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A comparative study of approximate symmetry and approximate homotopy symmetry to a class of perturbed nonlinear wave equations

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ABSTRACT

A comparative study of approximate symmetry and approximate homotopy symmetry to a class of perturbed nonlinear wave equations is performed. First, complete infinite-order approximate symmetry classification of the equation is obtained by means of the method originated by Fushchich and Shtelen. An optimal system of one-dimensional subalgebras is derived and used to construct general formulas of approximate symmetry reductions and similarity solutions. Second, we study approximate homotopy symmetry of the equation and construct connections between the two symmetry methods for the first-order and higher-order cases, respectively. The series solutions derived by the two methods are compared.

1. Introduction

Symmetry methods for differential equations, originally developed by Sophus Lie, have been evolved into one of the most explosive developments of mathematics and physics throughout the past century. There have been considerable important generalizations in this method which include nonclassical symmetry, Lie–Băcklund symmetry, potential symmetry, etc. [1–6]. Usually, with a continuous differential equation, we can study its invariance, symmetry properties and similarity reductions by means of the Lie symmetry method [1,3]. In particular, for the mathematical models described by differential equations containing arbitrary elements (parameters or functions) which have been found experimentally and so are not strictly fixed, the symmetry approach allows one to simplify them which make the models admit a symmetry group with certain properties or the most extensive symmetry group [7–9].

It is well known that there exist differential equations of physical interest with a small parameter possessing few exact symmetries or none at all and even if exist, the small parameter also disturbs symmetry group properties of the unperturbed equation [10,11]. Hence, two methods were introduced to study approximate symmetry of this type of equations. The first method due to Baikov et al. [10,11] represents a perturbation technique embedded into the standard procedure of the classical Lie symmetry method, which implements perturbation for symmetry generators. In 1989, Fushchich and Shtelen [12] proposed the second method which expand the dependent variables in terms of a small parameter (may be a physical parameter or artificially introduced) as the usual perturbation analysis and the method was later followed by Euler et al. [13–15]. In [16,17], two methods are applied to several equations and the comparisons are discussed.

Later, Liao [18,19] introduced the homotopy analysis method, which is a combination of the classical perturbation technique and homotopy concept as used in topology, to get series solutions of various types of nonlinear problems. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions of the

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governing equations. Quite recently, Jiao et al. [20] have proposed the approximate homotopy symmetry method (AHSM), which is an integration of the homotopy concept, perturbation analysis and the symmetry method, to study the sixth-order ill-posed Boussinesq equation [21].

In this paper, we study approximate symmetry and approximate homotopy symmetry of a class of perturbed nonlinear wave equations

$$u_{tt} + \epsilon u_t = [F(u)u_x]_x, \tag{1}$$

where ϵ is a small parameter, $F(u)$ is an arbitrary smooth function of u and make Eq. (1) nonlinear. Eq. (1) describes wave phenomena in shallow water, long radio engineering lines and isentropic motion of a fluid in a pipe etc. [11,22]. The perturbing term ϵu_t arises in the presence of dissipation and the function $F(u)$ is defined by the properties of the medium and the character of the dissipation. For Eq. (1), first- and second-order approximate symmetry analysis are discussed in [23,24], respectively.

Eq. (1) plays an important role in various applied problems. For example, when $F(u) = u$, Eq. (1) becomes

$$u_{tt} + \epsilon u_t = (uu_x)_x, \tag{2}$$

which attracts many researchers' attention. Eq. (2) arise from the one-dimensional gas dynamics [25] and the longitudinal wave propagation on a moving threadline [26] and one-dimensional wave propagation in nonlinear rate-dependent materials [27]. The approximate classical symmetries of Eq. (2) and the corresponding approximate solutions were discussed by Baikov et al. [11]. Solutions of Eq. (2) obtained by the approximate conditional symmetries were considered in [28].

In this paper, we solve the following three problems for Eq. (1).

1. Obtaining complete infinite-order approximate symmetry classification and constructing reduced equations and similarity solutions by a one-dimensional optimal system of subalgebras of approximate Lie algebra.

2. Performing complete infinite-order approximate homotopy symmetry classification and carrying out reductions by an optimal system of one-dimensional subalgebras of approximate homotopy Lie algebra.

3. Studying the connections between the approximate symmetry method (ASM) and the AHSM for different orders.

The rest of the paper is arranged as follows. After introducing some basic notions in Section 2, we concentrate on the complete infinite-order approximate symmetry classification of Eq. (1) by the method originated by Fushchich and Shtelen; the results are summarized in Section 3. Section 4 is devoted to construct approximate symmetry reductions and series solutions using the corresponding one-dimensional optimal system of subalgebras. Finally, complete approximate homotopy symmetry classification of Eq. (1) is performed and the connections between the two methods are established in Section 5. The last section contains a conclusion of our results.

2. Basic notions

We take the following nonlinear perturbed partial differential equation (PDE)

$$E(u) = E_0(u) + \epsilon E_1(u) = 0, \tag{3}$$

for example to review several concerning definitions, where E, E_0, E_1 are differential operators, ϵ is a perturbed parameter, $u = u(x, t)$ is an undetermined function, and x, t are independent variables.

2.1. Approximate symmetry method

Up until now, there exist the following two main methods to obtain approximate symmetry of perturbed PDE.

2.1.1. The method due to Fushchich and Shtelen

First, we consider an approximate symmetry of Eq. (3) by the method due to Fushchich and Shtelen. This method employs a perturbation of dependent variable and then the approximate symmetry of original equation is defined to an exact symmetry of the system corresponding to each order in the small parameter.

