_2,it_3+\epsilon_3)$, $\lambda_2=(-it_1,-it_2,-it_3)$. Then*

$$\lim_{\epsilon_1,\epsilon_2,\epsilon_3\to 0}\sum_{s\in\Omega(\mathbf{a}_0)}\frac{e^{s(\lambda_1+\bar\lambda_2)(C)}}{[s(\lambda_1+\bar\lambda_2)(\alpha_1^\vee)][s(\lambda_1+\bar\lambda_2)(\alpha_2^\vee)]}\langle M(s,\lambda_1),M(s,\lambda_2)\rangle$$

$$= 3c^2-2c\frac{R'}{R}(it_1-it_2)-2c\frac{R'}{R}(it_2-it_3)-2c\frac{R'}{R}(it_1-it_3)$$
$$+\frac{R'}{R}(it_1-it_2)\frac{R'}{R}(it_2-it_3)$$
$$+\frac{R'}{R}(it_1-it_3)\frac{R'}{R}(it_2-it_3)+\frac{R'}{R}(it_1-it_2)\frac{R'}{R}(it_1-it_3).$$

Proposition 7.3. *For any $\epsilon>0$ and $\lambda\in i\mathbf{a_0}^*$ with $\|\lambda\|$ large, we have that*

$$\int_{\mathcal F}|\Lambda^C E(P_0,x,\lambda)|^2dx = O_\epsilon(\|\lambda\|^\epsilon).$$

Proof: We start with some estimates on $\zeta(s)$. The following may be found in Chapter 3 of [Titchmarsh]: There is an absolute constant $\kappa>0$ such that

$$\frac{1}{\log(|t|+2)} \ll |\zeta(\sigma+it)| \ll \log(|t|+2) \tag{7.7}$$

and

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \ll \log(|t|+2)$$

in the region $\sigma \geq 1 - \frac{\kappa}{\log(|t|+2)}$. Thus

$$R(s) = \frac{Z(1-s)}{Z(1+s)} = \frac{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\pi^{-(1+s)/2}\Gamma\left(\frac{1+s}{2}\right)\zeta(1+s)}$$

$$\ll (1+|\operatorname{Im} s|)^{-\operatorname{Re} s}\log(|\operatorname{Im} s|+2)^2$$

and

$$\frac{R'}{R}(s) \ll \log(|\operatorname{Im} s|+2)$$

for

$$|\operatorname{Re} s| \leq \frac{\kappa}{\log(|\operatorname{Im} s|+2)}.$$

Of course both $R(s)$ and $\frac{R'}{R}(s)$ are analytic in this region because of the nonvanishing.

Fix $\lambda = (it_1, it_2, it_3)$. Then each of the six terms from Proposition 7.2 is trivially $O_\epsilon(\|\lambda\|^\epsilon)$. Of the remaining thirty terms (see the chart in the appendix), some may have singularities when either of the denominators

$$s(\lambda_1+\bar{\lambda}_2)(\alpha_1^\vee) \quad \text{or} \quad s(\lambda_1+\bar{\lambda}_2)(\alpha_2^\vee)$$

vanish. Nevertheless, the sum of all of the terms represents a holomorphic function for all $\lambda \in i\mathbf{a_0}^*$, so the poles cancel with other terms. If λ is such that each

$$|s(\lambda_1+\bar{\lambda}_2)(\alpha_j^\vee)| \geq \frac{\kappa}{2\log(\|\lambda\|+2)},$$

then each term is trivially $O_\epsilon(\|\lambda\|^\epsilon)$ as well (the numerators have modulus one when $\operatorname{Re}\ \lambda = 0$).

