STABLE COMPACTIFICATIONS OF POLYHEDRA

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ABSTRACT. We prove that if X is a locally finite n-dimensional polyhedron such that $X \times Q$ admits a $\mathcal{Z}$-compactification, then $X \times I^{2n+5}$ also admits a $\mathcal{Z}$-compactification. Our argument relies on an extension of Dierker's Lemma [6], [2] which says that if P^p and Q^q are locally finite polyhedra, $p \geq 3$ and $c : P \to Q$ is a PL surjection with contractible point-inverses, then $Q \times I^{2p+1}$ collapses to P. See Proposition 1.5 for details, including control on the collapse. In the last section, we give an example of a uniformly contractible manifold with bounded geometry which does not satisfy Chapman-Siebenmann's tameness condition at infinity and which therefore does not admit a $\mathcal{Z}$-compactification.

1. INTRODUCTION

To set the stage, we begin with some definitions.

Definition 1.1. i.) If X is a compact metric space and $Z \subset X$ is closed, Z is said to be a *Z-set* if there is an homotopy $h_t : X \to X,\ 0 \leq t \leq 1$, so that $h_0(x) = x$ for all x and $h_t(X) \subset X - Z$ for all $t > 0$. The model case is the case in which X is a topological manifold and $Z = \partial X$. Another interesting case is the visual compactification of a CAT(0) space.

ii.) A separable metric space X is said to be an *ANR* if X can be embedded in separable Hilbert space in such a way that there is an open neighborhood U of X which retracts to X. All locally contractible finite-dimensional metric spaces are ANR's.

iii.) The *Hilbert cube* I^∞ is defined to be the product $\prod_{i=1}^{\infty}[0,1]$. A *Hilbert cube manifold* X is a separable metric space such that each point in X has an open neighborhood which is homeomorphic to an open subset of the Hilbert cube. Fundamental work of Chapman and West says that every Hilbert cube manifold is the product of a locally finite polyhedron with I^∞ and that for a given Hilbert cube manifold the polyhedron is unique up to simple-homotopy.

iv.) If X is a locally compact ANR, a compact metric space $\bar{X}$ containing X is said to be a *$\mathcal{Z}$-compactification* of X if $Z = \bar{X} - X$ is a $\mathcal{Z}$-set in $\bar{X}$. It follows easily from the definition of $\mathcal{Z}$-set and Hanner's criterion for ANR'ness [10] that in this case $\bar{X}$ is also an ANR.

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v.) If $\{(K_i,\alpha_i)\}_{i=1}^{\infty}$ is a sequence of finite CW complexes K_i and maps $\alpha_i : K_i \to K_{i-1}$, the *inverse mapping telescope* $\mathrm{Tel}(K_i,\alpha_i)$ is obtained from the disjoint union of the mapping cylinders of the α_i's by identifying the top of the mapping cylinder of α_i with the base of the mapping cylinder of α_{i+1}.

In [4], Chapman and Siebenmann gave necessary and sufficient conditions for a noncompact Hilbert cube manifold X to admit a $\mathcal{Z}$-compactification. Stated geometrically, their condition said that X admits a $\mathcal{Z}$-compactification if and only if X is homeomorphic to the product of an inverse mapping telescope with the Hilbert cube. In the same paper, they asked whether a locally finite polyhedron X admits a $\mathcal{Z}$-compactification whenever $X \times Q$ admits a $\mathcal{Z}$-compactification.

In [9], Guilbault gave an example of a locally-finite two-dimensional polyhedron X such that $X \times Q$ is $\mathcal{Z}$-compactifiable, but such that X itself admits no $\mathcal{Z}$-compactification. In that paper, he asked whether $X \times I^k$ was $\mathcal{Z}$-compactifiable for any finite k. Our theorem answers his question in the affirmative. We note that there has been a good deal of interest in $\mathcal{Z}$-compactifications, particularly in the case of compactifications of universal covers of finite aspherical polyhedra. See [**?**] for a nice discussion of this topic.

