Fall 1999 Final Exam

1. (11 pts) Find the equation of the plane which passes through the points (1,2,3), (3,-2,1), and (-1,2,1). (Ans: $x + y - z = 0$.)

2. (12 pts) Find the curvature of the ellipse $x = 3\cos t,\ y = 4\sin t$ at the points (3,0) and (0,4). (Ans: $\kappa = 4$ and $\kappa = 3$.)

3. (11 points) Find the center and the radius of the sphere $x^2+y^2+z^2+4x+6y-10z+2 = 0$. (Ans: $(x+2)^2 + (y+3)^2 + (z-5)^2 = 36$, so center is $(-2, -3, 5)$ and radius is 6.)

4. (12 points) If $xy^2z^3 + x^3y^2z = x + y + z$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\left(\text{Ans: } \frac{\partial z}{\partial x} = -\frac{y^2\,z^3 + 3\,x^2\,y^2\,z - 1}{3\,x\,y^2\,z^2 + x^3\,y^2 - 1},\ \frac{\partial z}{\partial y} = -\frac{2\,x\,y\,z^3 + 2\,x^3\,y\,z - 1}{3\,x\,y^2\,z^2 + x^3\,y^2 - 1}\right)$$

5. (12 points) Find the equations of the tangent plane and normal line to the surface given by $x^2y + xz^2 + y^2z = -1$ at the point (1,2,-1). (Ans: $5(x-1) - 3(y-2) + 2(z+1) = 0$, $\mathbf{r}(t) = \langle 1,\ 2,\ -1\rangle + t\langle 5,\ -3,\ 2\rangle$.)

6. (12 points) Write the integral $\displaystyle\int_{-1}^{1}\int_{x^2}^{1} f(x,y)\,dy\,dx$ as an integral $dx\,dy$.

$$\left(\text{Ans:} \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy.\right)$$

7. (12 points) Find the area enclosed by one loop of the 4-leafed rose $r = \cos 2\theta$. (Ans: $\frac{\pi}{8}$.)

8. (12 points) Find the volume that lies under the paraboloid $z = x^2 + y^2$ and over the triangle with vertices (1,0,0), (0,1,0), and (0,0,0). (Ans: $\frac{1}{6}$.)

9. (11 points) Find the volume of the solid enclosed by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$. (Ans: $\frac{8}{15}$.)

10. (12 points) Convert the integral $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} f(x,y,z)\,dz\,dy\,dx$ to an equivalent integral in cylindrical coordinates.

$$\text{Ans:} \int_0^{2\pi}\int_{y^2}^{1}\int_{r^2}^{2-r^2} f(r\cos(\theta), r\sin(\theta), z)\,r\ \,dz\,dr\,d\theta.$$

11. (11 points) Evaluate $\int_C y \sin z \, ds$ where C is the helix given by $x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le 2\pi$. (ANS: $\sqrt{2}\pi$.)

12. (12 points) Find the local maxima, local minima, and saddles of $f(x, y) = x^3 + 3xy - y^3$. (ANS: Saddle at (0,0). Local min at (1,-1).)

13. (12 points) If $u = x^y$, $x = \sin t$, $y = \cos t$, find $\dfrac{du}{dt}$ when $t = \frac{\pi}{4}$.
ANS: $\sin(t)^{\cos(t)} \left(-\sin(t) \ln(\sin(t)) + \dfrac{\cos(t)^2}{\sin(t)} \right)$

14. (12 points) Let $\mathbf{F} = 2xy^3z^4 \, \mathbf{i} + 3x^2y^2z^4 \, \mathbf{j} + 4x^2y^3z^3 \, \mathbf{k}$. Find f with $\nabla f = \mathbf{F}$ and use it to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r} = t \, \mathbf{i} + t^2 \, \mathbf{j} + t^3 \, \mathbf{k},\ 0 \le t \le 2$. (ANS: 1048576.)

15. (12 points) Use Green's Theorem to find the work done by $\mathbf{F} = x(x + y) \, \mathbf{i} + 2xy^2 \, \mathbf{j}$ in moving a particle from the origin along the x-axis to (1,0), then along the line segment from (1,0) to (0,1), and then back to the origin along the y-axis. (ANS: 0.) For no particular reason I can see - it just comes out that way.

16. (12 points) Use the Divergence Theorem to calculate the integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = xy^2 \, \mathbf{i} + yz \, \mathbf{j} + zx^2 \, \mathbf{k}$ and S is the surface of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 3$. (ANS: 27π.)

17. (12 points) Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = e^y \, \mathbf{i} + ye^x \, \mathbf{j} + x^2y \, \mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, and has upward orientation. (ANS: $-\frac{5}{3}e + \frac{11}{6}$.)

BONUS PROBLEM: Find the volume of the intersection of the two cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. Can you find the volume of the triple intersection of these two and $y^2 + z^2 = 1$? (ANS: The first part was on midterm #2. The second part is a pain.)