Otherwise, if some denominator is small, we will take estimates further away and appeal to the maximum principle. Take a small neighborhood in

$$s_1 = s(\lambda_1+\bar{\lambda}_2)(\alpha_1^\vee) \quad , \quad s_2 = s(\lambda_1+\bar{\lambda}_2)(\alpha_2^\vee).$$

Now, suppose that

$$|s_1| \leq \frac{\kappa}{2\log(\|\lambda\|+2)} \leq |s_2|$$

(the other cases have almost-identical proofs). Then for $|s_1| = \frac{\kappa}{2\log(\|\lambda\|+2)}$, (7.7) shows that each term is $O_\epsilon(\|\lambda\|^\epsilon)$. The maximum modulus principle shows that this bound holds uniformly for $|s_1| \ll \log\|\lambda\|$. $\square$

Maximal Parabolic P_1

Now we turn to the sum in (7.2). We will first consider the simpler case that ϕ is a cusp form. If s is the lone permutation in $\Omega(\mathbf{a}_1, \mathbf{a}_2)$, then the intertwining operator acts as

$$M(s,\lambda)\phi = R(\lambda(\alpha^\vee),\phi)\phi',$$

where ϕ and ϕ' are cusp forms on M_1 and M_2 coming from the same even cusp form on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, and

$$R(s,\phi) = \frac{\Lambda(s,\phi)}{\Lambda(s+1,\phi)}$$

is a ratio formed from ϕ's completed standard L-function (7.3). As before with (7.6), since ϕ is everywhere unramified, this can be derived from Langlands' formula ([Lan2],[Lan4]). Then the Maass-Selberg relations take the following form:

Proposition 7.4. *The inner product*

$$\int_{\mathcal{F}} \left|\Lambda^C E(P,g,\lambda,\phi)\right|^2 dg$$

is a constant multiple of $3/2c - \frac{R'}{R}(\lambda(\alpha^\vee),\phi)$.

Proof: The inner product formula is a constant times

$$\lim_{\epsilon\to 0} \frac{e^{(2it+2\epsilon)c}}{2it+2\epsilon} - \frac{e^{-(2it+2\epsilon)c}}{2it+2\epsilon}\frac{R(2it+2\epsilon)}{R(2it)}.$$

$\square$

Proposition 7.5. *The integral*

$$\int_{t=T-1}^{T+1}\int_{\mathcal{F}}\left|\Lambda^C E(P,g,\lambda,\phi_j)\right|^2 dg dt = O(log(T+\|\lambda_j\|)).$$

Proof: Using the previous proposition, this requires only the estimate on $\int_{T-1}^{T+1}\frac{\Lambda'}{\Lambda}(1+it,\phi)dt$ in Proposition 7.1 (cf. (2.23)). □

Remark 7.6. (On summing over the eigenvalues) The formula (6.3) includes a sum over the $SL_2(\mathbb{Z})\backslash\mathbb{H}$ spectrum as well. Through a more-precise statement of Selberg's Theorem 1.1 such as

$$N(T) = \frac{1}{12}T^2 + O(T\log T),$$

we can bound the spectral points ν_j in the interval $[\nu-1,\nu+1]$ by $O(\sqrt{|\nu|}\log|\nu|)$.

The Constant Function on SL_2

There is only one Eisenstein series left to estimate,

$$E(P_1, g, \lambda, 1),$$

the Eisenstein series induced from the constant function on the maximal parabolic P_1.

The constant functions on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ may be viewed as multiples of

$$Res_{s=1}E_s(z) = \frac{1}{2}Res_{s=1}\sum_{(c,d)=1}\frac{y^s}{|cz+d|^{2s}}.$$

We may ignore the actual value of the constant since we are only trying to get an order-of-magnitude estimate. Thus, $E(P_1,g,\lambda,1)$is a constant multiple of the residue

$$Res_{\delta=0}E(P_0,g,(\frac{1}{2}+it+i\delta,-\frac{1}{2}+it-i\delta,-2it),1). \tag{7.8}$$

Taking the residue of the inner product

$$\int_{\mathcal{F}} |\Lambda^C E(P_1, g, \lambda, 1)|^2 dg$$

is more complicated, though a limiting value must exist since the Eisenstein series is meromorphic there. We will explicitly see this cancellation occurring. Slicker arguments are possible, but we will present a detailed proof for the sake of clarity.