Theorem. *If X is a locally finite n-dimensional polyhedron and $X \times Q$ admits a $\mathcal{Z}$-compactification, then $X \times I^{2n+5}$ admits a $\mathcal{Z}$-compactification.*

Definition 1.2. Let $f : X \to Y$ be a proper map with X and Y locally compact finite-dimensional ANR's. If $\bar{Y} = Y \cup B$ is a compactification of Y, we define $\bar{f} : \bar{X} = X \cup B \to \bar{Y}$ to be $f \coprod \mathrm{id}$ and give $\bar{X}$ the topology generated by the open subsets of X together with sets of the form $\bar{f}^{-1}(U)$, where $U \subset \bar{Y}$ is open. By a slight abuse of notation, we will denote $\bar{X}$ by $X \cup_f B$.

The theorem is a consequence of the following three propositions.

Proposition 1.3. *If P is a locally finite polyhedron of dimension $\leq n$ such that $P \times Q$ admits a boundary, then P is simple-homotopy equivalent to an inverse mapping telescope of n-dimensional polyhedra.*

Proposition 1.4. *If $f : X \to Y$ is a proper CE map between locally compact ANR's, and $\bar{Y} = Y \cup B$ is a $\mathcal{Z}$-compactification of Y, then $\bar{X} = X \cup_f B$ is a $\mathcal{Z}$-compactification of X.*

Proposition 1.5. *If P^n is a locally finite n-dimensional polyhedron, $n \geq 3$, and P collapses to a locally finite subpolyhedron Q, then $Q \times I^{2n+1}$ collapses to P. In fact, if $c : P \to Q$ is a proper PL surjection with contractible point-inverses, then given any function $\epsilon : Q \to (0,\infty)$, we can find a proper PL surjection with contractible point-inverses $k : Q \times I^{2n+1} \to P$ so that the composition $c \circ k : Q \times I^{2n+1} \to Q$ is ϵ-close to projection.*

Given these propositions, here's the proof of our theorem.

Proof. If X is a locally finite n-dimensional polyhedron such that $X \times Q$ admits a boundary, Proposition 1.3 says that X is simple-homotopy equivalent to an inverse mapping telescope $T = \mathrm{Tel}(K_i, \alpha_i)$, where the K_i are finite n-dimensional polyhedra and the α_i's are PL maps.

In [17], Wall shows that if K and L are simple-homotopy equivalent finite CW complexes of dimension $\leq n$, $n \geq 3$, then there is a finite CW complex P of dimension $\leq n+1$ such that P collapses to both K and L. Using the simple homotopy theory of [7], Wall's proof carries over to locally finite polyhedra. Given the PL version of this result for locally finite complexes, we get a locally finite polyhedron P of dimension $n+2$ with CE-PL maps to X and to T. By the Cylinder Completion Theorem on page 180 of [4], T admits a $\mathcal{Z}$-compactification. Since P has CE map to T, P also admits a $\mathcal{Z}$-compactification. Since P has a CE map to X, Proposition 1.5 shows that $X \times I^{2n+5}$ collapses to P and, by Proposition 1.4, that $X \times I^{2n+5}$ admits a $\mathcal{Z}$-compactification. □

We now proceed with the proofs of Propositions 1.3-1.5.

Proof. Except for the dimension estimate, this is the Geometric Characterization Theorem of [4], which says that $X \times Q$ admits a $\mathcal{Z}$-compactification if and only if X is infinite simple-homotopy equivalent to an inverse mapping telescope. We obtain the dimension estimate by examining the proof in [4]. If X is a locally finite n-dimensional polyhedron such that $X \times Q$ admits a $\mathcal{Z}$-compactification, choose a nested collection V_i of cocompact subpolyhedra of X with bicollared boundaries so that $\cap_{i=1}^{\infty} V_i = \emptyset$. Since $X \times Q$ admits a $\mathcal{Z}$-compactification, each of the V_i's has the homotopy type of some finite n-dimensional polyhedron K_i. The inclusion maps $V_{i+1} \to V_i$ induce maps $\alpha_{i+1} : K_{i+1} \to K_i$ which are well-defined up to homotopy. The argument on pages 204-206 of [4] shows that X is simple-homotopy equivalent near infinity to the inverse mapping telescope $\mathrm{Tel}(K_i, \alpha_i)$ and infinite simple-homotopy equivalent to a telescope which agrees with $\mathrm{Tel}(K_i, \alpha_i)$ everywhere except at the first stage. At the end of this paper, we will sketch a proof of this result. □

We begin the proof of Proposition 1.4 with a useful homotopy invariance result for $\mathcal{Z}$-sets .