Expanding the dependent variable with respect to the small parameter ϵ yields

$$u = \sum_{k=0}^{\infty} \epsilon^k u_k, \quad 0 < \epsilon \ll 1, \tag{4}$$

then inserting it into Eq. (3) and separating at each order of ϵ, one obtains a coupled system

$$\begin{aligned}
&O(\epsilon^0) : E_0(u_0) = 0,\\
&O(\epsilon^1) : \frac{\partial}{\partial \epsilon} E(u_0 + \epsilon u_1)|_{\epsilon=0} = 0,\\
&O(\epsilon^2) : \frac{\partial^2}{\partial \epsilon^2} E(u_0 + \epsilon u_1 + \epsilon^2 u_2)|_{\epsilon=0} = 0,\\
&\ldots,\\
&O(\epsilon^i) : \frac{\partial^i}{\partial \epsilon^i} E\left(\sum_{k=0}^{i} \epsilon^k u_k\right)|_{\epsilon=0} = 0, \ldots.
\end{aligned} \tag{5}$$

Definition 1 (*Approximate Symmetry [12]*)**.** The kth order approximate symmetry of the nonlinear equation (3) is defined to an exact symmetry of the system of the first $k+1$ equations in Eq. (5).

2.1.2. The method due to Baikov et al.

In this approach, there is no perturbation of the dependent variables but a perturbation of the symmetry generator [10,11].

An approximate symmetry, $X = X_0 + \epsilon X_1$, of Eq. (3) is obtained by solving for X_1 in

$$X_1(E_0(u)) \mid_{E_0(u)=0} + H = 0, \tag{6}$$

where the auxiliary function H is obtained by

$$H = \frac{1}{\epsilon} X_0(E(u)) \mid_{E(u)=0} . \tag{7}$$

X_0 is an exact symmetry of unperturbed PDE $E_0(u) = 0$.

2.2. Approximate homotopy symmetry method

For approximate homotopy symmetry of Eq. (3), we consider the following homotopy model

$$H(u, q) = 0, \tag{8}$$

with $q \in [0, 1]$ an embedding homotopy parameter. The above homotopy model has the property

$$H(u, 0) = H_0(u), \qquad H(u, 1) = E(u), \tag{9}$$

where $H_0(u) = 0$ is a differential equation of which the solutions can be easily obtained.

Assuming that Eq. (3) has the homotopy series solutions of the form

$$u = \sum_{k=0}^{\infty} q^k u_k, \tag{10}$$

where u_k solve the following system

$$\begin{aligned}
&O(q^0) : H_0(u_0) = 0,\\
&O(q^1) : H_0'(u_0)u_1 + F_1(u_0) = 0,\\
&O(q^2) : H_0'(u_0)u_2 + F_2(u_0, u_1) = 0,\\
&\ldots\\
&O(q^i) : H_0'(u_0)u_i + F_i(u_0, u_1, \ldots, u_{k-1}) = 0,\\
&\ldots,
\end{aligned} \tag{11}$$

in which the operator $H_0'(u_0)$ is defined as

$$H_0'(u_0)f = \frac{\partial}{\partial \alpha} H_0(u_0 + \alpha f)|_{\alpha=0}, \tag{12}$$

with arbitrary function $f(x, t)$, and all F_i satisfy

$$F_i = \frac{1}{i!} \frac{\partial^i}{\partial q^i} E\left(\sum_{k \neq i} u_k q^k\right)|_{q=0}, \quad i = 1, 2, 3, \ldots. \tag{13}$$

Based on the above homotopy model, approximate homotopy symmetry is defined as follows.

Definition 2 (*Approximate Homotopy Symmetry [20]*)**.** The kth order approximate homotopy symmetry of the nonlinear equation (3) corresponds to an exact symmetry of the first $k+1$ equations in Eq. (11).

The homotopy model (8) can be freely chosen. Later for simplicity, the following simple homotopy model is exclusively taken

$$H(u, q) = (1 - q)E_0(u) + q\omega E(u) = 0, \tag{14}$$

where, thereinafter, $q \in [0, 1]$ is defined as in Eq. (8), ω denotes the convergence-control parameter.

3. Infinite-order approximate symmetry classification

In this section, we generalize the method in [12–15] to obtain complete infinite-order approximate symmetry classification of Eq. (1). Note that all symmetries in this paper are obtained by means of the differential characteristic set method [29,30].

Expanding the dependent variable with respect to ϵ as in expression (4), then one can expand $F(u)$ in a series in ϵ

$$F(u)=F\left(\sum_{k=0}^{\infty}\epsilon^k u_k\right)=\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\left[\left.\frac{\partial^k F(u)}{\partial\epsilon^k}\right|_{\epsilon=0}\right]. \tag{15}$$

Inserting the expansions into Eq. (1) and separating at each order of perturbation parameter, one has

$$u_{k,tt}+u_{k-1,t}=\left[\sum_{i=0}^{k}\frac{F^{(j)}(u_0)}{j_0!j_1!\cdots j_i!}u_{l_0}^{j_0}u_{l_1}^{j_1}\dots u_{l_i}^{j_i}u_{k-i,x}\right]_x,\quad k=0,1,\dots, \tag{16}$$

where, hereinafter, $u_{-1}=0$, $u_{l_i}^{j_i}=(u_{l_i})^{j_i}$, $j_0+j_1+\cdots+j_i=j$, $l_0j_0+l_1j_1+\cdots+l_ij_i=i$, $0\le j\le i$ and $l_0,l_1,\dots,l_i$ are not equal to zero and mutual inequivalent. j_i,l_i,j,k $(i=0,1,2,\dots)$ are nonnegative integers. The kth order approximate symmetries of Eq. (1) correspond to exact symmetries of the first $k+1$ equations in Eq. (16).

