Exam I Answer Key

1. (11 pts) Find the velocity and position vectors of a particle if $\mathbf{a}(t) = \mathbf{k},\ \mathbf{v}(0) = \mathbf{i}+\mathbf{j},\ \mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}$.

$\mathbf{v}(t) = t\mathbf{k} + \mathbf{c}$
$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} = \mathbf{c}$, so $\mathbf{v}(t) = t\mathbf{k} + \mathbf{i} + \mathbf{j}$
$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{d}$
$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} = \mathbf{d}$, so
$\mathbf{r}(t) = (t+2)\mathbf{i} + (t+3)\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

2. (11 pts) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xyz = \cos(x+y+z)$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

3. (11 pts) Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

$$\text{Area} = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}|\langle -3, 1, -7\rangle \times \langle 0, -5, -5\rangle| = \frac{1}{2}|\langle -40, -15, -15\rangle| = \frac{\sqrt{2050}}{2}$$

4. (11 pts) If $z = x^2 \sin y$, $x = s^2 + t^2$, and $y = 2st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial s} = 4xs\sin y + 2x^2 t\cos y$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial z}{\partial t} = 4xt\sin y + 2x^2 s\cos y$$

5. (11 pts) Find dz if $z = xe^{\frac{y}{x}}$.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$
$$= \left(e^{\frac{y}{x}} + xe^{\frac{y}{x}}\left(-\frac{y}{x^2}\right)\right)\,dx + e^{\frac{y}{x}}\,dy$$

6. (11 pts) Find the unit tangent vector and curvature of $\mathbf{r}(t) = (1+t)\mathbf{i} + (1-t)\mathbf{j} + 3t^2\mathbf{k}$ at the point where $t = 2$.

$$\mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 6t\mathbf{k}$$
$$\mathbf{r}'(2) = \mathbf{i} - \mathbf{j} + 12\mathbf{k}$$
$$\mathbf{r}''(t) = 6\mathbf{k}$$
$$\mathbf{T}(2) = \frac{\mathbf{i} - \mathbf{j} + 12\mathbf{k}}{|\mathbf{i} - \mathbf{j} + 12\mathbf{k}|} = \frac{\mathbf{i} - \mathbf{j} + 12\mathbf{k}}{\sqrt{146}}$$
$$\kappa(2) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|-6\mathbf{i} - 6\mathbf{j}|}{146^{3/2}} = \frac{72^{1/2}}{146^{3/2}}$$

7. (11 pts) Find the directional derivative of the function $f(x, y) = x^4y - x^2y^2$ at the point $(2, -3)$ in the direction of $\langle 1, 2\rangle$. In what direction does f increase most rapidly? What is the rate of change of f in that direction?

$$\nabla f = \langle 4x^3 - 2xy^2,\, x^4 - 2x^2y\rangle = \langle -132,\, 40\rangle \text{ at } (2, \text{-}3).$$
$$u = \langle \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}}\rangle$$
$$D_u f = -\frac{132}{\sqrt{5}} + \frac{80}{\sqrt{5}} = -\frac{52}{\sqrt{5}}$$

The direction of maximum change is the direction of ∇f and the maximum rate of change is $|\nabla f|$.

8. (12 pts) Find parametric equations of the line of intersection of the planes $x + y - 2z = 2$ and $3x - 4y + 5z = 6$.

$$\mathbf{v} = \langle 1, 1, -2\rangle \times \langle 3, -4, 5\rangle = \langle -3, -11, -7\rangle$$

To find a point in the intersection, set $z = 0$ and solve $x + y = 2,\ 3x - 4y = 6$, getting $(2, 0, 0)$ as a point on both planes. The vector equation is $\mathbf{r}(t) = \langle 2, 0, 0\rangle + t\langle -3, -11, -7\rangle$. The parametric equations are $x = 2 - 3t,\ y = -11t,\ z = -7t$.

An alternative is to find another point on the line by setting x equal to zero. This gives us the point (0,-22/3,-14/3). We could then find the equation of the line containing these two points in the usual way.

9. (11 pts) Find the distance between the lines given by $\mathbf{r}_1(t) = \langle 1, 0, 0\rangle + t\langle 1, 1, 1\rangle$, and $\mathbf{r}_2(s) = \langle 1, 1, 0\rangle + s\langle 0, 1, 1\rangle$.

$\langle 1, 1, 1\rangle \times \langle 0, 1, 1\rangle = \langle 0, -1, 1\rangle$ so $-y + z = 0$ is the equation of a plane which contains the first line and which is parallel to the second. The distance from (1,1,0) to this plane is $\frac{|-1+0|}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$. This is the distance between the lines. See Example 10 on page 696 for more details.