Practice problems for the final exam

1. A quadrilateral has one pair of opposite sides parallel and of equal length. Use vectors to prove that the other pair of opposite sides is parallel and of equal length.

2. Show that the vector $\mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b}$ is perpendicular to $\mathbf{a}$.

3. Given P(0,1,2), Q(2,4,5), R(-1,0,1), S(6,-1,4), find the volume of the parallelepiped with adjacent edges PQ, PR, and PS. (Ans: 4)

4. Find an equation of the plane passing through the points (-1,1,-1), (1,-1,2), and (4,0,3). (Ans: $-5(x+1)+7(y-1)+8(z+1)=0$)

5. Find the distance between the planes $z = x+2y+1$ and $3x+6y-3z=4$. (Ans: $\frac{7}{3\sqrt{6}}$).

6. Find parametric equations for the tangent line to the curve given by $x = t\cos 2\pi t,\ y = t\sin 2\pi t,\ z = 4t$ at the point (0, 1/4, 1). (Ans: $x = -(\pi/2)t,\ y = t+1/4,\ z = 4t+1$)

7. Find the length of the curve $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k},\quad 1 \le t \le e$. (Ans: e^2)

8. Find the curvature of $y = \sin x$. $\left(\text{Ans}: \frac{|\sin(x)|}{(1+\cos^2(x))^{3/2}}\right)$

9. Find the position vector of a particle that has the acceleration vector $\mathbf{a}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k}$ if $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(0) = \mathbf{j}$.
(Ans: $\mathbf{r}(t) = (t^3/6+t)\,\mathbf{i} + (1+t^4/12)\,\mathbf{j} + (-\frac{\cos(2t)}{4} + t + 1/4)\,\mathbf{k}$)

10. Find the limit if it exists or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

(Ans: Does not exist)

11. Find the first partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2+y^2-z^2 = 2x(y+z)$.
(Ans: $\frac{\partial z}{\partial x} = \frac{x-y-z}{x+z},\ \frac{\partial z}{\partial y} = \frac{y-x}{x+z}$.)

12. Use differentials to find an approximate value for $\sqrt{9(1.95)^2+(8.1)^2}$. (Ans: $f(x,y) = \sqrt{9x^2+y^2},\ x_0 = 2,\ y_0 = 8,\ \Delta x = -.05,\ \Delta y = .1, df = (\frac{9x}{\sqrt{9x^2+y^2}})\,dx + (\frac{y}{\sqrt{9x^2+y^2}})dy,\ \Delta f \cong -.01$ The approximate value is 10-.01=9.99.)

13. If $z = y^2\tan x,\ x = t^2uv,\ y = u+tv^2$, find $\frac{\partial z}{\partial v}$ when $t=2,\ u=1$ and $v=0$. (Ans: 4)

14. Find the equation of the tangent plane and normal line to the surface

$$x^2-2y^2+z^2 = 3$$

at the point (-1,1,-2).
(Ans: $-2(x+1)-4(y-1)-4(z+2)=0, r(t) = \langle -2t-1, -4t+1, -4t-2\rangle$)

15. Find the absolute maximum and minimum values of the function $f(x,y) = x^2+2xy+3y^2$ on the closed triangular region whose vertices are (-1,1), (2,1), and (-1,-2).
(Ans: min at f(0,0)=0 and max at f(-1,-2)=17)

16. Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$. (Ans: max is $\sqrt{2} = f(\pm 2^{-1/4}, \pm 2^{-1/4})$ and min is $1 = f(\pm 1, 0) = f(0, \pm)$)

17. Evaluate the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$ by reversing the order of integration.
(Ans: (1/12)(1-cos(1)))

18. Find the volume of the solid that lies under the plane $6x + 4y + z = 12$ and above the disk with boundary circle $x^2 + y^2 = y$. (Ans: $\frac{5\pi}{2}$)

19. Find the volume of the solid enclosed by the paraboloids $z = x^2+y^2$ and $z = 18-x^2-y^2$.
(Ans: 81π)

20. Evaluate $\int\int\int_E x\, e^{(x^2+y^2+z^2)^2}\, dV$, where E is the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant. (Ans: $\frac{\pi}{16}(e^{16} - e)$)

21. Use the change of variables $u = y-x$, $v = y+x$ to evaluate the integral $\int\int_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices (1,0), (2,0), (0,2), and (0,1).
(Ans: $\frac{3}{2}\sin 1$)

22. $\int_C xy^4 ds$, where C is the right half of the circle $x^2 + y^2 = 16$. (Ans: $1638\frac{2}{5}$)

23. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = x^2y\,\mathbf{i} - xy\,\mathbf{j}$ and C is the curve $\mathbf{r} = t^3\,\mathbf{i} + t^4\,\mathbf{j}$, $0 \le t \le 1$. (Ans: $-\frac{19}{143}$)

24. If $\mathbf{F}(x,y,z) = 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$, find a function f so that $\nabla f = \mathbf{F}$.
(Ans: $f(x,y,z) = x^2y^3z^4$)

25. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = x^3y\,\mathbf{i} + x^4\,\mathbf{j}$ and C is the curve $x^4 + y^4 = 1$.
(Ans: 0)

26. Prove that $\text{curl}(f\mathbf{F}) = f\,\text{curl}\,\mathbf{F} + (\nabla f) \times \mathbf{F}$. To simplify the algebra, you may assume that $\mathbf{F} = P\,\mathbf{i}$, that is, you may assume that $Q = R = 0$.

27. Find the equation of the tangent plane to the surface given by $x = u + v,\ y = 3u^2,\ z = u - v$ at the point (2,3,0).
(Ans: $3x - y + 3z = 3$)

28. Find the flux of the vector field $\mathbf{F}(x,y,z) = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ across the sphere $x^2+y^2+z^2 = 9$.
(Ans: 108π)

29. Use Stokes' Theorem to compute the integral $\int\int_s \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = yz^3\,\mathbf{i} + \sin(xyz)\,\mathbf{j} + x^3\,\mathbf{k}$, and S is the part of the paraboloid $y = 1 - x^2 - z^2$ that lies to the right of the xz-plane, oriented toward the xz-plane. (Ans: $\frac{3\pi}{4}$)

30. Use the Divergence Theorem to redo problem 28.

31. Assume that S and E satisfy the conditions of the Divergence Theorem and show that

$$\int\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0.$$

32. Assume that S and E satisfy the conditions of the Divergence Theorem and show that

$$\int\int_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \int\int\int_E (f\nabla^2 g - g\nabla^2 f)\, dV.$$