Let

$$\lambda_1 = (\frac{1}{2} + it + i\delta_1 + \epsilon, -\frac{1}{2} + it - i\delta_1 + \epsilon, -2it - 2\epsilon),$$

and

$$\lambda_2 = (\frac{1}{2} - it + i\delta_2\ ,\ \ -\frac{1}{2} - it - i\delta_2\ ,\ 2it).$$

Then we are interested in

$$\lim_{\delta_1, \delta_2 \to 0} \delta_1 \delta_2 \sum_{s_1, s_2 \in \Omega(\mathbf{a}_0)} \frac{e^{(s_1\lambda_1 + s_2\lambda_2)(C)}}{(s_1\lambda_1 + s_2\lambda_2)(\alpha_1^\vee)(s_1\lambda_1 + s_2\lambda_2)(\alpha_2^\vee)} \langle M(s_1, \lambda_1) M(s_2, \lambda_2) \rangle .$$

This expression is holomorphic in δ_1 and δ_2 near zero, so it does not matter how we take the limits $\delta_1, \delta_2 \to 0$. We will first take the residue in δ_1. Of the 36 terms, we will of course ignore those which do not have a pole at $\delta_1 = 0$. These can occur only in $M(s_1, \lambda_1)$, for the denominators do not yet vanish (note $\epsilon, \delta_2 \neq 0$ at this stage). Since

$$M(s_1, \lambda_1) = \prod_{\substack{i<j \\ s_1(i) > s_1(j)}} R(\lambda_{1_i} - \lambda_{1_j}),$$

these permutations s_1 must interchange 1 and 2, which forces

$$s_1 \in \{(12), (13), (321)\}.$$

Next, when the residue at $\delta_2 = 0$ is taken, poles can occur in two different ways: $M(s_2, \lambda_2)$ might have a pole, or a denominator might vanish. The former occurs for $s_2 \in \{(12), (13), (321)\}$ as well.

Proposition 7.7. *Let $s_1 \in \{(12), (13), (321)\}$. If one of the denominator terms*

$$(s_1\lambda_1 + s_2\lambda_2)(\alpha_1^\vee) = 0$$

or

$$(s_1\lambda_1 + s_2\lambda_2)(\alpha_2^\vee) = 0,$$

then

$$s_2 \in \{e, (23), (123)\} = S_3 - \{(12), (13), (321)\},$$

i.e. $M(s_2, \lambda_2)$ *has no pole at* $\delta_2 = 0$.

Proof: Now that

$$\lambda_1 = (\frac{1}{2} + it + \epsilon, -\frac{1}{2} + it + \epsilon, -2it - 2\epsilon)$$

and

$$\lambda_2 = (\frac{1}{2} - it, -\frac{1}{2} - it, 2it),$$

the only possible way to get consecutive entries of $s_1\lambda_1 + s_2\lambda_2$ to equal is if

$$s_1 = s_2(12).$$

$\square$

We see that only the following terms have poles:

$$\text{two from } M\ 's : s_1, s_2 \in \{(12), (13), (321)\}$$

or

$$\text{one from } M,\ \text{one from a denominator} : s_1 = (12), s_2 = e \quad , \quad s_1 = (13), s_2 = (123).$$

Note that $(321) \times (23)$ fails to have poles in the denominators. The 11 terms (see the chart in the appendix) are:

$$(12) \times (12) : \frac{e^{(-1+\epsilon,1+\epsilon,-2\epsilon)(c,0,-c)}}{(-2)(1+3\epsilon)}$$

$$(12) \times (13) : \frac{e^{(-\frac{1}{2}+3it+\epsilon,\epsilon,\frac{1}{2}-3it-2\epsilon)(c,0,-c)}}{(-\frac{1}{2}+3it)(-\frac{1}{2}+3it+3\epsilon)} R(\frac{1}{2} - 3it) R(-\frac{1}{2} - 3it)$$

