ON THE HIGSON-ROE CORONA

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Abstract. Higson-Roe compactifications first arose in connection with C^*-algebra approaches to index theory on noncompact manifolds. Vanishing and/or equivariant splitting results for the cohomology of these compactifications imply the integral Novikov Conjecture for fundamental groups of finite aspherical CW complexes. We survey known results on these compactifications and prove some new results – most notably that the n^{th} cohomology of the Higson-Roe compactification of hyperbolic space $\mathbf{H}^n$ consists entirely of 2-torsion for n even and that the rational cohomology of the Higson-Roe compactification of $\mathbf{R}^n$ is nontrivial in all dimensions $1 \leq k \leq n$.

§1. The Higson-Roe Compactification

Higson's compactification $\bar{X}$ first appeared in [H] in connection with a K-theoretic analysis of Roe's index theorem for noncompact Riemannian manifolds. Higson defined $\bar{X}$ to be to be the maximal ideal space of the commutative C^*-algebra of smooth functions whose gradient vanishes at infinity. In [R1], Roe modified Higson's definition to make sense for more general spaces. Here is Roe's definition:

Definition. If M is a space and $\phi : M \to \mathbf{C}$ is a continuous function, define $V_r(\phi) : M \to \mathbf{R}^+$ by

$$V_r(\phi) = \sup\{|\phi(y) - \phi(x)| : y \in B_r(x)\}$$

Then $C_h(M)$ is the space of all bounded continuous functions $\phi : M \to \mathbf{C}$ so that for each $r > 0$, $V_r(\phi) \to 0$ at infinity. Lemma 5.3 of [R1] proves that $C_h(M)$ is a C^*-algebra, so it makes sense to define the *Higson-Roe compactification*, $\bar{M}$ of M to be the maximal ideal space of $C_h(M)$.

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An equivalent alternative definition is to define a map

$$\iota : M \to \prod_{\phi \in C_h(M)} \mathbf{C}$$

by $\iota(m)_\phi = \phi(m)$ and declare $\bar{M}$ to be the closure of $\iota(M)$ in the infinite product. It clear that $\bar{M}$ is generally nonmetrizable and that $\bar{M}$ is characterized by the fact that a bounded continuous function $\phi : M \to \mathbf{C}$ extends to a continuous function $\bar{\phi} : \bar{M} \to \mathbf{C}$ if and only if $V_r(\phi) \to 0$ at infinity. Such functions will be called *slowly oscillating.*

The Higson compactification of a metric space is a close relative of the Stone-Čech compactification. It differs significantly from the Stone-Čech, though, in that $\bar{X}$ is not a topological invariant of the underlying space X. It is, however, functorial under uniformly continuous maps. The *Higson corona* is the space $\nu X = \bar{X} - X$. The corona is functorial under proper uniformly continuous maps between proper metric spaces. (Recall that a metric space X is *proper* if every finite metric ball in X has compact closure and that a map between metric spaces is proper if the inverse image of each compact set is compact.) The space νX is a coarse invariant of X in the sense of Gromov. For details, we refer the reader to chapters 2 and 5 of [R1].

While the Higson compactification of a noncompact metric space X is an interesting object in its own right, it gains additional interest because of its relationship with the Novikov and Gromov-Lawson Conjectures. In particular, the *Principle of Descent* says that the Novikov Conjecture for the fundamental group of a finite aspherical complex K follows from an appropriate Coarse Novikov Conjecture for the universal cover, $\tilde{K}$. Moreover, this Coarse Novikov Conjecture is known to be true for $\tilde{K}$ whenever $\tilde{K}$ has a compactification with nice properties.

Novikov's Conjecture. If M is a topological n-manifold with $n \geq 5$, the Sullivan-Wall surgery exact sequence of M is

$$\ldots \to L_{n+1}(\pi_1 M) \to \mathcal{S}(M, \partial M) \to H_n(M; \mathrm{G/TOP}) \xrightarrow{\mathcal{A}} L_n(Z\pi_1 M).$$

The map A in this sequence factors as

$$H_n(M; \mathrm{G/TOP}) \to H_n(B\pi_1 M; \mathrm{G/TOP}) \to H_n(B\pi_1 M; \mathbf{L}(e)) \xrightarrow{\mathcal{A}} L_n(Z\pi_1 M)$$

where $H_n(\ \cdot\ ; \mathrm{G/TOP})$ and $H_n(\ \cdot\ ; \mathbf{L}(e))$ denote homology with coefficients in the connective and periodic L-spectra, respectively, $M \to B\pi_1 M$ is the classifying map, and $\mathcal{A}$ is the *universal assembly map.* The map $\mathcal{A}$ depends only on $\pi = \pi_1 M$ and is otherwise independent of M. The classical *Novikov Conjecture* says that the map $\mathcal{A}$ is a rational monomorphism for all groups π.