To simplify our calculations, we use the following equivalence transformation of Eq. (16). In particular, one has Proposition 1.

Proposition 1. *Any transformation of the form*

$$x=c_1\tilde{x}+c_2,\qquad t=\tilde{t}+c_3,\qquad u_k=c_4\tilde{u}_k+c_{k+5},\qquad \tilde{F}^{(j)}(\tilde{u}_0)=c_4^jc_1^{-2}F^{(j)}(u_0),\quad j,k=0,1,\dots, \tag{17}$$

where $c_1c_4\neq 0$, is an equivalence transformation of Eq. (16), i.e., transformation (17) maps Eq. (16) into

$$\tilde{u}_{k,\tilde{t}\tilde{t}}+\tilde{u}_{k-1,\tilde{t}}=\left[\sum_{i=0}^{k}\frac{\tilde{F}^{(j)}(\tilde{u}_0)}{j_0!j_1!\cdots j_i!}\tilde{u}_{l_0}^{j_0}\tilde{u}_{l_1}^{j_1}\dots\tilde{u}_{l_i}^{j_i}\tilde{u}_{k-i,\tilde{x}}\right]_{\tilde{x}}. \tag{18}$$

Now, we consider a one-parameter Lie symmetry group of local transformations with an infinitesimal operator of the form

$$X=\xi(x,t,u_0,u_1\dots)\partial_x+\tau(x,t,u_0,u_1\dots)\partial_t+\sum_{i=0}^{\infty}\eta_i(x,t,u_0,u_1\dots)\partial_{u_i}, \tag{19}$$

which leaves Eq. (16) invariant. Consequently, using Lie infinitesimal criterion [1,3], acting on the first $k+1$ equations in (16) with the second prolongation of operator X, one has

$$X^{(2)}\left[u_{k,tt}+u_{k-1,t}-\left[\sum_{i=0}^{k}\frac{F^{(j)}(u_0)}{j_0!j_1!\cdots j_i!}u_{l_0}^{j_0}u_{l_1}^{j_1}\dots u_{l_i}^{j_i}u_{k-i,x}\right]_x\right]=0,\ k=0,1,\dots, \tag{20}$$

for any u_i $(i=0,1,\dots,k)$ solve the first $k+1$ equations in Eq. (16). $X^{(2)}$ is given by

$$X^{(2)}=X+\sum_{i=0}^{k}[\eta_i^x\partial_{u_{i,x}}+\eta_i^t\partial_{u_{i,t}}+\eta_i^{xx}\partial_{u_{i,xx}}+\eta_i^{tt}\partial_{u_{i,tt}}], \tag{21}$$

with

$$\begin{aligned}&\eta_i^x=D_x\eta_i-u_{i,x}D_x\xi-u_{i,t}D_x\tau,\qquad \eta_i^t=D_t\eta_i-u_{i,x}D_t\xi-u_{i,t}D_t\tau,\\&\eta_i^{xx}=D_x\eta_i^x-u_{i,xx}D_x\xi-u_{i,xt}D_x\tau,\qquad \eta_i^{tt}=D_t\eta_i^t-u_{i,xt}D_t\xi-u_{i,tt}D_t\tau,\end{aligned} \tag{22}$$

where D_x, D_t are total differential operators about x, t, respectively.

Until now, there is no general method to construct infinite-order approximate symmetry classification of perturbed PDE, especially for the PDE containing arbitrary functions. Here, we adopt the method of mathematical induction. Inducing from the first-, second- and third-order approximate symmetry classifications and enlarging the domain of k by degrees and repeating similar procedures, we discover the formal coherence of ξ, τ and η_i for infinite-order approximate symmetry of Eq. (1).

Table 1
Approximate symmetries obtained by the approach of Baikov et al.

$F(u)$	Approximate symmetry operators
Arbitrary	$X_1 = \epsilon(x\partial_x + t\partial_t)$
αe^u	$X_1, X_2 = x\partial_x + t\partial_t + \epsilon t\left(\frac{t}{2}\partial_t - 2\partial_u\right), X_3 = \epsilon(x\partial_x + 2\partial_t)$
αu^μ	$X_1, X_4 = t\partial_t - \frac{2}{\mu}u\partial_u + \epsilon\frac{\mu t}{\mu+4}t\left(\frac{t}{2}\partial_t + \frac{2}{\mu}u\partial_u\right), X_5 = \epsilon\left(x\partial_x + \frac{2}{\mu}u\partial_u\right)$
$\alpha u^{-4/3}$	$X_1, X_5, X_7 = \epsilon(x^2\partial_x - 3xu\partial_u)$
αu^{-4}	$X_1, X_5, X_7 = \epsilon(t^2\partial_t + tu\partial_u)$

Table 2
Optimal system of one-dimensional subalgebras of Eq. (1) by ASM.

$F(u)$	The operators of the optimal system
Arbitrary	$X_1, X_2 + cX_3, c \in R$
$e^{\lambda u}$	$X_1 + aX_4, X_2 + cX_3, aX_3 + X_4, -\lambda/2X_1 + X_4 + aX_2, a \in R$
$u^\lambda\ \left(u \neq -\frac{4}{3}\right)$	$X_1 + aX_5, X_2 + aX_3, X_1 + X_5 + aX_3, -\frac{1}{2}(\lambda - 2)X_1 + X_5 + aX_2, a \in R$
$u^{-\frac{4}{3}}$	In addition to the operators of $F(u) = u^\lambda$ in this table: $aX_3 + X_6, 5X_1 + 3X_5 + aX_6, X_3 + X_6 - bX_2, X_2 + bX_3 - X_6, X_5 + X_6 - bX_2, X_2 + X_5 - bX_6, X_1 + X_6 - bX_2, X_1 + X_2 - bX_6, a \in R, b \in R^+$

Case 1. $F(u)$ is arbitrary.