Proposition 1.6. *Let (X, Z) and (Y, Z) be compact metric pairs which are homotopy equivalent rel Z by maps and homotopies which are the identity on Z and which take the complement of Z to the complement of Z. Then Z is a $\mathcal{Z}$-set in X if and only if Z is a $\mathcal{Z}$-set in Y.*

Proof. We start the proof of the proposition by giving a more precise statement of the properties of the maps and homotopies described in its statement. Here is what we are given:

i.) A map $f : (X, Z) \to (Y, Z)$ with $f|Z = id$ and $f(X - Z) \subset Y - Z$.
ii.) A map $g : (Y, Z) \to (X, Z)$ with $g|Z = id$ and $g(Y - Z) \subset X - Z$.
iii.) A homotopy $h_t : X \to X$ with $h_0 = id$ and $h_t|Z = id$ for all t. Moreover, we have $h_t(X - Z) \subset X - Z$ for all t.

iv.) A homotopy $k_t : Y \to Y$ with $k_0 = id$ and $k_t|Z = id$ for all t. Moreover, we have $k_t(Y - Z) \subset Y - Z$ for all t.

v.) A homotopy $\alpha_t : Y \to Y$ with $\alpha_0 = id$ and $\alpha_t(Y) \subset Y - Z$ for all $t > 0$.

Our goal is to produce a homotopy $\beta_t : X \to X$ so that $\beta_0 = id$ and $\beta_t(X) \subset X - Z$ for all $t > 0$. This will show that Z is a $\mathcal{Z}$-set in X when it is a $\mathcal{Z}$-set in Y. The other half of the argument is completely symmetric.

We first show that we can construct $\bar{\alpha}$ having property (v) above and so that $\bar{\alpha}_t(y) = y$ whenever $d(y, Z) \geq t$. To get this, we define $\sigma : Y \times [0,1] \to [0,1]$ by the formula

$$\sigma(y,t) = \begin{cases} t - d(y,Z) & d(y,Z) \leq t \\ 0 & d(y,Z) \geq t \end{cases}$$

and then let $\bar{\alpha}_t(y) = \alpha_{\sigma(y,t)}(y)$. To conserve notation, we will drop the bar and assume that $\alpha_t(y) = y$ when $d(y,Z) \geq t$.

Next, let $\bar{\beta}_t(x) = g \circ \alpha_t \circ f : X \to X$. We see that $\beta_t(x) \subset X - Z$ for all $t > 0$ and that $\beta_t(x) = g \circ f(x)$ when $d(f(x), Z) \geq t$. Let $\tau : X \times (0,1] \to [0,1]$ be defined by the formula

$$\tau(x,t) = \begin{cases} 0 & d(f(x),Z) \geq 2t \\ 2 - \frac{d(f(x),Z)}{t} & t \leq d(f(x),Z) \leq 2t \\ 1 & d(f(x),Z) \leq t. \end{cases}$$

Let $\bar{h}_t(x) = h_{\tau(x,t)}(x)$. Strictly speaking, this function $\bar{h}_t$ is only defined for $t > 0$, but it extends over $t = 0$ by setting $\bar{h}_0(x) = x$ for all x. To prove continuity, we need to show that if $(x_i, t_i) \to (x^*, 0)$, then $\bar{h}_{t_i}(x_i) \to x^*$. We consider two cases: If $x^* \in X - Z$, then $\bar{h}_{t_i}(x_i) = x_i$ for large i and $\bar{h}_{t_i}(x_i) \to x^*$. If $x^* \in Z$, then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $d(x, x^*) < \delta$, then $d(h_t(x), x^*) < \epsilon$ for all t. It follows immediately that $\bar{h}_{t_i}(x_i) \to x^*$ in this case, as well.

