1. (10 points) Find the local maxima, minima, and saddles of $f(x,y) = x^3 - 3xy - y^3$.

$\partial f/\partial x = 3x^2 - 3y = 0$, $\partial f/\partial y = -3x - 3y^2 = 0 \implies y = x^2$ and $x = -y^2$. Therefore, $x = -x^4$ and $x = 0$ or $x = -1$ and the critical points are $(-1,1)$ and $(0,0)$. $D = -36xy - 9$, so the critical point at $(0,0)$ is a saddle and the critical point at $(-1,1)$ is a local maximum.

2. (10 points) Find the absolute maximum and minimum of $f(x,y) = xy$, subject to the constraint $9x^2 + y^2 = 4$. $\nabla f = \langle y, x\rangle$, and $\nabla g = \langle 18x, 2y\rangle$.The critical points occur when $\nabla f = \lambda g$, which happens when

$$\begin{vmatrix} y & x \\ 18x & 2y \end{vmatrix} = 2y^2 - 18x^2 = 0$$

This gives us $y = \pm 3x$. Substituting into $9x^2 + y^2 = 4$, we get $18x^2 = 4$, so $x = \pm\sqrt{2}/3, y = \pm\sqrt{2}$. It follows that the maximum value of $f(x,y) = xy$ is 2/3 and the minimum value is -2/3 when $9x^2 + y^2 = 4$.

3. (10 points) Change the order of integration in the integral $\int_0^3 \int_{y^2}^9 f(x,y)\,dx\,dy$. Do NOT try to evaluate the integral.

The answer is $\int_0^9 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$.

4. (10 points) find the volume bounded by the paraboloid $z = 3x^2 + 3y^2$ and the plane $z = 3$.

The integral is $\int\int_D 3 - 3x^2 - 3y^2 \; dydx$, where D is the region $x^2 + y^2 \leq 1$, so in polar coordinates we have $\int_0^{2\pi} \int_0^1 (3 - 3r^2)r \; drd\theta = 3\pi/2$.

5. (10 points) Set up the triple integral to find $\int\int\int_E (x + 2y)\,dV$, where E is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + 3z = 6$. Please do NOT try to integrate the integral!

$$\int_0^3 \int_0^{6-2x} \int_0^{\frac{6-y-2x}{3}} (x + 2y)\,dz\,dy\,dx.$$

6. (10 points) Show that the integral $\int_C (12y^2 - 12x^3y^3)dx + (24xy - 9x^4y^2)dy$ is independent of path and evaluate the integral if C is the path $x(t) = 1 + 2t,\; y(t) = 1 + t,\; 0 \leq t \leq 1$.

Let $P = 12y^2 - 12x^3y^3$ and $Q = 24xy - 9x^4y^2$. Then $\partial P/\partial y = 24y - 36x^3y^2 = \partial Q/\partial x$, so the integral is independent of path. $\langle P, Q\rangle = \nabla f$, where $f = 12xy^2 - 3x^4y^3$. By the Fundamental Theorem, the value of the integral is then $f(3,2) - f(1,1) = -1809$.

7. (10 points) Find the Jacobian of the transformation $x = u - v^2$, $y = u + v^2$.

$$\frac{\partial(x,y)}{\partial(u,v)} = 4v.$$

8. (10 points) Find $\int_C y\,dx + xy\,dy$, along the curve $x = y^3$ from $(1,1)$ to $(8,2)$.

Parameterize by $x = t^3,\ y = t,\ 0 \le t \le 2.$

$$\int_1^2 t(3t^2)dt + t^3(t)dt = 349/20.$$

9. (10 points) Use Green's Theorem to evaluate the line integral $\int_C (x^3 - y^3)dx + (x^3 + y^3)dy$, where C is the boundary of the region between the circles $x^2 + y^2 = 1$, and $x^2 + y^2 = 4$.

$\int_C (x^3 - y^3)dx + (x^3 + y^3)dy = \int\int_D 3x^2 + 3y^2\,dA = \int_0^{2\pi}\int_1^2 3r^2\,r\,dr\,d\theta = 45\pi/2.$

10. (10 points) Find the centroid of the solid which lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.

volume $= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 10\pi.$

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho\cos(\phi)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 21\pi.$

$\bar{x} = \bar{y} = 0$ by symmetry. $\quad \bar{z} = 21\pi/10\pi = 21/10.$