Coarse Novikov and Borel Conjectures. In case the universal cover of M is contractible, $H_n(M; \mathbf{L}(e)) \xrightarrow{\cong} H_n(B\pi_1 M; \mathbf{L}(e))$ and we write the assembly map as

$\mathcal{A} : H_n(M; \mathbf{L}(e)) \to L_n(Z\pi_1 M)$. In this case, the *(rational) Coarse Novikov Conjecture* says that the bounded assembly map, see [F-P],

$$H_n^{\ell f}(\tilde{M}; \mathbf{L}(e)) \to L_{n,M}^{bdd}(e)$$

is a (rational) monomorphism. The (rational) *Coarse Borel Conjecture* says that this map is a (rational) isomorphism. The *Coarse Baum-Connes Conjecture*, (6.28) of [R1], is an analogous isomorphism statement in the language of C^*-algebras. The relationship between the topological and C^*-algebra versions of the Novikov Conjecture is discussed extensively in [Ros2]. These coarse conjectures invite generalizations to larger categories of spaces. Such generalizations will be discussed later in this paper but for now we will stick with universal covers of finite aspherical polyhedra.

§2. Principle of Descent

Let M^n and N^n be closed[1] aspherical manifolds and let $f : M \to N$ be a homotopy equivalence. Since tangentiality is not affected if we cross both manifolds with S^1, we can assume that we are in dimension ≥ 5 and that both manifolds are covered by euclidean space. We wish to show that f is topologically tangential. We pass to universal covers and form the diagram:

$$\begin{array}{ccc} \tilde{N} \times_\Gamma \tilde{N} & \xrightarrow{\tilde{f} \times_\Gamma \tilde{f}} & \tilde{M} \times_\Gamma \tilde{M} \\ {\scriptstyle proj_1} \downarrow & & {\scriptstyle proj_1} \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

Here $\Gamma = \pi_1 M = \pi_1 N$ acts diagonally on $\tilde{M} \times \tilde{M}$ and $\tilde{N} \times \tilde{N}$ and the maps $proj_1$ are induced by projection onto the first factor. One can show that that $\tilde{N} \times_\Gamma \tilde{N}$ and $\tilde{M} \times_\Gamma \tilde{M}$ are bundles with fiber $\tilde{N}$ and $\tilde{M}$ over N and M which are equivalent to the topological tangent bundles of N and M, respectively. To show that f is tangential, it suffices to show that $\tilde{f} \times_\Gamma \tilde{f}$ is proper homotopic to a fiber-preserving map which restricts to a homeomorphism on each fiber. This approach goes back to Farrell-Hsiang [F-H].

One approach to this problem is via bounded surgery theory [F-P]. The map $\tilde{f} \times_\Gamma \tilde{f}$ restricts to a copy of $\tilde{f}$ on each fiber. Thus, the problem of homotoping these maps to homeomorphisms can be viewed as a parameterized bounded surgery problem. We can proceed by induction on skeleta in N to boundedly homotop maps over each skeleton to homeomorphisms. Assuming that we have succeeded over $\partial\Delta^k$, the obstruction to succeeding over the interior lies in $\mathcal{S}^{bdd}\begin{pmatrix} \tilde{M} \times \Delta^k & \\ \downarrow & \text{rel } \partial(\tilde{M} \times \Delta^k) \\ \tilde{M} & \end{pmatrix}$. The bounded surgery

[1] By a result of M. Davis [D, p. 215], the closed manifold case implies the more general-looking case of groups Γ with $B\Gamma$ finite

sequence which computes this is:

$$\cdots \to \mathcal{S}^{bdd}\begin{pmatrix} \tilde{M}\times\Delta^k & \\ \downarrow & \text{rel } \partial(\tilde{M}\times\Delta^k) \\ \tilde{M} & \end{pmatrix} \to H^{\ell f}_{n+k}(\tilde{M};\mathrm{G/TOP}) \to L^{bdd}_{n+k,\tilde{M}}(e) \to \dots$$

These structure sets vanish if and only if the coarse assembly maps

$$H^{\ell f}_{n+k}(\tilde{M};\mathrm{G/TOP}) \to L^{bdd}_{n+k,\tilde{M}}(e)$$

are isomorphisms. Since $\tilde{M}$ is homeomorphic to $\mathbf{R}^n$, this amounts to showing that the assembly maps induce isomorphisms

$$\pi_k(\mathrm{G/TOP}) \to L^{bdd}_{n+k,\tilde{M}}(e)$$

for all k. The definitions of "bounded" and "L^{bdd}" depend on the metric on $\tilde{M}$, so we cannot simply replace $\tilde{M}$ by $\mathbf{R}^n$ on the right hand side.

If $\tilde{M}$ admits an equivariant compactification, we can follow [C-P] and use continuously controlled surgery theory [AnCFK], [C-P], [F-P] in place of bounded surgery theory in this construction. Suppose, for instance, that $M \cup X = \bar{M}$ is an L-acyclic metrizable compactification of $\tilde{M}$ such that compact subsets of $\tilde{M}$ become small near X – see [C-P] for a precise version of these conditions. For this argument only, we will use $\bar{M}$ to denote something other than the Higson-Roe compactification of M.