Finally, we define $\beta_t(x)$ by the formula

$$\beta_t(x) = \begin{cases} \bar{h}_t(x) & d(f(x),Z) \geq t \\ \bar{\beta}_t(x) & d(f(x),Z) \leq t. \end{cases}$$

It's easy to check that $\beta_t(x)$ is well-defined and satisfies property (v). When $d(f(x), Z) = t$, we have $\bar{h}_t(x) = \bar{\beta}_t(x) = g \circ f(x)$. When t=0, we have $\beta_0(x) = \bar{h}_0(x) = x$ for all x, and for $t > 0$, we have either

$$\beta_t(x) = g \circ \alpha_t \circ f(x) \subset g(Y - Z) \subset X - Z$$

or $\beta_t(x) = h_{\tau(x,t)}(x)$. We have $\beta_t(x) \in X - Z$ in this last case, since $x \notin Z$. (To clarify this last assertion, note that $x \in Z$ and $t > 0$ guarantees that $\beta_t(x) = \bar{\beta}_t(x)$.) It follows that $\beta_t(x) \subset X - Z$ for all $t > 0$, so Z is a $\mathcal{Z}$-set in X. $\square$

We are now in a position to prove Proposition 1.4.

Proof. Proposition 1.4 follows immediately from Proposition 1.6 using a general property of cell-like maps between ANR's: If $f : X \to Y$ is a cell-like map between locally compact ANR's, then for any open cover α of Y there is a map $g : Y \to X$ so that $f \circ g$ is α-homotopic to the identity and $g \circ f$ is $f^{-1}(\alpha)$-homotopic to the identity. (A homotopy $h_t : Z \to Z$ is a $\mathcal{U}$-homotopy, $\mathcal{U}$ an open cover of Z, if for each $z \in Z$, we have $\{h_t(z) \mid 0 \leq t \leq 1\} \subset U_z$ for some $U_z \in \mathcal{U}$. If $\mathcal{U}$ is an open cover of Y and $f : X \to Y$ is continuous, then $f^{-1}(\mathcal{U})$ is the cover of X consisting of sets $f^{-1}(U)$ with $U \in \mathcal{U}$.) See [12] for a proof in the finite-dimensional case and [11] for an extension to the infinite-dimensional case.

Adopting the notation of Proposition 1.4, it is not hard to use this fact to produce a map $\bar{g} : \bar{Y} \to \bar{X}$ and homotopies $h_t : \bar{X} \to \bar{X}$ and $k_t : \bar{Y} \to \bar{Y}$ which are the identity on B and which send complements of B to complements of B. Since we have given ourselves that B is a $\mathcal{Z}$-set in $\bar{Y}$, it follows that B is a $\mathcal{Z}$-set in $\bar{X}$ (and that $\bar{X}$ is an ANR). □

Finally, we prove Proposition 1.5.

Proof. Let $c : P^n \to Q$ be a PL map with contractible point-inverses. For simplicity, we will assume that n, the dimension of P, is at least 3. Choose a 1-1 (locally?) PL map $\iota : P \to \operatorname{int} I^{2n+1}$ and consider the diagram

$$c \times \iota : P \to Q \times I^{2n+1} \to Q$$

where the last map is the projection. To conserve notation, we will identify P with its image under $c \times \iota$.

Let σ be a simplex of Q in some (fixed) triangulation and denote the intersection of P with $Q_\sigma = \sigma \times I^{2n+1}$ by P_σ. Of course, P_σ is just $c^{-1}\sigma$. Now let N_σ be a regular neighborhood of $P_\sigma \cup (\partial\sigma \times I^{2n+1})$ in Q_σ. The inclusion $N_\sigma \to Q_\sigma$ is a homotopy equivalence, so by excision the inclusion $\partial N_\sigma \to (Q_\sigma - \operatorname{int}(N_\sigma))$ is a homology equivalence. Since P_σ is codimension-3 in $\sigma \times I^{2n+1}$, $\partial N_\sigma \to (Q_\sigma - \operatorname{int}(N_\sigma))$ is a homotopy equivalence, as well. By the relative h-cobordism theorem, $(Q_\sigma - \operatorname{int}(N_\sigma))$ is homeomorphic to $\partial N_\sigma \times [0, 1]$. It follows that there is a PL collapse from Q_σ to $P_\sigma \cup (\partial\sigma \times I^{2n+1})$. Inducting down from the top-dimensional simplices of Q gives a PL collapse from $Q \times I^{2n+1}$ to P. The ϵ-estimate in the statement of Proposition 1.5 follows immediately by taking a triangulation of Q with ϵ-small simplices. □