$$(12) \times (321) : \frac{e^{(-1+\epsilon,\frac{1}{2}+3it+\epsilon,\frac{1}{2}-3it-2\epsilon)(c,0,-c)}}{(-\frac{3}{2}-3it)(6it+3\epsilon)} R(\frac{1}{2} - 3it)$$

$$(13)\times(12): \frac{e^{(-\frac{1}{2}-3it-2\epsilon,\epsilon,\frac{1}{2}+3it+\epsilon)(c,0,-c)}}{(-\frac{1}{2}-3it-3\epsilon)(-\frac{1}{2}-3it)}R(\frac{1}{2}+3it+3\epsilon)R(-\frac{1}{2}+3it+3\epsilon)$$

$$(13)\times(13): \frac{e^{(-2\epsilon,-1+\epsilon,1+\epsilon)(c,0,-c)}}{(1-3\epsilon)(-2)}R(\frac{1}{2}+3it+3\epsilon)R(-\frac{1}{2}+3it+3\epsilon)R(\frac{1}{2}-3it)R(-\frac{1}{2}-3it)$$

$$(13)\times(321): \frac{e^{(-\frac{1}{2}-3it-2\epsilon,-\frac{1}{2}+3it+\epsilon,1+\epsilon)(c,0,-c)}}{(-6it-3\epsilon)(-\frac{3}{2}+3it)}R(\frac{1}{2}+3it+3\epsilon)R(-\frac{1}{2}+3it+3\epsilon)R(-\frac{1}{2}-3it)$$

$$(321)\times(12): \frac{e^{(-1+\epsilon,\frac{1}{2}-3it-2\epsilon,\frac{1}{2}+3it+\epsilon)(c,0,-c)}}{(-\frac{3}{2}+3it+3\epsilon)(-6it-3\epsilon)}R(\frac{1}{2}+3it+3\epsilon)$$

$$(321)\times(13): \frac{e^{(-\frac{1}{2}+3it+\epsilon,-\frac{1}{2}-3it-2\epsilon,1+\epsilon)(c,0,-c)}}{(6it+3\epsilon)(-\frac{3}{2}-3it-3\epsilon)}R(\frac{1}{2}+3it+3\epsilon)R(\frac{1}{2}-3it)R(-\frac{1}{2}-3it)$$

$$(321)\times(321): \frac{e^{(-1+\epsilon,-2\epsilon,-1+\epsilon)(c,0,-c)}}{(-1+3\epsilon)(-1-3\epsilon)}R(\frac{1}{2}+3it+3\epsilon)R(\frac{1}{2}-3it).$$

The next two terms had limits taken in $\delta_1,\delta_2\to 0$ and, up to constants from the residues, are

$$(12)\times e: \frac{e^{(\epsilon,\epsilon,-2\epsilon)(c,0,-c)}}{3\epsilon}$$

$$(13)\times(123): \frac{e^{(-2\epsilon,\epsilon,\epsilon)(c,0,-c)}}{-3\epsilon}R(\frac{1}{2}+3it+3\epsilon)R(-\frac{1}{2}+3it+3\epsilon)R(\frac{1}{2}-3it)R(-\frac{1}{2}-3it).$$

Note that

$$R(-\frac{1}{2}+x)R(\frac{1}{2}+x)=\frac{Z(-\frac{1}{2}+x)}{Z(\frac{3}{2}+x)}=\frac{Z(\frac{3}{2}-x)}{Z(\frac{3}{2}+x)}$$

by the functional equation. Accordingly, we may set $\epsilon=0$ in the first 9 terms; the last two involve a limit at $\epsilon=0$ producing derivatives.

