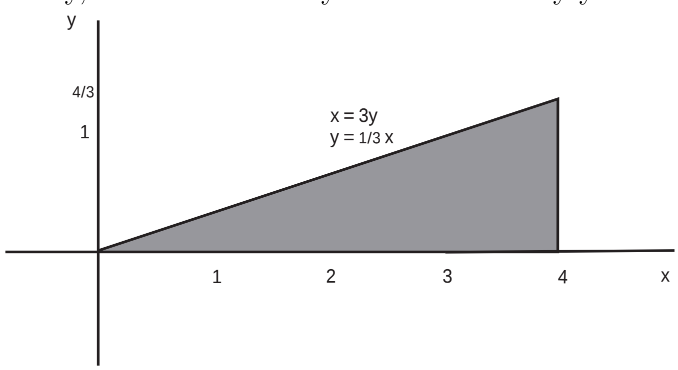
MA251 Midterm #2 November 14, 2002

1. (12 points) Find the absolute maxima and minima of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 = 1$.

$\nabla f = \langle 2xy,\, x^2\rangle$. Setting $g(x, y) = x^2 + y^2$, $\nabla g = \langle 2x,\, 2y\rangle$. $\begin{vmatrix} 2xy & x^2 \\ 2x & 2y \end{vmatrix} = 4xy^2 - x^3 = 2x(2y^2 - x^2) = 0$. Setting x=0 leads to points $(0, \pm 1)$ where $f = 0$. If $x^2 = 2y^2$, we have $3y^2 = 1$ and $y = \pm\sqrt{\frac{1}{3}}$, $x = \pm\sqrt{\frac{2}{3}}$. This gives a max of $\frac{2}{3}\sqrt{\frac{1}{3}}$ at $\left(\pm\sqrt{\frac{2}{3}},\ \sqrt{\frac{1}{3}}\right)$ and a min of $-\frac{2}{3}\sqrt{\frac{1}{3}}$ at $\left(\pm\sqrt{\frac{2}{3}},\ -\sqrt{\frac{1}{3}}\right)$.

2. (13 points) Please evaluate the integral $\displaystyle\int_0^{4/3}\int_{3y}^{4} e^{x^2}\,dx\,dy$ by reversing the order of integration.

Initially, x runs from x=3y to x=4 for every y between 0 and 4/3. This leads to the picture:



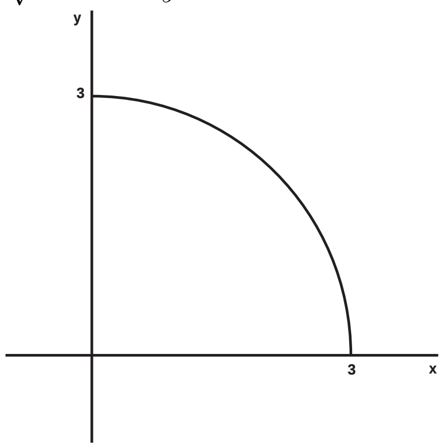
Reversing the order of integration leads to the integral $\int_0^4\int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx = \frac{1}{6}(e^{16} - 1)$.

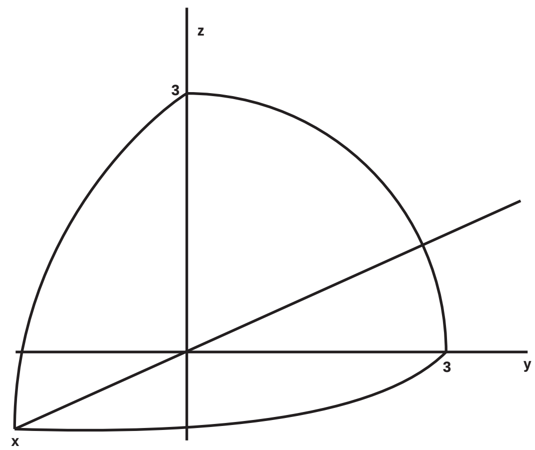
3. (13 points) Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

We do the problem as a double integral in polar coordinates. The equations become $z^2+r^2 = 16$ and $r = 2$. For each point in the annulus between $r = 2$ and $r = 4$, the volume we want runs from $-\sqrt{16 - r^2}$ to $\sqrt{16 - r^2}$, so the double integral is $\int_0^{2\pi}\int_2^4 2\sqrt{16 - r^2}\, r\, dr\, d\theta = 32\sqrt{3}\pi$.