Splitting of the determining system with respect to the arbitrary elements and their non-vanishing derivatives gives $\xi = c_1x + c_3, \tau = c_1t + c_2, \eta_i = c_1iu_i$. As a result, Eq. (1) has a three-dimensional Lie algebra spanned by the operators

$$X_1 = x\partial_x + t\partial_t + \sum_{i=0}^{\infty} iu_i\partial_{u_i}, \qquad X_2 = \partial_x, \qquad X_3 = \partial_t. \tag{23}$$

Studying all possible cases of Eq. (1) up to the extended equivalence group leads to the following cases.

Case 2. $F(u) = e^{\lambda u}$.

In this case, in addition to the infinitesimal operators X_1, X_2, X_3, Eq. (1) admits approximate Lie symmetry operator

$$X_4 = \frac{\lambda}{2}x\partial_x + \partial_{u_0}. \tag{24}$$

Case 3. $F(u) = u^\lambda(\lambda \neq -4/3)$.

We obtain X_1, X_2, X_3 and

$$X_5 = \frac{1}{2}(\lambda - 2)x\partial_x - t\partial_t + u_0\partial_{u_0} - \sum_{i=1}^{\infty} iu_{i+1}\partial_{u_{i+1}}. \tag{25}$$

Case 4. $F(u) = u^{-4/3}$.

The approximate invariance algebra of Eq. (1) is generated by the operators X_1, X_2, X_3, X_5 and

$$X_6 = -\frac{1}{3}x^2\partial_x + x\sum_{i=0}^{\infty} u_i\partial_{u_i}. \tag{26}$$

Remarks 1. Comparing with the method developed by Baikov et al. for Eq. (1), we obtain more new infinite-order approximate symmetries. Table 1 gives the approximate symmetries obtained by the approach of Baikov et al. [11].

4. An optimal system and approximate reductions

4.1. An optimal system of one-dimensional subalgebras

A Lie group (or Lie algebra) usually contains infinitely many one-dimensional subgroups (or subalgebras), it is impossible to use all of them to construct invariant solutions. Hence, a well-known standard procedure [1,2] allows us to classify all one-dimensional subalgebras into subsets of conjugate subalgebras, i.e. an optimal system. In this subsection, we investigate one-dimensional optimal system of subalgebras of approximate Lie algebra of Eq. (1).

Proposition 2. *For each case of approximate symmetry classification results of Eq. (1), the corresponding optimal system of one-dimensional subalgebras is given in Table 2.*

Proof. We take the case $F(u) = u^{-4/3}$ for example to present how to construct an optimal system of one-dimensional subalgebras, while the other three cases can be done with similar method.

Table 3 shows the Lie brackets of X_1, X_2, X_3, X_5, X_6.

Table 3
Lie brackets of X_1, X_2, X_3, X_5, X_6.

	X_1	X_2	X_3	X_5	X_6
X_1	0	$-X_2$	$-X_3$	0	X_6
X_2	X_2	0	0	$-\frac{5}{3}X_2$	X_1+X_5
X_3	X_3	0	0	$-X_3$	0
X_5	0	$\frac{5}{3}X_2$	X_3	0	$-\frac{5}{3}X_6$
X_6	$-X_6$	$-X_1-X_5$	0	$\frac{5}{3}X_6$	0

Table 4
Adjoint representation of X_1, X_2, X_3, X_5, X_6.

	X_1	X_2	X_3	X_5	X_6
X_1	X_1	$e^{\epsilon}X_2$	$e^{\epsilon}X_3$	X_5	$e^{-\epsilon}X_6$
X_2	$X_1-\epsilon X_2$	X_2	X_3	$X_5+\frac{5}{3}\epsilon X_2$	$X_6-\epsilon(X_1+X_5)-\frac{\epsilon^2}{3}X_2$
X_3	$X_1-\epsilon X_3$	X_2	X_3	$X_5+\epsilon X_3$	X_6
X_5	X_1	$e^{-\frac{5}{3}\epsilon}X_2$	$e^{-\epsilon}X_3$	X_5	$e^{\frac{5}{3}\epsilon}X_6$
X_6	$X_1+\epsilon X_6$	$X_2+\epsilon(X_1+X_5)-\frac{\epsilon^2}{3}X_6$	X_3	$X_5-\frac{5}{3}\epsilon X_6$	X_6

In Table 4, the adjoint representation of X_1, X_2, X_3, X_5, X_6 are presented, with the (i,j) entry indicating $\mathrm{Ad}(\exp(\epsilon X_i))X_j$ defined as

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \cdots. \tag{27}$$

Before proceeding with the classification scheme we need to identify invariants of the full adjoint action. These invariants place restrictions on how far we can expect to simplify a given arbitrary element spanned by X_1, X_2, X_3, X_5, X_6

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_5X_5 + a_6X_6, \tag{28}$$

where a_i $(i = 1, 2, 3, 5, 6)$ are arbitrary constants. Here, we only concentrate on the case $a_6 \neq 0$ because the case $a_6 = 0$ belongs to the cases $F(u) = u^{\lambda}\ \left(u \neq -\frac{4}{3}\right)$ in Table 2.

