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Approximate Symmetries for a Model Describing Dissipative Media

Antonino VALENTI

Dipartimento di Matematica e Informatica
Università di Catania
Viale A. Doria, 6 95125 Catania, Italy
E-mail: *valenti@dmi.unict.it*

Approximate symmetries of a mathematical model describing one-dimensional motion in a nonlinear medium with a small dissipation are studied. In a physical application, the approximate solution is calculated making use of the approximate generator of the first-order approximate symmetry.

1 Introduction

We consider the nonlinear wave equation with a small dissipation of the form

$$w_{tt} - f(w_x)\, w_{xx} = \varepsilon\, w_{xxt}, \tag{1}$$

where f is a smooth function, $w(t, x)$ is the dependent variable, $\varepsilon \ll 1$ is a small parameter and subscripts denote partial derivative with respect to the independent variables t and x.

The equation (1) can describe one-dimensional wave propagation in nonlinear dissipative media and some mathematical questions related to (1), as the existence, uniqueness and stability of weak solutions can be found in [1], moreover a study related to a generalized "shock structure" is showed in [2], while, for $\varepsilon = \lambda_0$ (λ_0 is the viscosity positive coefficient), a symmetry analysis is performed in [12].

As it is well known, a small dissipation is able to prevent the breaking of the wave profile allowing to study the so called "far field".

A technique widely used in studying nonlinear problems is the perturbation analysis performed by expanding the dependent variables in power series of a small parameter (may be a physical parameter or often artificially introduced).

Combination of the Lie group theory and the perturbation analysis give rise to the so-called approximate symmetry theories. The first paper on this subject is due to Baikov, Gazizov and Ibragimov [3]. Successively another method for finding approximate symmetries was proposed by Fushchich and Shtelen [4]. In the method proposed by Baikov, Gazizov and Ibragimov, the Lie operator is expanded in a perturbation series so that an approximate operator can be found. While in the method proposed by Fushchich and Shtelen the dependent variables are expanded in a perturbation series; equations are separated at each order of approximation and the approximate symmetries of the original equations are defined to be the exact symmetries of the system coming out from equating to zero the coefficients of the smallness parameter. Pakdemirli et al. in a recent paper [5] have made a comparison of those two methods. We summarize the main results of their analysis in the following two statements:

a) The expansion of the approximate operator assumed in the method proposed by Baikov, Gazizov and Ibragimov, does not reflect well an approximation in the perturbation sense; in fact, even if one uses a first order approximate operator, the corresponding approximate solution could contain higher order terms;

b) The method proposed by Fushchich and Shtelen is consistent with the perturbation theory and yields correct terms for the approximate solutions but it is impossible to work in hierarchy; in the searching of symmetries there is a coupled system between the equations at several order of approximation, therefore the algebra can increase enormously.

In this paper we follow the guide lines of the method proposed by Fushchich and Shtelen [4] and remove the "drawback" of the impossibility to work in hierarchy. We perform the group classification of the nonlinear function $f(w_x)$ through which equation (1) with the small parameter ε is approximately invariant and search for approximate solutions.

The plan of the paper is the following: the approximate symmetry method is introduced in the next section; the group classification via approximate symmetries is performed in Sec.3; in Sec.4, in a physical application, the approximate solution is calculated by means of the approximate generator of the first-order approximate group of transformations.

2 Approximate Symmetry Method

In general, any solution of (1) will be of the form $w = w(t, x, \varepsilon)$and the one-parameter Lie group of infinitesimal transformations in the (t, x, w)-space of the equation (1), can be considered in the following form:

$$\begin{aligned} \hat{t} &= t + a\,\xi^1(t, x, w(t, x, \varepsilon), \varepsilon) + \mathcal{O}(a^2), \\ \hat{x} &= x + a\,\xi^2(t, x, w(t, x, \varepsilon), \varepsilon) + \mathcal{O}(a^2), \\ \hat{w} &= w + a\,\eta(t, x, w(t, x, \varepsilon), \varepsilon) + \mathcal{O}(a^2), \end{aligned} \tag{2}$$

where a is the group parameter.