4. (13 points) Change $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xz\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$ to spherical coordinates. **Please do not integrate!!**

Working from the outside in, the x limits are 0 to 3 and the y limits are 0 to $\sqrt{9-x^2}$. This gives the 2-dimensional plot below. Moving to 3 dimensions, the z limits are 0 and $\sqrt{9-x^2-y^2}$.





The integral in spherical coordinates is $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 (\rho \sin\phi \cos\theta)(\rho\cos\phi)(\rho)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

5. (12 points) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = x^2y^3\,\mathbf{i} - y\,\mathbf{j}$ and $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $0 \le t \le 1$.

$$\int_0^1 \langle x^2y^3,\, -y\rangle \cdot \langle 2t,\, -3t^2\rangle\, dt = \int_0^1 \langle -t^{13},\, t^3\rangle \cdot \langle 2t,\, -3t^2\rangle\, dt = \int_0^1 -2t^{14} - 3t^5\, dt = -\frac{19}{30}.$$

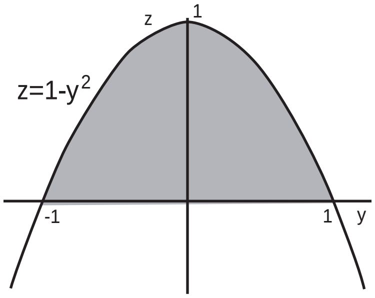
6. (10 points) Evaluate the integral $\int_C 2x\sin y\, dx + (x^2\cos y - 4y^3)\, dy$, where C is any path from $(-1,\, 0)$ to $(5,\, 1)$. (Hint: Use the fundamental theorem for line integrals).

We want to find f with $\nabla f = \langle 2x\sin y,\, x^2\cos y - 4y^3\rangle$. $\frac{\partial f}{\partial x} = 2x\sin y \implies f(x,y) = x^2\sin y + g(y)$. Then $\frac{\partial f}{\partial y} = x^2\cos y + g'(y) = x^2\cos y - 4y^3$, so $g'(y) = -4y^3$ and $g(y) = -y^4 + C$, so $f(x,y) = x^2\sin y - y^4$. The line integral is then $f(5,\, 1) - f(-1,\, 0) = 25\sin(1) - 1$.

7. (12 points) Find $\iint_R y^2\, dA$, where R is bounded by the ellipse $9x^2 + 4y^2 = 36$. Use the transformation $x = 2u$, $y = 3v$.

The new region is bounded by $9(2u)^2 + 4(3v)^2 = 36$, or $u^2 + v^2 = 1$. The Jacobian is 6, so the integral is $\iint_S (3v)^2(6)\, dA = 54\int_0^{2\pi} (r\sin\theta)^2\, r\, dr\, d\theta = \frac{27\pi}{2}$. Here, we switched to polar coordinates in the u-v plane to take advantage of the fact that the domain of integration is the unit circle.

8. (13 points) Set up the integrals to find the centroid of the solid region bounded by $z = 1 - y^2$, $x + z = 1$, $x = 0$ and $z = 0$. **Please do not integrate!!!**.

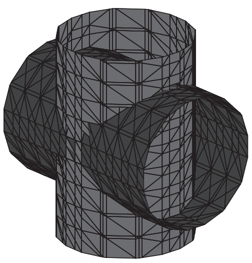


The equations $z = 0$ and $z = 1 - y^2$ define the region in the y-z plane pictured on the left. The only constraints on x are $x = 0$ and $x = 1 - z$, so we have

$$mass = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} dx\,dz\,dy, \quad M_{yz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} x\,dx\,dz\,dy$$

$$M_{xz} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} y\,dx\,dz\,dy, \quad M_{xy} = \int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{1-z} z\,dx\,dz\,dy$$

9. (10 points extra credit!!) A round termite that is two inches in diameter eats its way directly through a round post that is two inches in diameter. How much does the termite eat? Less poetically, find the volume of the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.



In the xy plane, we have $x^2 + y^2 = 1$. The z-direction is constrained by $z = \pm\sqrt{1 - x^2}$, so the volume may be expressed as the triple integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\,dy\,dx$.
Normally, one would expect that such an integral would work best in polar coordinates, but this one just happens to work better in rectangular coordinates. The answer is 16/3.