The adjoint representation group is generated (via Lie equations) by Lie algebra X_1, X_2, X_3, X_5, X_6 spanned by the following symmetries (see [10, vol. 2])

$$\Delta_i = c_{ij}^k e^j \frac{\partial}{\partial e^k}, \quad i, j = 1, 2, 3, 5, 6, \tag{29}$$

where c_{ij}^k are the structure constants in Table 3. Explicitly, we have

$$\begin{aligned}
\Delta_1 &= -a_2\frac{\partial}{\partial a_2} - a_3\frac{\partial}{\partial a_3} + a_6\frac{\partial}{\partial a_6},\\
\Delta_2 &= a_1\frac{\partial}{\partial a_2} - \frac{5}{3}a_5\frac{\partial}{\partial a_2} + a_6\left(\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_5}\right),\\
\Delta_3 &= (a_1 - a_5)\frac{\partial}{\partial a_3},\\
\Delta_5 &= \frac{5}{3}a_2\frac{\partial}{\partial a_2} + a_3\frac{\partial}{\partial a_3} - \frac{5}{3}a_6\frac{\partial}{\partial a_6},\\
\Delta_6 &= -a_1\frac{\partial}{\partial a_6} - a_2\left(\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_5}\right) + \frac{5}{3}a_5\frac{\partial}{\partial a_6}.
\end{aligned} \tag{30}$$

If a function $\rho(a_1, a_2, a_3, a_5, a_6)$ is an invariant of the full adjoint action, then one has

$$\Delta_i(\rho) = 0, \quad i = 1, 2, 3, 5, 6. \tag{31}$$

Using the method of characteristic equations [1], after direct computations, we get general solutions of Eq. (31)

$$\rho = f\left(a_1 - a_5, a_1a_5 - \frac{4}{3}a_5^2 - a_2a_6\right). \tag{32}$$

In particular,

$$\rho_1 = a_1 - a_5, \qquad \rho_2 = a_1a_5 - \frac{4}{3}a_5^2 - a_2a_6 \tag{33}$$

are two invariants of the full adjoint action given in Table 4.

The invariants ρ_1 and ρ_2 provide us a key condition to simplify X by the action of adjoint maps. For example, if $\rho_1 \neq 0$, we cannot simultaneously make a_1 and a_5 zero through adjoint maps.

To begin the classification process, we investigate the coefficients a_1, a_2, a_3, a_5 and a_6. If X is presented in (28), then

$$\widehat{X} = \mathrm{Ad}(\exp(\beta X_6)) \circ \mathrm{Ad}(\exp(\alpha X_2))X = \widehat{a_1}X_1 + \widehat{a_2}X_2 + \widehat{a_3}X_3 + \widehat{a_5}X_5 + \widehat{a_6}X_6 \tag{34}$$

has coefficients

$$\begin{aligned} \widehat{a_1} &= a_1 - \alpha a_6 + \beta\left(a_2 - a_1\alpha + \frac{5}{3}a_5\alpha - \frac{1}{3}a_6\alpha^2\right), \\ \widehat{a_2} &= a_2 - a_1\alpha + \frac{5}{3}a_5\alpha - \frac{1}{3}a_6\alpha^2, \\ \widehat{a_3} &= a_3, \qquad \widehat{a_5} = a_5 - a_6\alpha + \beta\left(a_2 - a_1\alpha + \frac{5}{3}a_5\alpha - \frac{1}{3}a_6\alpha^2\right), \\ \widehat{a_6} &= a_6 + \beta\left(a_1 - \frac{5}{3}a_5 + \frac{2}{3}a_6\alpha\right) - \frac{1}{3}\beta^2\left(a_2 - a_1\alpha + \frac{5}{3}a_5\alpha - \frac{1}{3}a_6\alpha^2\right). \end{aligned} \tag{35}$$

In the proceeding, we discuss the coefficients of X_i $(i = 1, 2, 3, 5, 6)$ in $\widehat{X}$ by virtue of the arbitrary real constants α and β. For the quadratic equation $\widehat{a_2} = 0$, the discriminant of it is $\Delta = (5a_5/3 - a_1)^2 + 4a_2a_6/3 = \rho_1^2 - 4\rho_2/3$, we therefore consider three cases for the classification of one-dimensional subalgebras which depend on the sign of Δ.

Case 1. $\Delta > 0$.

In this case, we choose $\alpha = (-3a_1 + 5a_5 + 3\sqrt{\Delta})/(2a_6)$, $\beta = -a_6/\sqrt{\Delta}$, then $\widehat{a_2} = \widehat{a_6} = 0$, $\widehat{X}$ becomes

$$\widehat{X} = \frac{5}{2}(\rho_1 + \sqrt{\Delta})X_1 + a_3X_3 + \frac{3}{2}(\rho_1 + \sqrt{\Delta})X_5. \tag{36}$$

Applying the adjoint map $\mathrm{Ad}(\exp(\gamma X_3))$ to $\widehat{X}$ and setting $\gamma = a_3/(\rho_1 + \sqrt{\Delta})$, we arrive at

$$\bar{X} = \frac{5}{2}(\rho_1 + \sqrt{\Delta})X_1 + \frac{3}{2}(\rho_1 + \sqrt{\Delta})X_5, \tag{37}$$

which is scaled to $\overline{X} = 5X_1 + 3X_5$.

Case 2. $\Delta = 0$.

In this case, we consider the following two subcases.

Subcase 1. $\rho_1 = 0$.

The case $\rho_1 = 0$ implies $\alpha = a_1/a_6$, then we obtain

$$\widehat{X} = a_3X_3 + a_6X_6, \tag{38}$$

which is scaled to $\bar{X} = aX_3 + X_6$, $a \in R$.

