Fall 2003 Exam # 2

1. (12 points) Find all maxima, minima, and saddles of the function $f(x, y) = x^2+y^2+x^2y+4$.

$f_x = 2x + 2xy = 2x(y + 1)$ and $f_y = 2y + x^2$. $f_x = 0$ if $x = 0$ or $y = -1$. Given this, setting $f_y = 0$ shows that the critical points are at $(0, 0)$ and $(\pm\sqrt{2}, -1)$. We have $f_{xx} = 2 + 2y$, $f_{yy} = 2$, $f_{xy} = 2x$, so $D = 4 + 4y - 4x^2$. This is negative at $(\pm\sqrt{2}, -1)$, so these critical points are saddles. D is positive at (0,0) and $f_{xx}(0, 0) > 0$, so that critical point is a local minimum.

2. (12 points) Find the maximum and minimum of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + 2y^2 = 6$.

$\nabla f = \langle 2xy,\, x^2\rangle$ and $\nabla g = \langle 2x,\, 4y\rangle$. $\nabla f \times \nabla g = \langle 0,\, 0,\, 8xy^2 - 2x^3\rangle$. This is zero when $x = 0$ or $4y^2 = x^2$. Solving this together with $x^2 + 2y^2 = 6$ gives $(0,\, \pm\sqrt{3})$ and $(\pm 2,\, \pm 1)$. The maximum is then 4 at $(\pm 2,\, 1)$ and the minimum is -4 at $(\pm 2,\, -1)$.

3. (12 points) Calculate $\displaystyle\int\int_D y\, dA$, where D is bounded by $y = x - 1$ and $y^2 = 2x + 6$.

This is exactly like Example 3 in section 15.3, except that I changed xy under the integral to y to simplify the arithmetic. The answer is 18.

4. (13 points) Set up the integrals to find the x-coordinate of the center of mass of the solid of constant density that is bounded by $x = y^2$, $x = z$, $z = 0$, and $x = 1$. DO NOT EVALUATE THE INTEGRALS!!!

$$\bar{x} = \frac{\int_{-1}^{1} \int_{y^2}^{1} \int_0^x x\, dz\, dx\, dy}{\int_{-1}^{1} \int_{y^2}^{1} \int_0^x dz\, dx\, dy}$$

or

$$\bar{x} = \frac{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^x x\, dz\, dy\, dx}{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^x dz\, dy\, dx}$$

You could also integrate with respect to y first, but that seems needlessly complex. Integrating with respect to x first is much more complicated because it requires two or three separate integrals.

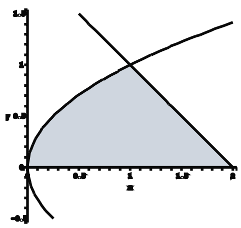
5. (12 points) Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} - y\sqrt{x}\,\mathbf{j}$.

$d\mathbf{r} = \langle 2t,\, -3t^2\rangle dt$, so $\mathbf{F} \cdot d\mathbf{r} = (2x^2y^3t - 3y\sqrt{x}t^2)dt = (2(t^2)^2(-t^3)^3 + 3(-t^3)tt^2)dt = (-2t^{14} - 3t^6)dt$. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-2t^{14} - 3t^4)dt = -2/15 - 3/7 = -59/105$.

6. (14 points) Let $\mathbf{F}(x,y) = e^{2y}\,\mathbf{i} + (1+2xe^{2y})\,\mathbf{j}$. Find a function $f(x,y)$ such that $\nabla f = \mathbf{F}$ and use it to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1+t^3)\,\mathbf{j}$, $0 \le t \le 1$.

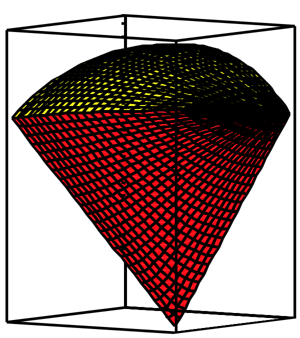
$\frac{\partial f}{\partial x} = e^{2y}$, so $f(x,y) = xe^{2y} + g(y)$.
$\frac{\partial f}{\partial y} = 1 + 2xe^{2y} = 2xe^{2y} + g'(y)$, so $g(y) = y$. We have $f(x,y) = xe^{2y} + y$. $\mathbf{r}(0) = (0,\,1)$ and $\mathbf{r}(1) = (1,\,2)$. By the fundamental theorem for line integrals, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,\,2) - f(0,\,1) = e^4 + 1$.

7. (13 points) Sketch the region of integration and change the order of integration in $\int_0^1 \int_{y^2}^{2-y} f(x,y)\,dx\,dy$.



$$\int_0^1 \int_{y^2}^{2-y} f(x,y)\,dx\,dy = \int_0^1 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dy\,dx$$

8. (12 points) Change the integral $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy$ to spherical coordinates. DO NOT EVALUATE THE INTEGRAL!!!



This is the region between a cone and a sphere in the first octant. The intersection of the cone and sphere is $x^2+y^2 = 9$ in the horizontal plane $z = 3$. In spherical coordinates, the integral is

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta$$