1. (10 pts) Find the equation of the plane which passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$. Find the distance from the point $(1,2,3)$ to this plane.

Solution: $PQ := \langle -3,1,-7\rangle$ and $PR = \langle 0,-5,-5\rangle$. We have $PQ \times PR = \langle -40,-15,15\rangle$. The equation of the plane is then

$$-40(x-1) - 15(y+1) + 15(z-1) = 0 \text{ or } -40x - 15y + 15z + 10 = 0.$$

The distance form the point to the plane is

$$\frac{|-40(1) - 15(2) + 15(3) + 10|}{\sqrt{40^2 + 15^2 + 15^2}} = \frac{3}{\sqrt{82}}.$$

2. (10 pts) Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

Solution: As in Problem #1 we have $PQ := \langle -3,1,-7\rangle$, $PR = \langle 0,-5,-5\rangle$, and $PQ \times PR = \langle -40,-15,15\rangle$. The area is $\frac{1}{2}|\langle -40,-15,15\rangle| = \frac{1}{2}\sqrt{40^2+15^2+15^2} = \frac{5\sqrt{82}}{2}$.

3. (10 pts) Find the length of the curve $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4t\mathbf{j} + 3\cos t\mathbf{k}$ from $t = 2$ to $t = 4$.

Solution: $r'(t) = 3\cos(t)i + 4j - 3\sin(t)k$, so $|r'(t)| = 5$.

$$L = \int_2^4 |r'(t)|\, dt = \int_2^4 5\, dt = 10.$$

4. (10 pts) Find parametric equations for the line passing through the point $(2,-7,5)$ which is parallel to the line $x = 2t+7,\ y = -3t+5,\ z = t+1$.

Solution: $x = 2t+2,\ y = -3t + -7,\ z = t+5$.

5. (10 pts) Find the equation of the tangent plane to the surface $z = \sin(x^2+y)$ at the point $(1,-1,0)$.

Solution: The normal vector to the surface is $\langle -2,-1,1\rangle$, so the equation of the plane is

$$-2(x-1) - (y+1) + z = 0.$$

6. (10 pts) Find the tangential and normal components of the acceleration vector if

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$$

Your answers should be vectors.

Solution: $r'(t) = \langle 1 - \cos(t), \sin(t)\rangle$ and $r''(t) = \langle \sin(t), \cos(t)\rangle$.

$$a_{\mathbf{N}} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{|cos(t) - 1|}{\sqrt{2 - 2\cos(t)}} = \frac{1}{2}\sqrt{1 - \cos(t)}.$$

$$a_{\mathbf{T}} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}.$$

The "vector" part was just that I wanted you to write

$$\left(\frac{1}{2}\sqrt{1 - \cos(t)}\right)\mathbf{N} \quad \text{and} \quad \left(\frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}\right)\mathbf{T}.$$

7. (10 pts) At what point does the curve $y = e^x$ have maximum curvature?

Solution: $f(x) = e^x, f'(x) = e^x,$ and $f''(x) = e^x$, so

$$k = \frac{e^x}{(1 + e^{2x})^{\frac{3}{2}}}, \text{ so } k'(x) = -\frac{e^x\left(-1 + 2\,e^{(2\,x)}\right)}{(1 + e^{(2\,x)})^{(5/2)}}$$

$k'(x) = 0$ when $-1 + 2\,e^{(2\,x)} = 0$, which means that $e^{2x} = 1/2$ or $x = -\frac{1}{2}\ln(2)$. $e^{-\frac{1}{2}\ln(2)} = \frac{\sqrt{2}}{2}$, so the point we want is $\left(-\frac{1}{2}\ln(2), \frac{\sqrt{2}}{2}\right)$.

8. (10 pts) If $xyz = \sin(x + y + z)$, find $\dfrac{\partial z}{\partial x}$.

Solution: Let $F(x, y, z) = xyz - \sin(x + y + z) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz - \cos(x + y + z)}{xy - \cos(x + y + z)}.$$

9. (10 pts) The temperature atna point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in degrees centigrade and x, y, and z are measured in meters.

a. (7 points) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

$$\nabla T = \langle -400\,x\,e^{(-x^2-3\,y^2-9\,z^2)}, -1200\,y\,e^{(-x^2-3\,y^2-9\,z^2)}, -3600\,z\,e^{(-x^2-3\,y^2-9\,z^2)}\rangle$$

$\langle 3,-3,3\rangle - \langle 2,-1,2\rangle = \langle 1,-2,1\rangle$, so the direction is $u = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$. At $(2,-1,2)$, $\nabla T = \langle -800\,e^{(-43)},\ 1200\,e^{(-43)},\ -7200\,e^{(-43)}\rangle$.

b. (3 points) In which direction does the temperature increase the fastest at P?

In the direction of the gradient or $\dfrac{\langle -800\,e^{(-43)},\ 1200\,e^{(-43)},\ -7200\,e^{(-43)}\rangle}{|\langle -800\,e^{(-43)},\ 1200\,e^{(-43)},\ -7200\,e^{(-43)}\rangle|}$.

10. (10 pts) Find the angle between the main diagonal of a cube and the diagonal of one of its faces.