We can form $\tilde{M}\times_\Gamma \bar{M}$ and $\tilde{N}\times_\Gamma \bar{N}$ with projections to M and N. Here $\bar{N}$ is the induced compactification of N with remainder X. These are analogs of the closed tangent disk bundles of M and N. To show tangentiality, we work through a similar induction using continuously controlled surgery theory over X. In this case, the crucial assembly assembly map turns out to be

$$H^{\ell f}_{n+k}(\tilde{M};\mathrm{G/TOP}) \to L^{cc}_{X,n+k}(e).$$

The advantage here is that the continuously controlled L-groups can be computed. It turns out that $L^{cc}_{X,n+k}(e) \cong \bar{H}^{st}_{n+k-1}(X;\mathbf{L}(e))$, where $\bar{H}^{st}$ denotes reduced Steenrod homology, and that the coarse assembly map is the composition

$$H^{\ell f}_{n+k}(\tilde{M};\mathrm{G/TOP}) \cong \bar{H}^{st}_{n+k}(\bar{M},X;\mathrm{G/TOP}) \xrightarrow{\ \partial\ } \bar{H}^{st}_{n+k-1}(X;\mathbf{L}(e)).$$

That ∂ is an isomorphism follows immediately from the contractibility of $\bar{M}$ and the long exact sequence of $(\bar{M},X)$ in Steenrod homology. Thus, in case $\tilde{M}$ has a nice compactification, the integral Novikov Conjecture holds for $Z\pi_1 M$.

Suppose that $\tilde{M} \cup N$ is a metrizable compactification of $\tilde{M}$ so that the "identity" map $\tilde{M} \to \tilde{M} \cup N$ is slowly oscillating. Then there is a map $\bar{M} \to \tilde{M} \cup N$ taking $\nu\tilde{M}$ to N. If $\tilde{M} \cup N$ is L-acyclic, we have a commutative diagram

$$\begin{array}{ccc} H_n^{\ell f}(\tilde{M};\mathbf{L}(e)) & \xrightarrow{\partial} & H_{n-1}^{st}(\nu(\tilde{M});\mathbf{L}(e)) \\ \cong\downarrow & & \downarrow \\ H_n^{\ell f}(\tilde{M};\mathbf{L}(e)) & \underset{\cong}{\xrightarrow{\partial}} & H_{n-1}^{st}(N;\mathbf{L}(e)). \end{array}$$

We call these conditions – that an equivariant compactification $\tilde{M} \cup N$ be metrizable, that the "identity" map $\tilde{M} \to \tilde{M} \cup N$ be slowly oscillating and that $\tilde{M} \cup N$ be L-acyclic – the *Carlsson-Pedersen Conditions.* If these conditions are satisfied, then diagram above shows that the boundary map

$$(*) \qquad H_n^{\ell f}(\tilde{M};\mathbf{L}(e)) \xrightarrow{\partial} H_{n-1}^{st}(\nu(\tilde{M}));\mathbf{L}(e))$$

must be equivariantly split. This motivates the study of this boundary map in connection with the Novikov Conjecture, with special interest in determining conditions under which $\bar{M}$ is L-acyclic, or under which the boundary map $(*)$ is equivariantly split.

Unfortunately, the Higson-Roe compactification of $\tilde{M}$ is never acyclic for closed aspherical M^n with $\pi_1(M) \neq 1$. An argument of Keesling, [K], shows that the 1-dimensional Čech cohomology of $\bar{M}$ *must* have infinite rank. Since his argument for nontriviality is simple, we sketch it here: Choose a point $m_0 \in \tilde{M}$ and let $f : \tilde{M} \to S^1 \subset \mathbf{C}$ be the function

$$f(m) = e^{i\sqrt{d_{\tilde{M}}(m,m_0)}}$$

The function f is slowly oscillating, so f extends continuously to $\bar{f} : \bar{M} \to S^1$. If $\bar{f}$ were nullhomotopic, $\bar{f}$ would have to lift via the standard cover to a function $f^* : \bar{M} \to \mathbf{R}$. Since no lift of f to $\mathbf{R}$ has compact image, this is impossible and $\bar{f}$ must be essential.

This leaves room for hope, since Keesling's argument also shows that the first cohomology of the Stone-Čech compactification of $\tilde{M}$ must be nontrivial. In the case of the Stone-Čech compactification, however, a theorem of Calder and Siegel says that the higher cohomology of $\beta\tilde{M}$ always vanishes for aspherical M. Also, an extension of the descent argument above (see [F-W]) shows that to prove the integral Novikov Conjecture it suffices to find a metrizable equivariant compactification $\tilde{M} \cup N$ of $\tilde{M}$ such that compact sets get small at infinity and such that the boundary map

$$(**) \qquad \bar{H}_{n+k}^{st}(M \cup N, N; \mathrm{G/TOP}) \xrightarrow{\partial} \bar{H}_{n+k-1}^{st}(N;\mathbf{L}(e))$$

has an equivariant splitting. This is a mild, but potentially useful, extension of the Carlsson-Pedersen result quoted above. Moreover, in order to prove the rational Novikov conjecture for $\pi_1 M$ with M a closed aspherical manifold, it suffices to prove this same statement rationally.