Let us suppose that $w(t, x, \varepsilon)$ and $\hat{w}(\hat{t}, \hat{x}, \varepsilon)$, analytic in ε, can be expanded in power series of ε, i.e.

$$w(t, x, \varepsilon) = w_0(t, x) + \varepsilon\, w_1(t, x) + \mathcal{O}(\varepsilon^2), \tag{3}$$

$$\hat{w}(\hat{t}, \hat{x}, \varepsilon) = \hat{w}_0(\hat{t}, \hat{x}) + \varepsilon\, \hat{w}_1(\hat{t}, \hat{x}) + \mathcal{O}(\varepsilon^2), \tag{4}$$

where: w_0 and w_1 are some smooth functions of t and x; $\hat{w}_0$ and $\hat{w}_1$, are some smooth functions of $\hat{t}$ and $\hat{x}$.

Upon formal substitution of (3) in (1), equating to zero the coefficients of zero and first degree powers of ε we arrive at the following system of PDEs

$$L_0 := w_{0tt} - f(w_{0x})\, w_{0xx} = 0, \tag{5}$$

$$L_1 := w_{1tt} - f(w_{0x})\, w_{1xx} = g(w_{0x})\, w_{0xx}\, w_{1x} + w_{0xxt}, \tag{6}$$

where we have set

$$f(w_{0x}) = f(w_x)\,|_{\varepsilon=0}, \quad g(w_{0x}) = \frac{d\, f(w_x)}{d\, w_x}\Big|_{\varepsilon=0}.$$

Hence, w_0 is a solution of the nonlinear wave equation (5) which we call *unperturbed equation*, while w_1 can be determined from the linear equation (6).

In order to have an one-parameter Lie group of infinitesimal transformations of the system (5)–(6), which is consistent with the expansions of the dependent variables (3) and (4), we introduce these expansions in the infinitesimal transformations (2). Upon formal substitution, equating to zero the coefficients of zero and first degree powers of ε, we get the following one-parameter Lie group of infinitesimal transformations in the (t, x, w_0, w_1)-space

$$\begin{aligned} \hat{t} &= t + a\, \xi_0^1(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{x} &= x + a\, \xi_0^2(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{w}_0 &= w_0 + a\, \eta_0(t, x, w_0) + \mathcal{O}(a^2), \\ \hat{w}_1 &= w_1 + a\, [\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1] + \mathcal{O}(a^2), \end{aligned} \tag{7}$$

where we have set

$$\xi_0^i(t, x, w_0) = \xi^i(t, x, w(t, x, \varepsilon), \varepsilon)\,|_{\varepsilon=0}, \quad i = 1, 2$$

$$\eta_0(t, x, w_0) = \eta(t, x, w(t, x, \varepsilon), \varepsilon)\,|_{\varepsilon=0},$$

$$\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1 = \frac{d\,\eta}{d\,\varepsilon}\Big|_{\varepsilon=0}.$$

Similarly to Fushchich and Shtelen [4], we give the following definition:

Definition 1. We call *approximate symmetries* of equation (1) the (exact) symmetries of the system (5)–(6) through the one-parameter Lie group of infinitesimal transformations (7).

Consequently, the one-parameter Lie group of infinitesimal transformations (7) the associated Lie algebra and the corresponding infinitesimal operator

$$\begin{aligned} X = {} & \xi^1(t, x, w_0)\frac{\partial}{\partial t} + \xi^2(t, x, w_0)\frac{\partial}{\partial x} + \eta(t, x, w_0)\frac{\partial}{\partial w_0} \\ & + [\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1]\frac{\partial}{\partial w_1}, \end{aligned} \tag{8}$$

are called the approximate Lie group, the approximate Lie algebra and the approximate Lie operator of the equation (1), respectively.

Moreover, after putting

$$X_0 = \xi_0^1(t, x, w_0)\frac{\partial}{\partial t} + \xi_0^2(t, x, w_0)\frac{\partial}{\partial x} + \eta_0(t, x, w_0)\frac{\partial}{\partial w_0}, \tag{9}$$

the approximate Lie operator (8) can be rewritten as

$$X = X_0 + [\eta_{10}(t, x, w_0) + \eta_{11}(t, x, w_0)\, w_1]\frac{\partial}{\partial w_1} \tag{10}$$

and X_0 can be regarded as the infinitesimal operator of the unperturbed equation (5).

