MA251 Midterm #1 October 10, 2002 Answer Key

Where the test problems came from...

#1 is problem 1 in section 12.3. It was assigned for homework.

#2 is problem 44 in section 12.5. Problem 43 is similar and was a homework problem.

#3 is example 5 on p. 846. It was also a workshop problem.

#4 is problem 11 on p. 856. It was assigned for homework.

#5 is example 2 on p. 890.

#6a is problem 28 on p. 906. #6b is problem 44c on p. 907.

#7 is problem 42 on p. 938. Problem 41 is similar and was assigned for homework.

#8 is problem 7 on p. 947. It was assigned for homework.

#9 is problem 31 on p. 962. It is similar to problem 19 on p. 916, which was a homework problem.

1. (11 points) Which of the following expressions are meaningful? Which are meaningless? The dots in this problem are dot products.

   i. $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ meaningless - you can't dot a scalar and a vector
   ii. $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ meaningful
   iii. $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ meaningful
   iv. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ meaningful
   v. $|a| \cdot (\mathbf{b} + \mathbf{c})$ meaningless - you can't dot a scalar and a vector
   vi. $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ meaningless - you can't add a scalar and a vector

2. (11 points) Determine whether the planes $2x + 2y - z = 4$ and $6x - 3y + 2z = 5$ are parallel, perpendicular, or neither. If neither, find the angle between them.

$\mathbf{n}_1 = \langle 2, 2, -1 \rangle$, $\mathbf{n}_2 = \langle 6, -3, 2 \rangle$. $\mathbf{n_1} \cdot \mathbf{n}_2 = 4$, $\mathbf{n_1} \times \mathbf{n}_2 = \langle 1, -10, -18 \rangle$. $\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{4}{21}$, $\theta = \cos^{-1}(\frac{4}{21})$.

3. (11 points) Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

$|\mathbf{r}(t)| = c$, so $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$. Differentiating, we have $2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, so $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$.

4. (11 points) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and the curvature of the curve $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$.

$\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle$. $|\mathbf{r}'(t)| = \sqrt{4 + 25} = \sqrt{29}$, so $\mathbf{T}(t) = \langle \frac{2}{\sqrt{29}} \cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}} \sin t \rangle$. $\mathbf{T}'(t) = \langle -\frac{2}{\sqrt{29}} \sin t, 0, -\frac{2}{\sqrt{29}} \cos t \rangle$, $|\mathbf{T}'(t)| = \frac{2}{\sqrt{29}}$, so $\mathbf{N}(t) = \langle -\sin t, 0, -\cos t \rangle$. $\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{2}{29}$.

5. (11 points) If $f(x,y) = \dfrac{xy}{(x^2+y^2)}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Explain your answer.

If we let $x = t$, $y = 0$, we have $\lim_{t\to 0} \frac{0}{t^2+0^2} = 0$. If we let $x = t$, $y = t$, the we get $\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \frac{1}{2}$. Since these are unequal, the limit does not exist.

6a. (5 points) If $f(x,y) = x^{\frac{y}{z}}$, find f_x.

$f_x = \frac{y}{z} x^{\frac{y}{z}-1}$

6b. (6 points) If $z = f\left(\dfrac{x}{y}\right)$, find $\dfrac{\partial z}{\partial y}$.

Set $u = \dfrac{x}{y}$. Then $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} = f'(u)\left(\dfrac{-x}{y^2}\right) = f'\left(\dfrac{x}{y}\right)\left(\dfrac{-x}{y^2}\right)$.

7. (11 points) Find the equations of the tangent plane and normal line to the surface $xe^{yz} = 1$ at the point (1, 0, 5).

Set $F(x,y,z) = xe^{yz}$. Then $\nabla F = \langle e^{yz},\, xze^{yz},\, xye^{yz}\rangle$ and this is $\langle 1,\, 5,\, 0\rangle$ at $(1,0,5)$. The tangent plane is then $1(x-1) + 5(y-0) + 0(z-5) = 0$ and the normal line is $\mathbf{r}(t) = \langle 1,0,5\rangle + t\langle 1,5,0\rangle$.

8. (11 points) Find all local maxima, local minima, and saddles of the function $f(x,y) = x^2 + y^2 + x^2y + 4$.

$\nabla f = \langle 2x + 2xy,\, 2y + x^2\rangle$. If $2x + 2xy = 2x(1+y) = 0$, then $x = 0$ or $y = -1$. If $x^2 = -2y$, then the critical points are $(0,0)$ and $(\pm\sqrt{2},\, -1)$. We have $f_{xx} = 2 + 2y$, $f_{xy} = 2x$, and $f_{yy} = 2$. It's easy to check that $(0,0)$ is a local min and that $(\pm\sqrt{2},\, -1)$ are saddles.

9. (11 points) Find the linear approximation of the function $f(x,y,z) = x^3\sqrt{y^2+z^2}$ at the point $(2,3,4)$ and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$.

$$\begin{aligned} L(x,\, y,\, z) &= f(2,3,4) + f_x(2,3,4)(x-2) + f_y(2,3,4)(y-3) + f_z(2,3,4)(z-4) \\ &= 40 + 60(x-2) + \frac{24}{5}(y-3) + \frac{32}{5}(z-4) \end{aligned}$$

$L(1.98,\, 3.01,\, 3.97) = 40 + 60(-.02) + \frac{24}{5}(.01) + \frac{32}{5}(-.03) = 38.656$.