To recapitulate, in order to prove the Novikov Conjecture it suffices to find a *metrizable* equivariant compactification $\tilde{M} \cup N$ so that fundamental domains get small near infinity and so that $(**)$ is equivariantly split. The existence of such a splitting for any compactification of $\tilde{M}$ satisfying the Carlsson-Pedersen conditions implies that the analogous boundary map for the Higson compactification is equivariantly split, as well.

§3. Large Riemannian Manifolds

The Gromov-Lawson conjecture states that a closed aspherical manifold cannot carry a metric of a positive scalar curvature [G-L]. This conjecture is a special case of the Novikov conjecture discussed in the previous section. Large Riemannian manifolds come into the picture when we consider universal covers of aspherical manifolds.

We recall that a metric space X, d is called *uniformly contractible* if for any number $R > 0$ there is a greater number S such that the R-ball $B_R(x)$, centered at x can be contracted to a point in the ball $B_S(x)$ of radius S for any point $x \in X$.

Example. *Let M be closed aspherical manifold with Riemannian metric d and let X be its universal covering space, $p : X \to M$. Then X with the induced metric p^*d is uniformly contractible.*

Proof. Let $Z \subset X$ be a compact set with $p(Z) = M$ and let d_1 be the diameter of Z. For any given R we consider a point $x_0 \in Z$ and the ball $B_{R+d_1}(x_0)$. Since M is aspherical, X is contractible, and there is an $S' > 0$ such that the ball $B_{R+d_1}(x_o)$ is contractible in $B_{S'}(x_0)$. Then for any $x \in X$ the ball $B_R(x)$ is contractible in $B_S(x)$ for $S = S' + d_1$. Indeed, there is an element $g \in \pi_1(M)$ such that $g(x) \in Z$. Then $B_R(g(x))$ is contained in $B_{R+d_1}(x_0)$ and hence is contractible in $B_{S'}(x_0) \subset B_S(g(x))$. Since the metric p^*d is $\pi_1(M)$-invariant, $B_R(x) = g^{-1}(B_R(g(x)))$ is contractible in $B_S(x) = g^{-1}(B_S(g(x)))$. □

Definition. An open Riemannian n-manifold M is called *hypereuclidean* (rationally hypereuclidean) if there exists a Lipschitz map $f : M \to \mathbf{R}^n$ of degree one (nonzero degree).

The Gromov-Lawson conjecture is proved in [G-L] for manifolds with hypereuclidean universal covers. The following natural question is due to Gromov [G2]:

Problem. Is every uniformly contractible manifold hypereuclidean?

A positive answer to this question would imply the Gromov-Lawson conjecture. It turns out that the answer is negative [D-F-W]: there is an uniformly contractible Riemannian metric on R^8 which is not hypereuclidean. Nevertheless that metric is rationally hypereuclidean. Since the rational hypereuclideaness suffices for the Gromov-Lawson conjecture, the following conjecture is of a great importance.

Conjecture. Every uniformly contractible manifold is rationally hypereuclidean.

It is possible that we should restrict ourselves here to uniformly contractible manifolds with bounded geometry. This would also suffice for Gromov-Lawson. See [H-R]. In this paper we will refer to this conjecture as to the Gromov Conjecture. We compare the Gromov Conjecture with the following:

Weinberger Conjecture [Ro1]. For every uniformly contractible metric space X with a proper metric the boundary homomorphism $\partial : H^{*-1}(\nu X; \mathbf{Q}) \to H^*_c(X; \mathbf{Q})$ is an epimorphism.

When X is a manifold of dimension n, the Weinberger conjecture states that

$$\partial : H^{n-1}(\nu X; \mathbf{Q}) \to H^n_c(X; \mathbf{Q}) = \mathbf{Q}$$

is an epimorphism provided X is uniformly contractible. The Weinberger conjecture implies the rational injectivity of a coarse assembly map [Ro1] and, in particular, the Gromov-Lawson Conjecture.

One way to prove the Weinberger conjecture would be to show that the Higson compactification of X is rationally acyclic, but this is not the case even when X is Euclidean space $\mathbf{R}^n$ by the argument of Keesling quoted above. On the other hand, in [D-K-U] the Weinberger Conjecture was checked for Γ-invariant metrics on contractible manifolds for a broad class of finitely presented groups Γ. The argument of the last section shows that the Weinberger Conjecture holds for Euclidean spaces and for hyperbolic spaces, since they have nice compactifications.