It is worthwhile noticing that, thanks to the functional dependencies of the coordinates of the approximate Lie operator (8) (or (10)), now we are able to work in hierarchy in finding the invariance conditions of the system (5)–(6): firstly, by classifying the unperturbed equation (5) through the operator (9) and after by determining η_{10} and η_{11} from the invariance condition that follows by applying the operator (10) to the linear equation (6). In fact the invariance condition of the system (5)–(6) reads:

$$X_0^{(2)}(L_0)\Big|_{L_0=0} = 0, \tag{11}$$

$$X^{(3)}(L_1)\Big|_{L_0=0,\ L_1=0} = 0, \tag{12}$$

where $X_0^{(2)}$ and $X^{(3)}$ are the second and third extensions of the operators X_0 and X, respectively.

Finally, the procedure outlined above is a variant of that developed by Donato and Palumbo [7, 8] and successively by Wiltshire [9].

3 Group Classification via Approximate Symmetries

The classification of the equation (5) is well known (see for details Ibragimov [6] and bibliography therein). From (11), we arrive at the following result:

$$\begin{aligned}
&\xi_0^1 = a_5\, t^2 + a_3\, t + a_1, \quad \xi_0^2 = a_4\, x + a_2,\\
&\eta_0 = (a_5\, t + a_6)\, w_0 + a_7\, t\, x + a_8\, t + a_9\, x + a_{10},\\
&[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\, f(w_{0x})}{d\, w_{0x}} - 2\,(a_4 - a_3)\, f(w_{0x}) = 0,\\
&(a_5\, w_{0x} + a_7)\, \frac{d\, f(w_{0x})}{d\, w_{0x}} + 4\, a_5\, f(w_{0x}) = 0,
\end{aligned} \tag{13}$$

where a_i, $i = 1, 2, \ldots, 10$ are constants.

Taking (13) into account, from (12) we obtain the following additional conditions:

$$a_5 = a_7 = 0, \tag{14}$$

$$\eta_{10} = a_{11}\,t + a_{12}, \quad \eta_{11} = a_3 - 2\,a_4 + a_6, \tag{15}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\,g(w_{0x})}{d\,w_{0x}} + (2\,a_3 - 3\,a_4 + a_6)\,g(w_{0x}) = 0, \tag{16}$$

with a_{11} and a_{12} constants.

After observing that conditions (14) impose restrictions upon to X_0, summarizing we have to manage the following relations:

$$\xi_0^1 = a_3\,t + a_1, \quad \xi_0^2 = a_4\,x + a_2, \quad \eta_0 = a_6\,w_0 + a_8\,t + a_9\,x + a_{10}, \tag{17}$$

$$\eta_{10} = a_{11}\,t + a_{12}, \quad \eta_1 = a_3 - 2\,a_4 + a_6, \tag{18}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\,f(w_{0x})}{d\,w_{0x}} - 2\,(a_4 - a_3)\,f(w_{0x}) = 0, \tag{19}$$

$$[(a_6 - a_4)\ w_{0x} + a_9]\ \frac{d\,g(w_{0x})}{d\,w_{0x}} + (2\,a_3 - 3\,a_4 + a_6)\,g(w_{0x}) = 0. \tag{20}$$

For f an arbitrary function we obtain $a_6 = a_4 = a_3$, $a_9 = 0$, from which it follows that g is also an arbitrary function.

We call the associate seven-dimensional Lie algebra the *Approximate Principal Lie Algebra* of equation (1). It is spanned by the seven operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial w_0}, \quad X_4 = t\,\frac{\partial}{\partial w_0},$$

$$X_5 = t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} + w_0\,\frac{\partial}{\partial w_0}, \quad X_6 = \frac{\partial}{\partial w_1}, \quad X_7 = t\,\frac{\partial}{\partial w_1}$$

and we denote it by $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$.

Otherwise, from (19) and (20) we obtain that f and g are linked by the relation

$$g(w_{0x}) = \frac{d\,f(w_{0x})}{d\,w_{0x}},$$

as we hoped and expected in order to be consistent with the perturbation theory.

The classification of $f(w_{0x})$ and the corresponding extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$ arising from (17)–(19), are reported in Table 1.

Table 1. Classification of $f(w_{0x})$ and corresponding extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$. f_0, p and q are constitutive constants with $f_0 > 0$, $p \neq 0$.