Remark 1.7. i.) For experts, the estimates – both the dimension estimate and the $\epsilon(x)$-estimate – in Proposition 1.5 will probably be the most interesting novelties in this paper. Dierker's original idea was to note that if $X \nearrow Y$, then $Y \subset X \times [0, 1]$ and $X \times [0, 1] \searrow Y$. Iterating this construction, one gets a proof that if X and Y are finite polyhedra and $X \searrow Y$, then $Y \times I^q \searrow X$ for some q. There is no estimate on the q in terms of $\dim X$ and $\dim Y$ and there is no hint as to whether a similar result should hold for locally finite polyhedra. Brown and Cohen [2] modified Dierker's construction to obtain a somewhat different $\epsilon(x)$-estimate for finite polyhedra. Dierker's dimension estimate remained unchanged. They used their improved Dierker's Lemma to give a short proof that if X and Y are simple-homotopy equivalent polyhedra, then $X \times Q$

and $Y \times Q$ are homeomorphic Hilbert cube manifolds. Proposition 1.5 leads to such a proof for locally finite polyhedra.

ii.) Proposition 1.6 gives a quick proof that if K and L are homotopy equivalent finite aspherical polyhedra and $\tilde{K}$ admits a $\mathcal{Z}$-structure in the sense of [1], then so does $\tilde{L}$. This is also proven in [1] – it's a design criterion for the definition of $\mathcal{Z}$-structure – but it's occasionally useful[1] to have proofs of such facts which come directly from formulas, rather than relying on Hurewicz and Whitehead-type theorems.

2. An expanded proof of Proposition 1.3

We begin with some further discussion of Proposition 1.3.

Proof. If X is a finite-dimensional polyhedron such that $X \times Q$ admits a $\mathcal{Z}$-compactification, choose cocompact subpolyhedra $V_i \subset X$ so that $X = V_1 \supset V_2 \supset \ldots$ and $\cap_{i=1}^{\infty} V_i = \emptyset$. The compactification of $X \times Q$ induces compactifications of the $V_i \times Q$'s. These are compact ANR's, so by West's theorem [18], they have the homotopy types of finite complexes K_i. For $n \geq 3$, Wall [16] showed that an n-dimensional complex which is homotopy equivalent to a finite complex is homotopy equivalent to a finite n-dimensional complex, so we can assume that each K_i has dimension equal to $\max(n, 3)$. Let $\alpha_i : K_i \to K_{i-1}$ and $j_i : K_i \to V_i$ be maps such that the diagrams

$$\begin{array}{ccc} K_i & \xleftarrow{\alpha_{i+1}} & K_{i+1} \\ \cong \downarrow j_i & & \cong \downarrow j_{i+1} \\ V_i & \hookleftarrow & V_{i+1} \end{array}$$

homotopy commute for all i. There is an obvious map from $\mathrm{Tel}(K_i, \alpha_i)$ to X which is equal to j_i on each K_i. It is easy to verify that this map satisfies the conditions of the proper Whitehead theorem of [8], so the map is a proper homotopy equivalence. This uses the finite-dimensionality of both X and $\mathrm{Tel}(K_i, \alpha_i)$. It remains to show that this homotopy equivalence is a simple-homotopy equivalence near infinity.

By the Geometric Characterization Theorem of [4], we know that X is proper homotopy equivalent to $\mathrm{Tel}(L_i, \beta_i)$ for some finite polyhedra L_i and maps β_i, so it suffices to prove that proper homotopy equivalent telescopes are simple equivalent near infinity. Our argument is extracted from an old argument of Siebenmann [14].

First, note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of finite polyhedra and maps, then there is a simple homotopy equivalence rel $X \coprod Z$ from $M(f) \cup_Y M(g)$ to $M(g \circ f)$. Here, $M(f)$ denotes the mapping cylinder of f. Also, if $f, g : X \to Y$ are homotopic maps, then there is a simple homotopy equivalence rel $X \coprod Y$ from $M(f)$ to $M(g)$. These lemmas can be

[1] For instance, one might someday want a parameterized version of the theorem.

found in [5]. One consequence of this is that an inverse mapping telescope is infinite simple-homotopy equivalent to a telescope obtained by "passing to subsequences," i.e., by passing to a subsequence of the polyhedra and composing the appropriate bonding maps.