Theorem 3.1. *For open n-manifolds M with n even, the Weinberger Conjecture is equivalent to the Gromov Conjecture.*

Definition. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be a positive function tending to zero as x approaches infinity. Denote by $C_f(M)$ the algebra of bounded functions ϕ on a metric space M with the variation tending to zero as f or faster, i.e. for every $\phi \in C_f(M)$ and for every $R > 0$ there exists a constant C such that $Var_R\phi(x) \le Cf(d(x, x_0))$ where $x_0 \in M$ is a fixed point. Then the Higson-Roe compactification of growth f of a given space M is the maximal ideal space $\bar{M}_f$ for $C_F(M)$. The remainder $\nu_f M = \bar{M}_f \setminus M$ is called the Higson-Roe corona of M of growth f.

We recall that the Higson corona of M is the corona corresponding to the algebra $C_1(M)$ of bounded functions on M with variation tending to zero at infinity. It is clear that $C_f(M) \subset C_1(M)$ and that there is therefore a map $f : \bar{M} \to \bar{M}_f$ extending the identity on M.

The open cone CY on a geodesic compact space Y with weight function $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ is the standard quotient space $Y \times [0, \infty)/\sim$ with the metric

$$d_\Phi((y, t), (z, s)) = \inf_\gamma l_\Phi(\gamma)$$

where γ is a 'rectangular' path, defined by vertices $(y_0,t_0) = (y,t)$, (y_1,t_1),...,(z,s) such that $t_i = T_{i+1}$ for even i and $y_i = y_{i+1}$ for odd i, joining (y,t) with (z,t), and $l_\Phi(\gamma) = \sum_i \mid t_{i+1} - t_i \mid + \Phi(x_i)d(y_i, y_{i+1})$.

Proposition 3.2. **[Ro1, Example 5.28].** *Let $M = O_\Phi N$ be an open cone over N with a weight function Φ that tends to infinity. Then there exists a map to a closed cone $q : \bar{M}_{1/\Phi} \to C_\Phi N$ such that the restriction of q on M is the identity map.*

Let Z be the remainder of a compactification of an open oriented n-manifold, then the degree of a map $f : Z \to S^{n-1}$ is the degree of the following homomorphism $\mathbf{Z} = H^{n-1}(S^{n-1}) \xrightarrow{f^*} \check{H}^{n-1}(Z) \xrightarrow{\partial} H^n_c(M) = \mathbf{Z}$.

The proof of Theorem 3.1 is based on a characterization of hypereuclidean manifolds which is a modification of a characterization due to J. Roe [Ro1].

Lemma 3.3. *For an n-dimensional open manifold M the following conditions are equivalent:*

(1) *M is hypereuclidean,*
(2) *there is a map $g : \nu_{1/x}M \to S^{n-1}$ of degree one,*
(3) *there is a map $g' : \nu M \to S^{n-1}$ of degree one.*

Proof.
1)⇒2). Let $f : M \to \mathbf{R}^n$ be a Lipschitz map of degree one. Then f induces a map $C_{1/x}(\mathbf{R}^n) \to C_{1/x}(M)$ and hence a map $\bar{f} : \bar{M}_{1/x} \to \overline{\mathbf{R}^n_{1/x}}$. Since $\mathbf{R}^n$ is a weighted open cone $O_x S^{n-1}$ over S^{n-1}, by Proposition 3.2 we have a map $q : \overline{\mathbf{R}^n_{1/x}} \to C_x S^{n-1}$. Consider $g = q \circ \bar{f} \mid_{\nu_{1/x}M} : \nu_{1/x}M \to S^{n-1}$. The diagram

$$\begin{array}{ccc} \check{H}^{n-1}(\nu_{1/x}M) & \longrightarrow & H^n_c(M) \\ g^*\uparrow & & =\uparrow \\ H^{n-1}(S^{n-1}) & \longrightarrow & H^n_c(\mathbf{R}^n) \end{array}$$

implies that $deg(g) = 1$.

2)⇒3). There is a map $h : \nu M \to \nu_{1/x}M$ such that $h \mid_M = id$. Define $g' = g \circ h$, then the diagram

$$\begin{array}{ccc} \check{H}^{n-1}(\nu M) & \longrightarrow & H^n_c(M) \\ \uparrow & & id\uparrow \\ \check{H}^{n-1}(\nu_{1/x}M) & \longrightarrow & H^n_c(M) \\ g^*\uparrow & & =\uparrow \\ H^{n-1}(S^{n-1}) & \longrightarrow & H^n_c(\mathbf{R}^n) \end{array}$$

implies the proof.

3)$\Rightarrow$ 1). Lemmas 6.3, 6.4, 6.5 and Remark after in [Ro1] imply the proof. □

For rationally hypereuclidean spaces one can similarly prove the following

Lemma 3.4. *For an n-dimensional open manifold M the following conditions are equivalent:*

(1) *M is rationally hypereuclidean,*
(2) *there is a map $g : \nu_{1/x} M \to S^{n-1}$ of nonzero degree,*
(3) *there is a map $g' : \nu M \to S^{n-1}$ of nonzero degree.*

Proof of Theorem 3.1. If the Gromov Conjecture holds for n-dimensional manifold M, then by Lemma 3.4 there is a map $f : \nu M \to S^{n-1}$ of nonzero degree. By the definition of this degree it follows that $\partial(f^*(e))$ rationally generates the group $H^n_c(M; \mathbf{Q})$, where e is the fundamental cohomology class on the sphere S^{n-1}.

