MA251 Exam #1 Oct. 13, 2000

1. (11 points) a. Find a vector perpendicular to the plane through the points $A(1, 0, 0)$, $B(2, 0, -1)$, $C(1, 4, 3)$. b. Find the area of $\triangle ABC$.

a. $\vec{AB} = \langle 1, 0, -1\rangle$, $\vec{AC} = \langle 0, 4, 3\rangle$, $\vec{AB} \times \vec{AC} = \langle 4, -3, 4\rangle$.

b. $\frac{1}{2}|\langle 4, -3, 4\rangle| = \frac{1}{2}\sqrt{41}$.

2. (11 points) Find the velocity and position of a particle that starts at the origin at time $t = 0$ with velocity $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and has acceleration $\mathbf{a}(t) = t\,\mathbf{i} + \mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{v} = \langle t^2/2 + 1, t + 2, t^3/3 + 1\rangle$, $\mathbf{r} = \langle t^3/6 + t, t^2/2 + 2t, t^4/12 + t\rangle$

3. (11 points) For the curve $\mathbf{r}(t) = \langle t^3/3,\ t^2/2,\ t\rangle$, find the unit tangent vector and the curvature.

$$\mathbf{v} = \langle t^2, t, 1\rangle,\ \mathbf{T} = \frac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$$

$\mathbf{a} = \langle 2t, 1, 0\rangle$

$|\mathbf{v} \times \mathbf{a}| = |\langle -1, 2t, -t^2\rangle|$

$$\kappa = \frac{(1 + 4t^2 + t^4)^{\frac{1}{2}}}{(1 + t^2 + t^4)^{\frac{3}{2}}}$$

4. (11 points) a. Find the angle between the planes $x + y + 2z = 1$ and $x - 2y + z = 1$.

b. Find parametric equations for the line of intersection of these planes.

a. $\cos\theta = \dfrac{\langle 1, 1, 2\rangle \cdot \langle 1, -2, 1\rangle}{\sqrt{6}\sqrt{6}} = \dfrac{1}{6}$, so $\theta = 80.4°$

b. If $x = 0$, we solve and find that $(1, 0, 0)$ is in the intersection of the planes. The cross product of the normal vectors is $\langle 1, 1, 2\rangle \times \langle 1, -2, 1\rangle = \langle 5, 1, -3\rangle$, so $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle 5, 1, -3\rangle$ and $x = 5t + 1,\ y = t,\ z = -3t$ is a set of parametric equations for the intersection.

5. (11 points) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x^2 + y^2}$ or prove that the limit doesn't exist.

If $x = t,\ y = 0$, the limit is 1 and if $x = y = t$, the limit is 2. Therefore, the limit doesn't exist.

6. (11 points) The radius of a cylinder is increasing at .2 m/s and its height is decreasing at a rate of .1 m/s. How fast is the cylinder's volume changing when $r = 2$ and $h = 3$? ($V = \pi r^2 h$.)

$\dfrac{dV}{dt} = 2\pi r h\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt}$, so $\dfrac{dV}{dt} = 2.4\pi - .4\pi = 2\pi$ at the time specified.

7. (11 points) Find $\dfrac{\partial z}{\partial x}$ if $xy^3z^2 + x^3y^2z = x + y^2 + z$.

Set $F(x, y, z) = xy^3z^2 + x^3y^2z - x + y^2 - z$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{y^3z^2 + 3x^2y^2z - 1}{2xy^3z + x^3y^2 - 1}$.

8. (11 points) Find the directional derivative of the function $f(x, y, z) = x^2y + x\sqrt{1 + z}$ at the point $(1, 2, 3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find the maximum rate of change of $f$ at this point.

$\nabla f = \langle 2xy + \sqrt{1 + z}, x^2, x(1 + z)^{-\frac{1}{2}}\frac{1}{2}\rangle = \langle 6, 1, \frac{1}{4}\rangle$. $D_u f = \langle 6, 1, \dfrac{1}{4}\rangle \cdot \dfrac{\langle 2, 1, -2\rangle}{3} = \dfrac{25}{6}$.

The maximum rate of change is $|\nabla f| = \sqrt{37\frac{1}{16}}$.

9. (12 points) Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^3 - 6xy + 8y^3$.

$\nabla f = \langle 3x^2 - 6y, -6x + 24y^2\rangle = 0$ gives us $x^2 = 2y$ and $x = 4y^2$, so $16y^4 = 2y$ and $y = 0$ or $y = \frac{1}{2}$. The critical points are therefore $(0, 0)$ and $(1, \frac{1}{2})$. $D = 144xy - 36$, so $(0, 0)$ is a saddle and $(1, \frac{1}{2})$ is a local minimum.