If $\mathrm{Tel}(K_i, \alpha_i)$ and $\mathrm{Tel}(L_i, \beta_i)$ are proper homotopy equivalent, we can pass to subsequences and, retaining our original notation, obtain a homotopy commuting diagram:

$$\begin{array}{ccccccccc} K_1 & \xleftarrow{\alpha_2} & K_2 & \xleftarrow{\alpha_3} & K_3 & \xleftarrow{\alpha_4} & K_4 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle f_1} & \nwarrow{\scriptstyle g_2} & \downarrow{\scriptstyle f_2} & \nwarrow{\scriptstyle g_3} & \downarrow{\scriptstyle f_3} & \nwarrow{\scriptstyle g_4} & \downarrow{\scriptstyle f_4} & & \\ L_1 & \xleftarrow[\beta_2]{} & L_2 & \xleftarrow[\beta_3]{} & L_3 & \xleftarrow[\beta_4]{} & L_4 & \longleftarrow & \cdots \end{array}$$

Using the simple-homotopy lemmas mentioned above, one sees that $\mathrm{Tel}(K_i, \alpha_i)$ is infinite simple-homotopy equivalent to the inverse telescope of the sequence

$$K_1 \xleftarrow{g_2} L_2 \xleftarrow{f_2} K_2 \xleftarrow{g_3} L_3 \xleftarrow{f_3} K_3 \longleftarrow \cdots$$

and that $\mathrm{Tel}(L_i, \beta_i)$ is infinite simple-homotopy equivalent to the inverse telescope of the sequence

$$L_1 \xleftarrow{f_2} K_1 \xleftarrow{g_2} L_2 \xleftarrow{f_2} K_2 \xleftarrow{g_3} L_3 \xleftarrow{f_3} K_3 \longleftarrow \cdots$$

The map f_1 is a homotopy equivalence, since K_1 and L_1 are both homotopy equivalent to X. The last mapping telescope is therefore infinite simple-homotopy equivalent to the mapping telescope of the sequence

$$K'_1 \xleftarrow{g'_2} L_2 \xleftarrow{f_2} K_2 \xleftarrow{g_3} L_3 \xleftarrow{f_3} K_3 \longleftarrow \cdots$$

where K'_1 is an n-dimensional complex simple-homotopy equivalent to L_1. This, in turn is infinite simple-homotopy equivalent to the mapping telescope of the sequence

$$K'_1 \xleftarrow{g'_2 \circ f_2} K_2 \xleftarrow{g_3 \circ f_3 \sim \alpha_3} K_3 \xleftarrow{g_4 \circ f_4 \sim \alpha_4} K_4 \longleftarrow \cdots$$

which shows both that X is infinite simple-homotopy equivalent to the mapping telescope of a sequence of finite n-dimensional polyhedra, as desired, and that the telescope can be taken to be $\mathrm{Tel}(K_i, \alpha_i)$, except for a possible change in the first term of the sequence. □

On page 207 of [4], the authors refer to an unpublished theorem of Ferry. Since the result has never been published, it seems reasonable to include the original proof in this paper. The result is also an immediate corollary of Torunczyk's characterization [15] of Hilbert cube manifolds.

Theorem. *If M is a Hilbert cube manifold and $\bar{M} = M \cup B$ is a $\mathcal{Z}$-compactification of M, then $\bar{M}$ is a Hilbert cube manifold.*

Proof. $\bar{M}$ is ϵ-dominated by M for every $\epsilon > 0$, so by Hanner's criterion [10], $\bar{M}$ is an ANR. By a well-known theorem of Edwards [3], $\bar{M} \times Q$ is a Hilbert cube manifold. Z-set unknotting shows that the cell-like map $\bar{M} \times Q \to (\bar{M} \times Q)/\sim$ obtained by shrinking out factors of Q in $B \times Q$ is shrinkable, so $(\bar{M} \times Q)/\sim$ is a Hilbert cube manifold. But the projection $M \times Q \to M$ is a can be approximated arbitrarily closely by homeomorphisms, so $\bar{M} \times Q$ is homeomorphic to $\bar{M}$ and $\bar{M}$ is a Hilbert cube manifold. □

3. A sobering example

Recently, there has been a resurgence of interest in the problem of $\mathcal{Z}$-compactifying polyhedra. Much of this interest involves the case in which the polyhedron in question is the universal cover of a finite aspherical polyhedron.[2] There is a nice discussion of this in [1].