Case	Forms of $f(w_{0x})$	Extensions of $\mathcal{A}pprox\mathcal{L}_{\mathcal{P}}$
I	$f(w_{0x}) = f_0\,e^{w_{0x}/p}$	$X_8 = x\,\frac{\partial}{\partial x} + (w_0 + 2\,p\,x)\,\frac{\partial}{\partial w_0} - w_1\,\frac{\partial}{\partial w_1}$
II	$f(w_{0x}) = f_0\ (w_{0x} + q)^{2/p}$	$X_8 = x\,\frac{\partial}{\partial x} + [(1+p)\,w_0 + p\,q\,x]\,\frac{\partial}{\partial w_0} + (p-1)\,w_1\,\frac{\partial}{\partial w_1}$

4 A Physical Application

Let us consider a homogeneous viscoelastic bar of uniform cross-section and assume the material to be a nonlinear Kelvin solid. This model is described by the classical equation of motion (the constant density is normalized to 1 and the mass forces are neglected)

$$w_{tt} = \tau_x \tag{21}$$

and by assuming a stress-strain relation of the following form:

$$\tau = \sigma(w_x) + \lambda_0 \, w_{xt}, \tag{22}$$

where τ is the stress, x the position of a cross-section in the homogeneous rest configuration of the bar, $w(t,x)$ the displacement at time t of the section from its rest position, $\sigma(w_x)$ the elastic tension (w_x is the strain), λ_0 the viscosity positive coefficient. Taking (22) into account and setting

$$\frac{d\,\sigma(w_x)}{d\,w_x} = f, \quad \lambda_0 = \varepsilon,$$

the equation (21) reduces to (1).

Let us consider the following form of the tension $\sigma(w_x)$:

$$\sigma(w_x) = \sigma_0 \, \log(1 + w_x), \tag{23}$$

which was suggested by G. Capriz [10,11].

So, we fall in the Case II of Table 1 with the following identifications:

$$f_0 = \sigma_0, \quad p = -2, \quad q = 1.$$

In this case, the approximate Lie operator X_8 assumes the form

$$X_8 = x \, \frac{\partial}{\partial x} - (w_0 + 2\,x) \, \frac{\partial}{\partial w_0} - 3 \; w_1 \, \frac{\partial}{\partial w_1}$$

and from the corresponding invariant surface conditions we obtain the following representation for the different terms in the expansion of w:

$$w_0 = \frac{\psi(t)}{x} - x, \quad w_1 = \frac{\chi(t)}{x^3}, \tag{24}$$

which give the form of an invariant solution approximate at the first order in ε.

The functions ψ and χ must satisfy the following system of ODEs to which, after (23), the system (5) is reduced through (24):

$$\psi_{tt} + 2\,\sigma_0 = 0, \quad \chi_{tt} + \frac{6\,\sigma_0}{\psi}\,\chi - 2\,\psi_t = 0. \tag{25}$$

After solving (25) and taking (24) into account, we have

$$w_0 = -\,\sigma_0\,\frac{t^2}{x} - x, \quad w_1 = -\,\frac{(40\,\sigma_0\,\log t - 8\,\sigma_0 - 25)\,t^5 - 25}{50\,t^2\,x^3}\,.$$

Therefore, the invariant solution up to the first order in ε is

$$w(t,x,\varepsilon) = -\,\sigma_0\,\frac{t^2}{x} - x - \varepsilon\,\frac{(40\,\sigma_0\,\log t - 8\,\sigma_0 - 25)\,t^5 - 25}{50\,t^2\,x^3} + \mathcal{O}(\varepsilon^2).$$

We have an unperturbed state represented by a stretching modified by the viscosity effect. For large time this latter becomes dominant and the linear expansion is not longer valid. This can be probably ascribed to the stress-strain relation (22) which is linear in the viscosity. More sophisticated model with a non linear viscosity are currently under investigation by the author and will be the subject of a future paper.

5 Conclusions

In this paper we perform the group analysis of the nonlinear wave equation with a small dissipation (1) in the framework of the approximate symmetries.

We follow the guide lines of the method proposed by Fushchich and Shtelen [4], expanding in a perturbation series the dependent variables and removing the "drawback" of the impossibility to work in hierarchy in calculating symmetries.

In order to remove that "drawback", we introduce, according to the perturbation theory, the expansions of the dependent variables in the one-parameter Lie group of infinitesimal transformations of the equation (1). Equating to zero the coefficients of zero and first degree powers of ε, we obtain an approximate Lie operator which permits to solve in hierarchy the invariance condition of the system (5)–(6) starting from the classification of the unperturbed non linear wave equation (5).

The proposed strategy is consistent with the perturbation point of view and can be generalized in a simple way to the higher orders of approximation in ε.

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