Proposition 7.8.

$$(i) \quad R(\frac{1}{2}+it) \ll 1 \tag{7.9}$$

$$(ii) \qquad \left| R(-\frac{1}{2}+it)R(\frac{1}{2}+it) \right| = 1 \tag{7.10}$$

$$(iii) \qquad \frac{d}{d\epsilon}\frac{Z(\frac{3}{2}+it+\epsilon)}{Z(\frac{3}{2}+it-\epsilon)} \, |_{\epsilon=0} = O(\log t). \tag{7.11}$$

Proof: (ii) follows from the facts that $Z(\bar{s}) = \overline{Z(s)}$ and $R(s)R(-s) = 1$. For (i), recall

$$R(s) = \sqrt{\pi}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{\zeta(s)}{\zeta(s+1)}.$$

By Stirling's formula

$$\left| \frac{\Gamma(\frac{1}{4}+\frac{it}{2})}{\Gamma(\frac{3}{4}+\frac{it}{2})} \right| \sim \sqrt{\frac{2}{t}} \text{ as } t \to \infty,$$

and by the convexity bound,

$$\zeta(\frac{1}{2}+it) = O_\epsilon(t^{1/4+\epsilon}) \tag{7.12}$$

(better bounds can be obtained – see [Titchmarsh]). Also, taking the logarithm of the Euler product of $\zeta(s)$ we find

$$-\log\zeta(\frac{3}{2}+it) = \sum_{p \text{ prime}} \log(1-p^{-3/2-it}) = O(1),$$

which proves (i). Differentiating,

$$\frac{\zeta'}{\zeta}(\frac{3}{2}+it) = -\sum_p \frac{p^{-3/2-it}\log p}{1-p^{-3/2-it}} = O(1)$$

also. So

$$\frac{Z'}{Z}(\frac{3}{2}+it) = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{3}{4}+\frac{it}{2}) + \frac{\zeta'}{\zeta}(\frac{3}{2}+it) = O(\log t),$$

proving (iii). □

Summarizing these pointwise bounds,

Proposition 7.9. *For T large,*

$$\int_{T-1}^{T+1}\int_{\mathcal{F}} |\Lambda^C E(P_1, g, (it, it, -2it), 1)|^2 dg dt = O_\epsilon(T^\epsilon).$$

Assembling the Bounds

Proof of (5.6): Choose c large enough so that $\mathcal{F}_C$ is contained in $\{x \mid \hat{\tau}_{P_1}(H_0(x-C)), \hat{\tau}_{P_2}(H_0(x-C)) \le 0\}$. Then the truncation does not affect $\mathcal{F}_C$ and

$$\int_{\mathcal{F}_C} |E(g,\lambda)|^2 dg \le \int_{\mathcal{F}} |\Lambda^C E(g,\lambda)|^2 dg$$

for each Eisenstein series $E(g,\lambda)$ on $\mathcal{H}$. In the propositions we bounded local integrals of all of the Eisenstein series as growing slower than any polynomial, with the exception of the maximal parabolic Eisenstein series (Remark 7.6). For this we must sum over the $SL_2(\mathbb{Z})\backslash\mathbb{H}$ spectrum as well, so the contribution of the left-hand side of (7.2) near the point λ is bounded by $O_\epsilon(\|\lambda\|^{1+\epsilon})$.

We note in comparison with Proposition 2.7 that (4.4) shows

$$\hat{g}_{t\Sigma}(\lambda) = \begin{cases} 1 + O_m(\mathrm{dist}(\lambda, \partial(t\Sigma))^{-m}), & \lambda \in t\Sigma, \\ O_m(\mathrm{dist}(\lambda, \partial(t\Sigma))^{-m}), & \lambda \in t\Sigma \end{cases}, \qquad m \ge 0.$$

Since we have shown that all the Eisenstein series grow polynomially, we may switch the order of integration, ignore the tail, and conclude (see (5.5))

$$\int_{\mathcal{F}_C} Eis_{t\Sigma}(x,x)dx \ll \int_{t\Sigma} (1+\|\lambda\|)^{1+\epsilon} d\lambda.$$

Since $\beta(\lambda)$ grows at the rate of $\|\lambda\|^3$ in almost all directions, we have completed the proof that

$$\int_{\mathcal{F}_C} Eis_{g_{t\Sigma}}(x,x)dx = o\left(\int_{t\Sigma} \beta(\lambda)d\lambda\right).$$

$\square$

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