MA251 Exam #2 Apr. 12, 1999

1. (11 pts) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 2x^2y\,\mathbf{i} + xy\,\mathbf{j}$ and $\mathbf{r}(t) = t^3\,\mathbf{i} + t^4\,\mathbf{j}, \quad 0 \le t \le 1.$

$$\int \mathbf{F}\cdot d\mathbf{r} = \int 2x^2y\,dx + xy\,dy = \int_0^1 6t^{12} + 4t^{10}\,dt = 6/13 + 4/11 = 118/143.$$

2. (11 pts) Find all local maxima, minima, and saddles of $f(x,y) = x^3 - 6xy - y^3$.

$f_x = 3x^2 - 6y,\quad f_y = -6x - 3y^2$. Now, $3x^2 - 6y = 0$ implies $y = x^2/2$ and $-6x - 3y^2 = 0$ implies $x = -y^2/2$. Substituting, we have $2x + x^4/4 = 0$, so $x(x^3+8) = 0$ and $x = 0$ or $x = -2$. Thus, $y = 0$ or $y = 2$, so the critical points are $(0,0)$ and $(-2,2)$.

We have $f_{xx} = 6x,\ f_{yy} = -6y,\ f_{xy} = -6$, so $D = -36xy - 36$. At $(0,0)$, D=-36, so we have a saddle. At $(-2,2)$, $D = 108$ and $f_{xx} = -12$, so we have a local maximum.

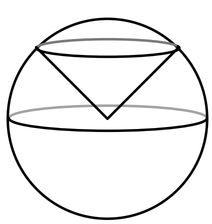
3. (11 points) Find $\int\int_D x\cos y\,dA$ where D is bounded by $y = 0,\ y = x^2$, and $x = 1$.

$$\int_0^1\int_0^{x^2} x\cos y dy\,dx = \int_0^1 x\sin y\Big|_0^{x^2} dx = \int_0^1 x\sin x^2\,dx = -\frac{\cos x^2}{2}\Big|_0^1 = \frac{1-\cos 1}{2}.$$

4. (11 points) Find the absolute maximum and minimum of the function $f(x,y) = x^3 - y^3$ subject to the condition that $x^2 + y^2 = 1$.

Solution #1: $\begin{vmatrix} 2x & 2y \\ 3x^2 & -3y^2 \end{vmatrix} = -6xy^2 - 6x^2y = 0$ when $x = 0, y = 0$, or $x = -y$. This leads to critical points at $(0,\pm 1)$, $(\pm 1, 0)$, $\left(\frac{\pm 1}{\sqrt{2}}, \frac{\mp 1}{\sqrt{2}}\right)$. The max is at $(1,0)$ and $(0,-1)$, while the min is at $(-1,0)$ and $(0,1)$. I'll put up a solution with λ's sometime later.

5. (11 points) Set up integrals in spherical coordinates to find the centroid of the solid above $z = \sqrt{x^2+y^2}$ and below $x^2+y^2+z^2 = 1$.



By symmetry, we have $\bar{x} = \bar{y} = 0$. It remains to find $\bar{z} = \frac{M_{xy}}{mass}$.

We have $mass = \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ and

$$M_{xy} = \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 z\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi.$$

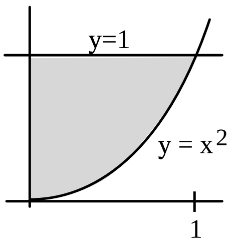
6. (11 points) Calculate the line integral $\int_C 2x\,\sin y\,dx + (x^2\cos y - 3y^2)\,dy$, where C is any path from $(-1,0)$ to $(5,1)$.

This integral is evidently independent of path, so we look for an f with $\nabla f = 2x\sin y\,\mathbf{i} + x^2\cos y - 3y^2\,\mathbf{j}$. From $\frac{\partial f}{\partial x} = 2x\sin y$, we see that $f(x,y) = x^2\sin y + g(y)$. From $\frac{\partial f}{\partial y} = x^2\cos y - 3y^2$, we see that $x^2\cos y - 3y^2 = x^2\cos y + g'(y)$, so $g'(y) = -3y^2$. It follows that $g(x) = -y^3$ and that $f(x,y) = x^2\sin y - y^3$. By the Fundamental Theorem for line integrals, $\int_C 2x\,\sin y\,dx + (x^2\cos y - 3y^2)\,dy = f(5,1) - f(-1,0) = 25\sin 1 - 1.$

7. (11 points) Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$.

$$\int\int_D x^2 + y^2\, dA = \int_0^{2\pi}\int_0^3 r^3\, dr\, d\theta = \frac{81\pi}{2}.$$

8. (12 points) Evaluate the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$ by reversing the order of integration.



$$\int_0^1 \int_0^{\sqrt{y}} x^3 \sin y^3\, dx\, dy = \int_0^1 \frac{x^4}{4} \sin y^3 \Big|_0^{\sqrt{y}} dy = \frac{1}{4}\int_0^1 y^2 \sin y^3\, dy = \frac{1}{4}\frac{\cos y^3}{3}\Big|_0^1$$
$$= \frac{1}{12}(1 - \cos 1).$$

9. (11 points) Use the transformation $x = 5u,\ y = 2v$ to evaluate the integral $\int\int_R xy\, dA$ where R is bounded by the ellipse $4x^2 + 25y^2 = 100$.

The Jacobian $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is equal to 10, so the change of variables formula tells us that

$$\int\int_R xy\, dA = \int\int_S (5u)(2v)10\, dA.$$

Since R is the interior of the region bounded by $4x^2 + 25y^2 = 100$, S is the interior of the region bounded by $4(5u)^2 + 25(2v)^2 = 100$ or $u^2 + v^2 = 1$. The easiest way to evaluate the integral is to switch to polar coordinates $u = r\cos\theta,\ v = r\sin\theta$, whence the integral becomes $100\int_0^{2\pi}\int_0^1 r^3 \sin\theta\cos\theta dr\, d\theta = 0$.