Subcase 2. $\rho_1 \neq 0$.

Acting on $\widehat{X}$ by the adjoint map $\mathrm{Ad}(\exp(\kappa X_3))$, we have

$$\overline{X} = \frac{5}{2}\rho_1X_1 + (a_3 - \rho_1\kappa)X_3 + \frac{3}{2}\rho_1X_5 + a_6X_6, \tag{39}$$

assuming $\kappa = a_3/\rho_1$, we obtain

$$\overline{X} = \frac{5}{2}\rho_1X_1 + \frac{3}{2}\rho_1X_5 + a_6X_6, \tag{40}$$

which is scaled to $\overline{X} = 5X_1 + 3X_5 + aX_6$, $a \in R$.

Case 3. $\Delta < 0$.

Since $\Delta < 0$, we cannot make two of the coefficients $\widehat{a_1}, \widehat{a_2}, \widehat{a_5}$ and $\widehat{a_6}$ vanish simultaneously but make one of the coefficients vanish. $\Delta < 0$ means that the curve of quadratic function $f(\alpha) = \widehat{a_2}$ has no intersection point with the α-axis, i.e. $a_6 > 0, f(\alpha) < 0$ and $a_6 < 0, f(\alpha) > 0$.

Acting on X with the adjoint map $\mathrm{Ad}(\exp(\alpha X_2))$, we obtain

$$\widehat{X} = (a_1 - \alpha a_6)X_1 + \left(a_2 - a_1\alpha + \frac{5}{3}a_5\alpha - \frac{1}{3}a_6\alpha^2\right)X_2 + a_3X_3 + (a_5 - a_6\alpha)X_5 + a_6X_6.$$

Next, we only consider the case $a_6 > 0, f(\alpha) < 0$, while the case $a_6 < 0, f(\alpha) > 0$ gives the same classification results.

Subcase 1. Setting $\alpha = a_1/a_6$, we get

$$\widehat{X} = -\frac{\rho_2}{3a_6}X_2 + a_3X_3 + \rho_1X_5 + a_6X_6. \tag{41}$$

Since $\Delta < 0$ implies $\rho_2 > 3\rho_1^2/4 \geq 0$, then one has $\rho_2/(3a_6) > 0$. Hence, we consider the following two cases.

(I) If $\rho_1 = 0$, $\widehat{X}$ becomes $-\rho_2/(3a_6)X_2 + a_3X_3 + a_6X_6$, then applying $\mathrm{Ad}(\exp(\gamma X_1))$ to it, we get

$$\overline{X} = -\frac{\rho_2}{3a_6}\mathrm{e}^{\gamma}X_2 + a_3\mathrm{e}^{\gamma}X_3 + a_6\mathrm{e}^{-\gamma}X_6. \tag{42}$$

To proceed, two different cases arise.

(a) $a_3 > 0$.
In this case, $\overline{X}$ is equivalent to $X_3 + X_6 - aX_2$ after scaling the coefficients of it, where $a \in R^+$.

(b) $a_3 < 0$.
Here, $\overline{X}$ is equivalent to $X_2 - X_6 + aX_3$ after scaling the coefficients of it, where $a \in R^+$.

(II) If $\rho_1 \neq 0$, by the adjoint map $\mathrm{Ad}(\exp(\kappa X_3))$, we have

$$\overline{X} = \mathrm{Ad}(\exp(\kappa X_3))\widehat{X} = -\frac{\rho_2}{3a_6}X_2 + (a_3 + \kappa\rho_1)X_3 + \rho_1X_5 + a_6X_6. \tag{43}$$

Assume $\kappa = a_3/\rho_1$, Eq. (43) becomes $\overline{X} = -\rho_2/(3a_6)X_2 + \rho_1X_5 + a_6X_6$. Again, with adjoint map $\mathrm{Ad}(\exp(\gamma X_1))$, we obtain

$$\widetilde{X} = \mathrm{Ad}(\exp(\gamma X_1))\widehat{X} = -\frac{\rho_2}{3a_6}\mathrm{e}^{\gamma}X_2 + \rho_1X_5 + a_6\mathrm{e}^{-\gamma}X_6. \tag{44}$$

(a) $\rho_1 > 0$.
In this case, $\overline{X}$ is equivalent to $X_5 + X_6 - aX_2$ after scaling the coefficients of it, where $a \in R^+$.

(b) $\rho_1 < 0$.
Here, $\overline{X}$ is equivalent to $X_2 + X_5 - aX_6$ after scaling the coefficients of it, where $a \in R^+$.

Subcase 2. Setting $\alpha = a_5/a_6$, we get

$$\widehat{X} = \rho_1X_1 - \frac{\rho_2}{3a_6}X_2 + a_3X_3 + a_6X_6. \tag{45}$$

The case $\rho_1 = 0$ is same to (I) of subcase 1 in *Case* 3. If $\rho_1 \neq 0$, with the adjoint map $\mathrm{Ad}(\exp(a_3/\rho_1X_3))$, we have

$$\widetilde{X} = \rho_1X_1 - \frac{\rho_2}{a_6}X_2 + a_6X_6. \tag{46}$$

Again, with adjoint map $\mathrm{Ad}(\exp(\lambda X_1))$, we obtain

$$\widetilde{X} = \mathrm{Ad}(\exp(\lambda X_1))\widehat{X} = -\frac{\rho_2}{a_6}\mathrm{e}^{\lambda}X_2 + \rho_1X_1 + a_6\mathrm{e}^{-\lambda}X_6. \tag{47}$$

Hence, we consider the following two cases.

