Name ______________________________________ MA251 Exam #2 Nov. 22, 2000

1. (13 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = x^2y$ on the ellipse $x^2 + 2y^2 = 6$.

$\nabla f = \langle 2xy, x^2 \rangle$ and $\nabla g = \langle 2x, 4y \rangle$. $\begin{vmatrix} 2xy & x^2 \\ 2x & 4y \end{vmatrix} = 0 \Rightarrow 8xy^2 - 2x^3 = 0 \Rightarrow x = 0$ and $y = \pm\sqrt{3}$ or $\pm\, 2y = x$. In this last case, $y^2 = 1$ and $x = \pm 2$. The critical points are $(0, \pm\sqrt{3})$, and $(\pm 1, \pm 2)$. The max and min values of $f$ are $\pm 4$.

2. (12 points) Evaluate the integral $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$ by reversing the order of integration and integrating.

$$\int_0^2 \int_0^{x^3} e^{x^4} dy\, dx = \frac{e^{16} - 1}{4}$$

3. (12 points) Find the volume of the solid above the paraboloid $z = x^2 + y^2$ and below the half-cone $z = \sqrt{x^2 + y^2}$.

The picture is a cross-section because the 3D picture doesn't show much. The surfaces intersect where $z = z^2$, which implies $z = 0$ or $z = 1$. $z = 1$ is the one we want, so $x^2 + y^2 = 1$ is the intersection.

The volume is $\int\int_D \sqrt{x^2 + y^2} - (x^2 + y^2)\, dA$, where $D$ is the region $x^2 + y^2 \leq 1$. The integral is easy in polar coordinates. We have $\displaystyle\int_0^{2\pi} \int_0^1 (r - r^2) r\, dr\, d\theta = \frac{\pi}{6}$.

4. (13 points) Use spherical coordinates to evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^2 dz\, dy\, dx$.

Working from the outside in, the projection onto the $x$-axis is $[0,1]$, the projection onto the $xy$-plane is a quarter circle, and the entire object is the intersection of the unit ball with the first octant. The integral is $\displaystyle\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^4 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \frac{\pi}{14}$.

5. (13 points) Find the mass and center of mass of a thin wire in the shape of a quarter-circle $x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0$ if the density function is $\rho(x, y) = x + y$.

Mass $= \int_C \rho\, ds$. $C$ is given by $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq \pi/2$, so $ds = dt$ and the integral is $\int_0^{\pi/2} (\cos(t) + \sin(t))\, dt = 2$.

$M_y = \int_C x\rho\, ds = \int_0^{\pi/2} \cos(t)(\cos(t) + \sin(t))\, dt = \pi/4 + 1/2$. We have $\bar{x} = M_y/\text{Mass} = \pi/8 + 1/4$. By symmetry, $\bar{y} = \pi/8 + 1/4$, as well.

6. (12 points) Find $\displaystyle\int_C (2y^2 - 12x^3y^3)\, dx + (4xy - 9x^4y^2)\, dy$, where $C$ is any path from $(1, 1)$ to $(3, 2)$.

$(2y^2 - 12x^3y^3)\, dx + (4xy - 9x^4y^2)\, dy = df$, where $f(x, y) = 2xy^2 - 3x^4y^3$, so the integral is $f(3, 2) - f(1, 1) = -1919$.

7. (13 points) Evaluate $\oint_C y^2\, dx + 3xy\, dy$, where $C$ is the boundary of the semi-annular region $D$ in the upper half plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\oint_C y^2\, dx + 3xy\, dy = \int\int_D 3y - 2y\, dA = \int_0^{\pi} \int_1^2 r\sin(\theta) r\, dr\, d\theta = \frac{7\pi}{3}.$$

8. (12 points) Find the curl and divergence of the vector field

$$\mathbf{F} = e^x \sin y\, \mathbf{i} + e^x \cos y\, \mathbf{j} + z\, \mathbf{k}.$$

$\operatorname{div} \mathbf{F} = 1$ and $\operatorname{curl} \mathbf{F} = 0$.