MA251 Exam #2 Nov. 16, 2001

Problem 1. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x + y^2$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 4$.

Solution: $\nabla f := \langle 1, 2y \rangle$ and $\nabla g := \langle 2x, 4y \rangle$. $\begin{vmatrix} 1 & 2y \\ 2x & 4y \end{vmatrix} = 4y - 4xy = 0$. So $4y(1 - x) = 0$ and $y = 0$ or $x = 1$. If $y = 0$, substituting into $x^2 + 2y^2 = 4$ gives $x = \pm 2$ and $f = \pm 2$. If $x = 1$, $2y^2 = 3$, so $y = \pm\sqrt{3/2}$ and $f = 5/2$. Thus, the minimum is -2 at $(-2, 0)$ and the maximum is 5/2 at $(1, \pm\sqrt{3/2})$.

Problem 2. Evaluate the iterated integral $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$.

Solution: $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv = \int_0^1 v\sqrt{1 - v^2}\, dv$. Let $w = 1 - v^2$. Then $dw = -2v\, dv$ and $\int_0^1 v\sqrt{1 - v^2}\, dv = -\frac{1}{2} \int_1^0 \sqrt{w} dw = \frac{1}{3}$. Don't forget to change the u-limits to w-limits!

Problem 3. Change the integral $\int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2)^{\frac{3}{2}}\, dx\, dy$ to polar coordinates. **Do not evaluate the integral.**

Solution: The region of integration is the part of $x^2 + y^2 = a^2$ in the first quadrant, so the integral is $\int_0^a \int_0^a r^3\, r\, dr\, d\theta$.

Problem 4. Evaluate the integral $\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$ by reversing the order of integration.

Solution: The reversed integral is $\int_0^9 \int_0^{\sqrt{x}} y \cos(x^2)\, dy\, dx = \int_0^9 \frac{x}{2} \cos(x^2)\, dx = \frac{1}{4} \sin x^2 \Big|_{x=0}^{x=9} = \frac{\sin 81}{4}$.

Problem 5. Set up a double integral in polar coordinates to find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy$-plane, and inside the cylinder $x^2 + y^2 = 2x$. **Do not evaluate the integral**.

Solution: In polar coordinates, $x^2+y^2 = 2x$ becomes $r^2 = 2r\cos(\theta)$. Dividing out one $r$ gives $r = 2\cos(\theta)$. In polar coordinates, $z = x^2+y^2 = r^2$, so the integral is $\int_{-\frac{\Pi}{2}}^{\frac{Pi}{2}} \int_0^{2\cos(\theta)} r^2\, r\, dr\, d\theta$.

Problem 6. Set up the triple integrals to find the $x$-coordinate of the center of mass of a solid of constant density that is bounded by the parabolic cylinder $y^2 = x$ and the planes $x = z,\ z = 0$, and $x = 1$. **Do not evaluate the integrals**.

DUE TO A TYPOGRAPHICAL ERROR ON THE EXAM, PROBLEM 6 WAS NOT GRADED.

Solution: To find the mass, we integrate $m = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x k\, dz\, dx\, dy$. To find $M_{yz}$, we integrate $\int_{-1}^{1} \int_{y^2}^{1} \int_0^x kx\, dz\, dx\, dy$. $\bar{x} = \dfrac{M_{yz}}{m}$.

Problem 7. Set up a triple integral in spherical coordinates to find $\int\int\int_E x^2\, dV$, where $E$ lies between the spheres $\rho = 1$ and $\rho = 3$ and above the cone $\phi = \pi/4$. **Do not evaluate the integral**.

Solution: $\int_0^{2\pi} \int_0^{\pi/4} \int_1^3 (\rho \sin\phi \cos\theta)^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Problem 8. Evaluate $\int_C (x - 2y^2)\, dy$, where $C$ is the arc of the parabola $y = x^2$ from $(-2, 4)$ to $(1, 1)$.

Solution: $x(t) = t,\ y(t) = t^2$, so the integral is $\int_{-2}^{1} (t - 2(t^2)^2)2t\, dt = 48$.

Problem 9. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^2 y^3\,\mathbf{i} - y\sqrt{x}\,\mathbf{j}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $0 \leq t \leq 1$.

Solution: $\displaystyle\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_0^1 \langle (t^2)^2(-t^3)^3, -(-t^3)\sqrt{t^2}\rangle \cdot \langle 2t, -3t^2\rangle\, dt = \int_0^1 -2t^{14} - 3t^6\, dt = \frac{-59}{105}.$