Assume that the Weinberger conjecture holds for M. Then there exists a map $f : \nu M \to K(\mathbf{Q}, n-1)$ which defines an element $\alpha \in \check{H}^{n-1}(\nu M; \mathbf{Q})$ with nontrivial image $\delta(\alpha) \in H^n_c(M) = \mathbf{Q}$. By virtue of Serre's theorem on the finiteness of higher homotopy groups of odd-dimensional spheres, an Eilenberg-MacLane complex $K(\mathbf{Q}, n-1)$ can be chosen as a telescope of spheres. Since νM is compact, $f(\nu M)$ lies in a finite stage sphere in the telescope. Thus, we have nonzero degree map of the Higson corona onto $n-1$-sphere. By Lemma 3.4, M is rationally hypereuclidean. □

§4. Cohomology of the Higson-Roe compactification of Euclidean space

Although the Weinberger conjecture holds for $\mathbf{R}^n$, the n-dimensional cohomology group $\check{H}^n(\overline{\mathbf{R}^n}; \mathbf{Q})$ of the Higson compactification is nontrivial. It follows immediately that $\check{H}^n(\nu \mathbf{R}^n; \mathbf{Q}) \neq 0$, as well.

Theorem 4.1. *For every n, $\check{H}^n(\overline{\mathbf{R}^n}; \mathbf{Q}) \neq 0$.*

For the proof we need the following fact.

Proposition 4.2. *For every $n \geq 1$ there is a locally trivial bundle $p : E \to S^{n+1}$ with $(n+1)$-connected the total space ($\pi_k(E) = 0$ for $k \leq n+1$) with fiber a CW complex F containing a homotopy equivalent subcomplex M with $(n+1)$-dimensional skeleton $M^{(n+1)}$ homeomorphic to the n-sphere S^n.*

Proof. For $n = 1$, the Hopf bundle $h : S^3 \to S^2$ satisfies all the conditions. For $n > 1$ we can take Milnor's model [Ad] for the Serre fibration $* \xrightarrow{\Omega S^{n+1}} S^{n+1}$. In that model the fiber $F = FS^n$ is the free nonabelian topological group generated by the sphere S^n.

This bundle is defined by the twisting map $\xi : S^n \times FS^n \to FS^n$ defined by the formula $\xi(x, w) = x^{-1}w$. Here $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$ is a word in the alphabet S^n and their inverses (so, $x_i \in S^n$, $\epsilon_i = 1$ or -1) and $x^{-1}w$ is an element of FS^n represented by a word obtained from w by adding the letter x^{-1} from the left. It is clear that for every $x \in S^n$ the multiplication by x^{-1} defines a homeomorphism of FS^n to itself. Thus, the map ξ defines a bundle by gluing the two natural charts on S^{n+1} over the equator S^n.

It is possible to show that the total space E of this bundle is contractible. Therefore the fiber FS^n is homotopy equivalent to the loop space ΩS^{n+1}. The free topological group FS^n contains the free topological monoid MS^n. By James' Theorem [J], MS^n is homotopy equivalent to $\Omega\Sigma S^n = \Omega S^{n+1}$ and hence, to FS^n. Moreover the inclusion $MS^n \subset FS^n$ induces that equivalence. We note that the $(n+1)$-skeleton of MS^n (James infinite reduced product of S^n) is homeomorphic to S^n for $n \geq 2$. □

Proposition 4.3. *If $f : S^n \to S^n$ is a piecewise smooth degree m map from the unit sphere to itself, then $V_\epsilon(f) \geq \frac{\epsilon m^{\frac{1}{n}}}{2}$.*

Proof. For each n and ϵ, let $\mathrm{vol}(\epsilon, n)$ denote the volume of a ball of radius ϵ in S^n. Given n and ϵ, choose points $x_1, \ldots, x_\ell \in S^n$ so that the open balls of radius $\frac{\epsilon}{2}$ in S^n form a maximal disjoint collection. We have an inequality:

$$\mathrm{vol}(S^n) \geq \ell \cdot \mathrm{vol}\left(\frac{\epsilon}{2}, n\right)$$

Since the collection is maximal, the ϵ-balls with the same centers cover S^n and since f has degree m, the volume of the image of some ϵ-ball centered at some x_i must be at least

$$\frac{m \cdot \mathrm{vol}(S^n)}{\ell} \geq \left(\frac{m \cdot \mathrm{vol}(S^n)}{\ell}\right)\left(\frac{\ell \cdot \mathrm{vol}\left(\frac{\epsilon}{2}, n\right)}{\mathrm{vol}(S^n)}\right) = m \cdot \mathrm{vol}\left(\frac{\epsilon}{2}, n\right)$$

