FALL 2003 FINAL EXAM

1. (12 points) If $\mathbf{v} = \langle 1, 2, 3\rangle$ and $\mathbf{w} = \langle -1, 1, 2\rangle$

   a. Find $\mathbf{v}\cdot\mathbf{w}$. (Ans. 7)

   b. Find $\mathbf{v}\times\mathbf{w}$. (Ans. $\langle 1,\ -5,\ 3\rangle$.)

2. (14 points) Find the equation of the plane through $(6, 0, -2)$ which contains the line

$$\mathbf{r}(t) = \langle 4, 3, 7\rangle + t\langle -2, 5, 4\rangle.$$

(Ans. $33(x-6) + 10y + 4(z+2) = 0$.)

3. (16 points) Find the tangential and normal components of the acceleration vector if $\mathbf{r}(t) = \langle 1+t,\ t^2-2t\rangle$. You get a two point bonus if your (correct) answers are vectors. (Ans. $\mathbf{a}_T = \frac{4t-4}{1+(2t-2)^2}\langle 1,\ 2t-2\rangle,\ \mathbf{a}_N = \frac{-2}{1+(2t-2)^2}\langle 2t-2,\ -1\rangle$.)

4. (14 points) If $e^{xz} - e^{zx^2} = 1$, find $\dfrac{\partial z}{\partial x}$. (Ans. $-\frac{ze^{xz}-2xze^{x^2z}}{xe^{xz}-x^2e^{x^2z}}$.)

5. (16 points) Find the equations of the tangent plane and normal line to the surface $z = \ln(x+2y)$ at the point $(3,\ -1,\ 0)$. (Ans. $-(x-3) - 2(y+1) + z = 0,\ \mathbf{r}(t) = \langle 3,\ -1,\ 0\rangle + t\langle -1,\ -2,\ 1\rangle$.)

6. (14 points) Find the local maxima, minima, and saddle points for $f(x, y) = x^2+y^2+x^2y+4$. (Ans. Local min (0,0), saddles $(\pm\sqrt{2},\ -1)$.)

7. (14 points) Change the order of integration in $\displaystyle\int_1^2\int_0^{\ln x} f(x,\ y)\,dy\,dx$. (Ans. $\int_0^{\ln 2}\int_{e^y}^2 f(x, y)\,dx\,dy$.)

8. (14 points) Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$, above the $xy-$plane, and inside the cylinder $x^2+y^2 = 2x$. (Ans. $\frac{3\pi}{2}$.)

9. (16 points) If $\mathbf{F}(x, y) = \langle 3+2xy,\ x^2-3y^2\rangle$, find a function f such that $\mathbf{F} = \nabla f$. Evaluate the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = e^t\sin(t)\,\mathbf{i} + e^t\cos(t)\,\mathbf{j},\ 0 \le t \le \pi$. (Ans. $e^{3\pi}+1$.)

10. (14 points) Set up a triple integral to find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$. YOU DO NOT NEED TO EVALUATE THIS INTEGRAL!! (Ans. $\int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{1-x} dz\, dx\, dy$ or $\int_{0}^{1} \int_{-\sqrt{x}}^{\sqrt{x}} \int_{0}^{1-x} dz\, dx\, dy$.)

11. (14 points) Find $\displaystyle\int\!\!\int_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 9$. (Ans. $\frac{4\pi \cdot 3^6}{5}$. Use the Divergence Theorem.)

12. (14 points) Evaluate $\displaystyle\oint_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy$, where C is the boundary of the rectangle $0 \le x \le 2,\ 0 \le y \le 3$. (Ans. 24.)

13. (14 points) Find $\displaystyle\int\!\!\int_S z\, dS$, where S is the surface with parametric equations $x = \cos(u),\ y = \sin(u),\ z = v,\ 0 \le u \le 2\pi,\ 0 \le v \le 2$. (Ans. 4π.)

14. (14 points) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle 2z,\ 4x,\ 5y \rangle$ and C is the curve of intersection of $z = x + 4$ and $x^2 + y^2 = 4$. C is oriented clockwise as viewed from above. (Ans. 4π. There's an extra "-" sign coming from the clockwise orientation.)

BONUS PROBLEM (5 POINTS) If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g.$$