(a) $\rho_1 > 0$.
In this case, $\widetilde{X}$ is equivalent to $X_1 + X_6 - aX_2$ after scaling the coefficients of it, where $a \in R^+$.

(b) $\rho_1 < 0$.
Here, $\widetilde{X}$ is equivalent to $X_1 + X_2 - aX_6$ after scaling the coefficients of it, where $a \in R^+$.

This proves it. □

4.2. *Approximate symmetry reduction*

The group properties are very useful for the construction of invariant solutions of the differential equation under study. Using the invariants of the subgroups associated with the generators, one can reduce the original equation (1) to ordinary differential equations and then construct approximate solutions. In this subsection, different subclasses are well investigated based on the optimal system of preceding subsection and affluent approximate symmetry reductions and invariant solutions are constructed. Note that c_1, c_2, c_3, c_4 are arbitrary constants and P_i exist for $i = 0, 1, 2, \ldots$ if no special notes are added in this section.

4.2.1. *Reduction by X_1*

The similarity variables of X_1 is derived from the determining equations

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}u_0}{0} = \frac{\mathrm{d}u_1}{u_1} = \cdots = \frac{\mathrm{d}u_k}{ku_k}, \tag{48}$$

which are $z = x/t$, $P_0 = u_0$. Viewing P_0 as a function of z, we have $u_0 = P_0(z)$. Similarly, we get other similarity variables

$$u_1 = tP_1(z), \qquad u_2 = t^2P_2(z), \qquad u_3 = t^3P_3(z), \ldots, u_k = t^kP_k(z). \tag{49}$$

Inserting these variables into Eq. (16), we have kth order reduced equations given by

$$z^2P_{k,zz} - zP_{k-1,z} + (k-1)[kP_k - 2zP_{k,z} + P_{k-1}] = \left[\sum_{i=0}^{k} \frac{F^{(j)}(P_0)}{j_0!j_1!\cdots j_i!} P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z. \tag{50}$$

In terms of Eq. (4), we obtain series solutions of Eq. (1) in the form

$$u = \sum_{k=0}^{\infty}(\epsilon t)^k P_k(z). \tag{51}$$

4.2.2. Reduction by $cX_2 + X_3$

When $c = 0$, the invariant solution generated by $cX_2 + X_3$ is a steady one and when $c \neq 0$ provides traveling wave solutions, where c is the speed of wave. We assume $c \neq 0$ and omit the procedure of calculating invariants thereinafter.

Following standard procedure, we integrate the characteristic equations for $cX_2 + X_3$ to get similarity variables $z = x - ct$, $u_i = P_i$. Then we get series solutions of Eq. (1) in the form

$$u = \sum_{k=0}^{\infty}\epsilon^k P_k(x - ct), \tag{52}$$

where P_k satisfy

$$c^2P_{k,zz} - cP_{k-1,z} = \left[\sum_{i=0}^{k} \frac{F^{(j)}(P_0)}{j_0!j_1!\cdots j_i!} P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z. \tag{53}$$

4.2.3. Reduction by $aX_3 + X_4$

Here, we consider two different cases.

(I) $a = 0$

By X_4, Eq. (16) with $F(u) = e^{\lambda u}$ can be reduced to

$$P_{k,zz} + P_{k-1,z} = -2e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-1}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\dots P_{l_i}^{j_i}, \tag{54}$$

where $z = t$, $P_0 = u_0 - 2\ln x/\lambda$, $P_i = u_i$, $(i = 1, 2, \dots)$.

(II) $a \neq 0$

By $aX_3 + X_4$, Eq. (16) with $F(u) = e^{\lambda u}$ can be reduced to

$$\begin{aligned}
&\frac{1}{a^2}P_{0,zz} = -\frac{2}{\lambda}e^{\lambda P_0}(P_{0,z} - 1) + \frac{4}{\lambda^2}[e^{\lambda P_0}(P_{0,z} - 1)]_z,\\
&\frac{1}{a^2}P_{1,zz} + \frac{1}{a}P_{0,z} = -\frac{2}{\lambda}e^{\lambda P_0}(P_{1,z} + \lambda P_1P_{0,z} - \lambda P_1) + \frac{4}{\lambda^2}[e^{\lambda P_0}(P_{1,z} + \lambda P_1P_{0,z} - \lambda P_1)]_z,\\
&\frac{1}{a^2}P_{k,zz} + \frac{1}{a}P_{k-1,z} = -2e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-1}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\cdots P_{l_i}^{j_i}P_{k-i,z} + 4\left[e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-2}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\dots P_{l_i}^{j_i}P_{k-i,z}\right]_z,
\end{aligned} \tag{55}$$

where $z = t/a - 2\ln x/\lambda$, $P_0 = u_0 - 2\ln x/\lambda$, $P_i = u_i$, $(i = 1, 2, \dots, k)$.

4.2.4. Reduction by $X_1 + aX_4 (a \neq 0)$

By $X_1 + aX_4$, Eq. (16) with $F(u) = e^{\lambda u}$ can be reduced to

$$\begin{aligned}
&\left(\frac{\lambda}{2}a + 1\right)^2 z^2P_{0,zz} - \left(\frac{\lambda}{2}a + 1\right)\left(\frac{\lambda}{2}a + 2\right)zP_{0,z} - a = [e^{\lambda P_0}P_{0,z}]_z,\\
&\left(\frac{\lambda}{2}a + 1\right)^2 [z^2P_{1,zz} + zP_{1,z}] - \left(\frac{\lambda}{2}a + 1\right)zP_{0,z} + a = [e^{\lambda P_0}(P_{1,z} - \lambda P_1P_{0,z})]_z,\\
&\left(\frac{\lambda}{2}a + 1\right)\left[\left(\frac{\lambda}{2}a + 1\right)z^2P_{k,zz} - (2k-1)zP_{k,z} - zP_{k-1,z}\right] + (k-1)(kP_k + P_{k-1})\\
&\quad = \left[e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^j}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\dots P_{l_i}^{j_i}P_{k-i,z}\right]_z,
\end{aligned} \tag{56}$$

where $z = xt^{-(1+\lambda a/2)}$, $P_0 = u_0 - a\ln t$, $P_i = t^{-i}u_i$, $(i = 1, 2, \dots, k)$.