Therefore, since volumes of balls grow more slowly in S^n than in $\mathbf{R}^n$, the image of some some ϵ-ball centered at x_i is not contained in the $(m^{1/n} \cdot \frac{\epsilon}{2})$-ball centered at $f(x_i)$. This implies that $V_\epsilon(f) > \frac{\epsilon m^{\frac{1}{n}}}{2}$, as desired. □

Denote by $B(m)$ the boundary of the cube in $\mathbf{R}^{n+1}$ which is centered at the origin and which has sides of length m parallel to the coordinate axes. Let $h : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be a radial contracting homeomorphism defined in the spherical coordinates by the formula $h(r, \phi) = (r^\alpha, \phi)$, where $\frac{n}{n+1} \leq \alpha < 1$ and consider the subset $M = h^{-1}(\cup_{m=1}^{\infty} B(2m)) \subset \mathbf{R}^{n+1}$ with the induced metric.

Proof of Theorem 4.1. For $n = 1$ the theorem is proven in [K].

Assume that $n > 1$ and let $q : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}/\mathbf{Z}^{n+1} = T^{n+1}$ be the quotient map onto the torus and let $\alpha : T^{n+1} \to S^{n+1}$ be the quotient map to the sphere. Consider the map $f = \alpha \circ q \circ h : \mathbf{R}^{n+1} \to S^{n+1}$, where h is as defined above. Let $z_0 \in S^{n+1}$ be the quotient point, so that $(\alpha \circ q)^{-1}(z_0)$ is the set of points in $\mathbf{R}^{n+1}$ with at least one integral coordinate. This means, in particular, that $(\alpha \circ q)^{-1}(z_0) \supset M$. Since $B(m)$ is centered at the origin, $B(m)$ is not contained in $(\alpha \circ q)^{-1}(z_0)$ for m odd.

The map f has an extension $\bar{f} : \overline{\mathbf{R}^{n+1}} \to S^{n+1}$, since the gradient of f tends to zero at infinity. Pulling back the fundamental class of S^{n+1}, $\bar{f}$ defines an element of integral Čech cohomology of the Higson compactification $\overline{\mathbf{R}^{n+1}}$. We will show that this element is of infinite order.

Assume the contrary. Then there exists a number d such that the composition $g_d \circ \bar{f}$ is nullhomotopic, where $g_d : S^{n+1} \to S^{n+1}$ is a map of degree d. Then there is a map $f' : \overline{\mathbf{R}^{n+1}} \to E$ such that $p \circ f' = g_d \circ \bar{f}$, where $p : E \to S^{n+1}$ is the bundle introduced in Proposition 4.2. This follows from the fact that nullhomotopic maps always lift.

We may assume that $g_d(z_0) = z_0$. Note that $f(M) = z_0$ and $g_d(f(M)) = z_0$, so $f'(M) \subset p^{-1}(z) = FS^n$. The exact sequence of the fibration p implies that the map f' restricted to $h^{-1}(B(2m))$ has the same degree as the quotient map

$$q_{2m} : h^{-1}D(2m)/h^{-1}B(2m) \to S^{n+1}$$

induced by $g_d \circ f \mid_{h^{-1}(D(m))}$. Here $B(2m)$ is the boundary of the cube $D(2m)$. The degree of q_{2m} equals $d \cdot (2m)^{n+1}$ which is equal to d times the number of unit cubes in $D(2m)$.

Since the inclusion $MS^n \subset FS^n$ is homotopy equivalence, there is a homotopy of $f' \mid_{\bar{M}}$ to a map $f'' : \overline{\mathbf{R}^{n+1}} \to E$ with $f''(\bar{M}) \subset MS^n$. By [D-K-U], the closure of M in $\overline{\mathbf{R}^{n+1}}$ is $\bar{M}$, the Higson-Roe compactification of M, so this notation is consistent. Since the $(n+1)$-skeleton of CW-complex MS^n is S^n and $\overline{\mathbf{R}^{n+1}}$ is $(n+1)$-dimensional, we can assume that $f''(\bar{M}) \subset S^n$, so $f'' \mid M$ must be slowly oscillating. We will see that this is impossible, a contradiction which will complete the proof of Theorem 4.1.

