Test #1 Fall 2003

1. Find the area of the parallelogram with vertices
$A(1,\ 1,\ 1),\ B(2,\ -1,\ 4),\ C(3,\ 5,\ -4),\ D(4,\ 3,\ -1)$.
Hint: Two of the sides of the parallelogram are are AB and AC.

$AB = \langle 1,\ -2,\ 3\rangle$ and $AC = \langle 2,\ 4,\ -5\rangle$. $AB \times AC = \langle -2,\ 11\,8\rangle$ and $|AB \times AC| = \sqrt{189}$.

2. Find the equation of the plane through $(6, 0, -2)$ which contains the line

$$\mathbf{r}(t) = \langle 4,\ 1,\ 6\rangle + t\langle -3, 4, 1\rangle.$$

Letting $t = 0$ and $t = 1$, we see that $A(4, 1, 6)$, $B(1, 5, 7)$, and $C(6, 0, -2)$ lie in the plane. $AB = \langle -3, 4, 1\rangle$ and $AC = \langle 2, -1, -8\rangle$, so $AB \times AC = \langle 31, 22, 5\rangle$. The equation of the plane is

$$31(x - 6) + 22(y - 0) + 5(z + 2) = 0$$

3. Find parametric equations for the tangent line to the curve
$\mathbf{r}(t) = \langle \cos t,\ 3e^{2t},\ 3e^{-2t}\rangle$ at the point $(1, 3, 3)$.

$$\mathbf{r}'(t) = \langle cos(t), 3e^{2t}, 3e^{-2t}\rangle$$

$\mathbf{r}(0) = (1, 3, 3)$ and $\mathbf{r}'(0) = (0, 6, -6)$, so the vector equation of the tangent line is

$$\mathbf{r}(t) = \langle 1, 3, 3\rangle + t\langle 0, 6, -6\rangle$$

and the parametric equations of the tangent line are $x = 1,\ y = 3 + 6t,\ z = 3 - 6t$.

4. Find the curvature of the parabola $x = t,\ y = 2t^2 + 1$ at the point $(2, 9)$.

$$\mathbf{r}(t) = \langle t,\ 2t^2 + 1,\ 0\rangle$$
$$\mathbf{r}'(t) = \langle 1,\ 4t,\ 0\rangle$$
$$\mathbf{r}''(t) = \langle 0,\ 4,\ 0\rangle$$
$$\kappa = \frac{|\mathbf{r}'(2) \times \mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \frac{4}{65^{3/2}}$$

5. If $f(x, y) = \dfrac{x^2y^2}{(x^4 + y^4)}$, does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist? Explain your answer.

If $x = 0,\ y = t,\ \frac{0^2t^2}{(0^4+t^4)} = 0$, so $\lim_{t->0} f(0, t) = 0$.

If $x = t,\ y = t,\ \frac{t^2t^2}{(t^4+t^4)} = \frac{1}{2}$, so $\lim_{t->0} f(t, t) = 1/2$.

Since these are different, the limit does not exist.

6a. If $z = xe^{\frac{y}{x}}$, find dz.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(e^{\frac{y}{x}} - \frac{y}{x}e^{\frac{y}{x}}\right)dx + e^{\frac{y}{x}}dy$$

6b. If $z = g(ax^2y,\ by^2)$, find $\dfrac{\partial z}{\partial y}$.

Set $u = ax^2y,\ v = by^2$. $\dfrac{\partial z}{\partial y} = \dfrac{\partial g}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial g}{\partial v}\dfrac{\partial v}{\partial y} = \dfrac{\partial g}{\partial u}(ax^2) + \dfrac{\partial g}{\partial v}(2by)$

7. Find the directional derivative of the function $f(x,\ y) = x^3y - 2x^2y^2$ at the point (1, 2) in the direction of $\langle 3,\ 4\rangle$. In what direction does f increase most rapidly? What is the rate of increase in that direction?

$\nabla f = \langle 3x^2y - 4xy^2,\ x^3 - 4x^2y\rangle$. At the point (1,2), this is $\langle -10,\ -7\rangle$. $\mathbf{u} = \langle 3/5,\ 4/5\rangle$ and $D_{\mathbf{u}}f = \langle -10,\ -7\rangle \cdot \langle 3/5,\ 4/5\rangle = -\frac{58}{5}$. The direction of most rapid increase is the direction of $\langle -10,\ -7\rangle$ and the rate increase in that direction is $\sqrt{149}$.

8. Find the linearization of the function $f(x, y) = x/y$ at the point $(100, 300)$ and use it to estimate the value of 101/301.

$$L(x,y) = f(100,300) + f_x(100,300)(x-100) + f_y(100,300)(y-300)$$

$$f_x = \frac{1}{y},\ \text{so } f_x(100,300) = \frac{1}{300}$$

$$f_y = -\frac{x}{y^2},\ \text{so } f_y(100,300) = -\frac{100}{300^2} = \frac{1}{900}$$

$$L(x,y) = \frac{1}{3} + \frac{1}{300}(x-100) - \frac{1}{900}(y-300)$$

$$L(101,301) = \frac{1}{3} + \frac{1}{300}(1) - \frac{1}{900}(1) = \frac{151}{450}$$