MA251 Exam #1 Oct. 15, 2001

1. a. (7 pts) Find the equation of the plane which contains the points $A(1,1,1)$, $B(2,-1,3)$ and $C(-1,2,1)$.
b. (4 pts) What is the area of $\triangle ABC$?

$\mathbf{AB} = \langle 1,-2,2\rangle$, $\mathbf{AC} = \langle -2,1,0\rangle$, $\mathbf{AB}\times\mathbf{AC} = \langle -2,-4,-3\rangle$, so the equation of the plane is $-2(x-1)-4(y-1)-3(z-1)=0$. The area of the triangle is $\dfrac{\sqrt{29}}{2}$.

2. (11 points) Find a vector equation for the line through $(1,3,-1)$ which is parallel to the line of intersection of the planes $x+y+z=1$ and $2x-y+3z=6$.

The line is parallel to the cross product of the two normal vectors. $\langle 1,1,1\rangle \times \langle 2,-1,3\rangle = \langle 4,-1,-3\rangle$, so the equation is $\mathbf{r}(t) = \langle 1,3,-1\rangle + t\langle 4,-1,-3\rangle$.

3. (11 points) Find the arc length of the curve $\mathbf{r}(t) = (\cos t + t\sin t)\,\mathbf{i} + (t\cos t - \sin t)\,\mathbf{j} + t^2\,\mathbf{k}$ from $t=0$ to $t=\pi/2$.

$\mathbf{r}'(t) = \langle t\cos t, -t\sin t, 2t\rangle$, so $|\mathbf{r}'(t)| = \sqrt{5}t$ and $L = \displaystyle\int_0^{\frac{\pi}{2}} \sqrt{5}t\,dt = \frac{\pi^2\sqrt{5}}{8}$.

4. (11 points) Find the tangential and normal components of the acceleration vector if $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$.

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t+18t^3}{\sqrt{1+4t^2+9t^4}}.$$

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{36t^4+36t^2+4}}{\sqrt{1+4t^2+9t^4}}.$$

5. a. (7 points) Find $\dfrac{\partial z}{\partial x}$ if $e^{x^2+y^3+z^5} = xyz$.

b. (4 points) Find $\dfrac{\partial z}{\partial x}$ if $z = (\sin y)^{\cos x}$.

a. $F(x,y,z) = e^{x^2+y^3+z^5} - xyz$, so $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2xe^{x^2+y^3+z^5} - yz}{5z^4e^{x^2+y^3+z^5} - xy}$.

b. $\dfrac{\partial z}{\partial x} = (\sin y)^{\cos x}\ln(\sin y)(-\sin x)$.

6. (11 points) Find $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$ or show that the limit does not exist.

If $x=0$, $y=t$, $\lim_{t\to 0} \frac{t^2}{t^2} = 1$.

If $x=t$, $y=t$, $\lim_{t\to 0} \frac{(t+t)^2}{t^2+t^2} = 2$.

Therefore, the limit does not exist.

7. (11 points) The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

$P = \frac{8.31\,T}{V}$, so $dP = 8.31\frac{dT}{V} - \frac{8.31\,T}{V^2}dV$ and $\triangle P \approx 8.31\left(\frac{-5}{12}\right) - \frac{8.31\cdot 310}{12^2}\cdot(.3) = -8.83$

8. (11 points) Find the equations of the tangent plane and normal line to the surface described by the function $f(x,y,z) = \frac{x}{y} + \frac{y}{z}$ at the point $(4,2,1)$.

$\nabla f = \langle \frac{1}{y}, -\frac{x}{y^2} + \frac{1}{z}, -\frac{y}{z^2}\rangle = \langle \frac{1}{2}, 0, -2\rangle$, so the equation of the tangent plane is $\frac{1}{2}(x-4) - 2(z-1) = 0$. The equation of the normal line is $\mathbf{r}(t) = \langle 4,2,1\rangle + t\langle \frac{1}{2}, 0, -2\rangle$.

9. (12 points) If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\end{aligned}$$