In this section, we will focus on Chapman-Siebenmann's tameness condition, which must be satisfied if a locally finite polyhedron X is to admit a $\mathcal{Z}$-compactification. Here is the statement of the condition.

Definition 3.1. A locally finite polyhedron X is *tame at infinity* if for every compact $A \subset X$ there is a larger compact B so that the inclusion $X - B \to X - A$ factors up to homotopy through a finite complex. Thus, we require that there exist a finite complex K and maps $j : X - B \to K$, $p : K \to M - A$ so that $\beta \circ \alpha$ is homotopic to the inclusion.

Question: If K is a finite aspherical polyhedron, must $\widetilde{K}$ be tame at infinity?[3]

In order to answer this question positively, one would presumably have to find some geometric or homotopy-theoretic property of universal covers of finite polyhedra which was strong enough to imply that any locally finite polyhedron possessing these properties is tame at infinity. Two popular properties of universal covers which have been abstracted to other spaces are that a space should be *uniformly contractible* and that it should have *bounded geometry*. Our goal in this section is to construct an example of a smooth 5-manifold having both of these properties but which is not tame at infinity.

Definition 3.2. i.) A metric space X is *uniformly contractible* if for every $R > 0$ there is an $S > 0$ so that for each $x \in X$, the ball of radius R centered at x contracts in the ball of radius S centered at x.

ii.) A smooth manifold X has *bounded geometry* if its sectional curvature is bounded above and below and if its injectivity radius is bounded below.

Let $D = \Sigma^3 - \mathrm{int}(D^3)$, where Σ^3 is the Poincaré sphere. D collapses to a 2-complex K which embeds in $\mathbb{R}^5$. Let N be a regular neighborhood of K in R^5. Do surgery on the interior of N to kill π_1 and do another surgery to kill the resulting π_2. The result is a contractible

[2] Recall that a polyhedron K is *aspherical* if its universal cover is contractible.

[3] There is no obvious relation between this question and the well-known question of whether $\widetilde{K}$ must have semistable fundamental group at infinity. An inverse mapping telescope of S^1's with degree two bonding maps is Chapman-Siebenmann tame, but not semistable at infinity.

5-manifold W whose boundary M is a 4-manifold M with fundamental group isomorphic to the binary icosahedral group.

Let Q be a copy of $M \times [0,1]$ in which the Riemannian metric is multiplied by $(1+t)$ on $M \times \{t\}$. Thus, distances are twice as large at the $M \times \{1\}$-end as they are at the $M \times \{0\}$-end. Take an interior connected sum of Q with $S^2 \times S^3$ and do surgery to kill the element $2\alpha z - z$, where z is the element of π_2 represented by $S^2 \times \{*\}$ and α is an element of order 2 in $\pi_1(Q)$. Call this new manifold P. The point of this construction is that $H_*(P, M \times \{0\}) = 0$, while $H_*(\widetilde{P}, \widetilde{M} \times \{0\})$ consists of 60 copies of $\mathbb{Z}/3\mathbb{Z}$.

Form a contractible open manifold Z by attaching a copy of W to $M \times \{0\} \subset P$ and $2P \cup 4P \cup 8P \cup \ldots$ to $M \times \{1\} \subset P$. By kP, we mean a copy of P in which distances have been multiplied by k. The open manifold Z is contractible because W is contractible and $H_*(P, M \times \{0\}) = 0$. We will denote by Z_k the union of W with $P \cup \cdots \cup 2^k P$.

The manifold Z does not satisfy the Chapman-Siebenmann tameness condition because the inclusion $Z - Z_k \to Z - W$ does not factor through a finite complex L for any k. If it did, we could take the universal cover $\widetilde{Z} - \widetilde{W}$ of $Z - W$ and pull back over p and j to get a finite-sheeted covering space $\hat{L}$ of L and maps $\tilde{j} : \widetilde{Z} - \widetilde{Z}_k \to \hat{L}$ and $\tilde{p} : \hat{L} \to \widetilde{Z} - \widetilde{W}$ whose composition was homotopic to the identity. This is impossible, since $\hat{L}$ is a finite complex and the image of $H_*(\widetilde{Z} - \widetilde{Z}_k)$ in $H_*(\widetilde{Z} - \widetilde{W})$ is infinitely generated.

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