Let $k : S^n \to h^{-1}(B(2))$ be a Lipschitz map (the radial projection will do) with Lipschitz constant L. For each m we have a composition c_m

$$S^n \xrightarrow{\ k\ } h^{-1}(B(2)) \xrightarrow{\times m^{\frac{1}{\alpha}}} h^{-1}(B(2m)) \xrightarrow{\ f'\ } S^n.$$

By Proposition 4.3, $V_\epsilon(c_m) > (d(2m)^{n+1})^{1/n} \cdot \epsilon/2$. This implies that

$$V_{Lm^{1/\alpha}\epsilon}(f'') > (d(2m)^{n+1})^{1/n} \cdot \epsilon/2$$

and we get

$$V_\epsilon(f'') > (d(2m)^{n+1})^{1/n} \cdot \frac{\epsilon}{2Lm^{\frac{1}{\alpha}}} = \frac{d^{\frac{1}{n}} 2^{\frac{1}{n}}}{L} m^{(\frac{n+1}{n} - \frac{1}{\alpha})} \epsilon$$

Since the exponent of m is non-negative, $V_\epsilon(f'' \mid M)$ does not go to zero with increasing m and f'' cannot be slowly oscillating. □

Corollary 4.4. *$H^k(\overline{\mathbf{R}^n}) \neq 0$ for all $1 \leq k \leq n$.*

Proof. The retraction $r : \mathbf{R}^n \to \mathbf{R}^{n-1}$ induces a retraction $\bar{r} : \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^{n-1}}$. The result follows by induction and Theorem 4.1. □

§5. Cohomology of the Higson-Roe compactification of hyperbolic space

We begin this section by proving

Theorem 5.1. *For the hyperbolic plane* $\mathbf{H}^2$, $\check{H}^2(\bar{\mathbf{H}}^2) = 0$.

Lemma 5.2. *Let* $S^3 \subset \mathbf{R}^4$ *be the unit 3-sphere and let* $S^1 \subset S^3$ *be a great circle. Let* $p^{\epsilon}_{S^1} : N_\epsilon S^1 \to S^1$ *denote the projection from an* ϵ*-tubular neighborhood of to* S^1*. Then* $p^{\epsilon}_{S^1}$ *is a Lipschitz map with Lipschitz constant* K_ϵ *where* $\lim_{\epsilon \to 0} K_\epsilon = 1$.

Proof. This follows from the proof of the tubular neighborhood theorem provided that the normal vector field is taken perpendicular to the great circle S^1. □

Proof of Theorem 5.1. Since $\dim \bar{\mathbf{H}}^2 = 2$ [D-K-U], every element of $\check{H}^2(\bar{\mathbf{H}}^2)$ can be represented by a map $f : \bar{\mathbf{H}}^2 \to S^2$. Since any map of a 2-dimensional compactum into S^3 is nullhomotopic, to show that f is nullhomotopic it suffices to lift f with respect to the Hopf bundle $h : S^3 \to S^2$. Hence, it suffices a slowly oscillating map $g : \bar{\mathbf{H}}^2 \to S^3$ such that $h \circ g = f$. Since g is nullhomotopic, its composition with h will be nullhomotopic, as well.

Let x_0 be a fixed point in $\mathbf{H}^2$ and let $S(n)$ be a sphere of radius n centered at x_0. Let $\xi_m : \mathbf{H}^2 \to B(m)$ be the geodesic retraction onto the ball $B(m)$. It is clear that ξ_m restricted to $B(m+1)$ moves points not further than by one. There is a constant $C < 1$ such that $\xi_m \mid_{S(m+1)}$ is a Lipschitz map with Lipschitz constant C for all m. (In fact, for large m, C is approximately $1/e$.) Choose $\epsilon > 0$ so that $C < 1/K_\epsilon$.

We define a lift $g : \bar{\mathbf{H}}^2 \to S^3$ of f with respect to h as follows:

- (i) Choose a ball $B(R)$ of radius R centered at x_0 so that for every two points $x, y \in \mathbf{H} \setminus B(R)$ with $\mathrm{dist}(x, y) \leq 1$, the great circle $h^{-1}(f(x))$ lies in an ϵ-neighborhood of the great circle $h^{-1}(f(y))$.
- (ii) We will define the lift g by induction. Begin with any Lipschitz lift g of f over $B(R)$.
- (iii) Assuming that g is already defined on $B(R+n)$, extend g to $B(R+n+1)$ by setting
$$g(x) = p^{\epsilon}_{h^{-1}(f(x))}(g(\xi_{R+n}(x))).$$

Denote the Lipschitz constant of g restricted to $S(r)$ by L_r and note that

$$L_{R+n+1} \leq L_{R+n} C K_\epsilon \leq L_R (C K_\epsilon)^{n+1}$$

This shows that the Lipschitz constant L_r goes to zero as $n \to \infty$. This implies that g is slowly oscillating at infinity, completing the proof. □

Extensions. Using Hopf fibrations $S^7 \to S^4$ and $S^{15} \to S^8$, the argument above shows that $H^n(\overline{\mathbf{H}^n}) = 0$ for $n = 4$ and $n = 8$. Replacing the Hopf fibration by the unit tangent bundle of S^n for n even shows that $H^n(\overline{\mathbf{H}^n})$ is at most 2-torsion for every even n. In fact, a slight extension of the argument (using, for instance, the fibrations $S^{2n+1} \to CP^n$ in the 2-dimensional case) shows that

$$H^2(\overline{\mathbf{H}^n}) = H^4(\overline{\mathbf{H}^n}) = H^8(\overline{\mathbf{H}^n}) = 0$$

for every n. □

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