4.2.5. Reduction by $-\lambda/2X_1 + aX_2 + X_4$

In the proceeding, two different cases arise.

Case 1: $a \neq 0$

By $-\lambda X_1/2 + aX_2 + X_4$, Eq. (16) with $F(u) = e^{\lambda u}$ can be reduced to

$$\begin{aligned}
&\frac{4}{\lambda^2}P_{0,zz} - \frac{2}{\lambda}P_{0,z} - \frac{2}{\lambda} = \frac{1}{a^2}[e^{\lambda P_0}P_{0,z}]_z,\\
&\frac{4}{\lambda^2}P_{1,zz} - \frac{2}{\lambda}P_{1,z} + \frac{2}{\lambda}(P_{0,z}+1) = \frac{1}{a^2}[e^{\lambda P_0}(P_{1,z} + \lambda P_1 P_{0,z})]_z,\\
&\frac{4}{\lambda^2}P_{k,zz} + \frac{2}{\lambda}(2k-1)P_{k,z} + \frac{2}{\lambda}P_{k-1,z} + (k-1)(kP_k + P_{k-1}) = \frac{1}{a^2}\left[e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^j}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z,
\end{aligned} \tag{57}$$

where $z = x/a + 2\ln t/\lambda$, $P_0 = u_0 - x/a$, $P_i = t^{-i}u_i$, $(i = 1, 2, \ldots, k)$.

Case 2: $a = 0$

By $-\lambda X_1/2 + X_4$, Eq. (16) with $F(u) = e^{\lambda u}$ can be reduced to

$$\begin{aligned}
&[e^{\lambda P_0}P_{0,z}]_z = \frac{1}{\lambda},\\
&[e^{\lambda P_0}(P_{1,z} + \lambda P_1 P_{0,z})]_z = -\frac{2}{\lambda},\\
&(k-1)[kP_k + P_{k-1}] = \left[e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^j}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z,
\end{aligned} \tag{58}$$

where $z = x$, $P_0 = u_0 + 2\ln t/\lambda$, $P_i = t^{-i}u_i$, $(i = 1, 2, \ldots, k)$.

From the first and second equations of Eq. (58), we obtain

$$P_0 = \frac{1}{\lambda}\ln\left(\frac{1}{2}z^2 + c_1 z + c_2\right), \qquad P_1 = e^{-\frac{1}{2}z\lambda(2c_1+z)}\left[\int_1^z \frac{2e^{\frac{1}{2}\lambda s(s+2c_1)}(c_3\lambda - 2s)}{\lambda(s^2 + 2c_1 s + 2c_2)}\,ds + c_4\right], \tag{59}$$

then the solutions of Eq. (1) are expressed in the form

$$\begin{aligned}
u = {} & \frac{1}{\lambda}\left[\ln\left(\frac{1}{2}x^2 + c_1 x + c_2\right) - 2\ln t\right]\\
& + \epsilon t e^{-\frac{1}{2}x\lambda(x+2c_1)}\left[\int_1^x \frac{2e^{\frac{1}{2}\lambda s(s+2c_1)}(c_3\lambda - 2s)}{\lambda(s^2 + 2c_1 s + 2c_2)}\,ds + c_4\right] + \sum_{k=2}^{\infty}(\epsilon t)^k P_k,
\end{aligned} \tag{60}$$

where P_k, $(k = 2, \ldots)$ satisfy the third equation of Eq. (58).

4.2.6. Reduction by X_5

In the proceeding, two different cases arise.

Case 1: $\lambda = 2$

By X_5, Eq. (16) with $F(u) = u^2$ can be reduced to

$$(k-2)[(k-1)P_k + P_{k-1}] = \left[\sum_{i=0}^{k}\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z, \tag{61}$$

where $z = x$, $P_i = t^{1-i}u_i$.

Case 2: $\lambda \neq 2$

By X_4, Eq. (16) with $F(u) = u^\lambda$ can be reduced to

$$\begin{aligned}
z^2P_{k,zz} + z(2kP_{k,z} + P_{k-1,z}) + (k-1)(kP_k + P_{k-1}) = {} & \frac{2(4-\lambda)}{(\lambda-2)^2}z^{3-\lambda}\left[\sum_{i=0}^{k}\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]\\
& + \frac{4}{(\lambda-2)^2}z^{4-\lambda}\left[\sum_{i=0}^{k}\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z,
\end{aligned} \tag{62}$$

where $z = tx^{2/(\lambda-2)}$, $P_i = t^{1-i}u_i$.

4.2.7. Reduction by $X_1 + aX_5$

Here, we consider the following two different cases.

Case 1. $a \neq 1$.

By $X_1 + aX_5$, Eq. (16) with $F(u) = u^\lambda$ can be reduced to

$$\left(1 + \frac{\lambda a}{2(1-a)}\right)^2 z^2 P_{k,zz} + \left(k + \frac{2a-1}{1-a}\right)\left[\left(k + \frac{a}{1-a}\right)P_k + P_{k-1}\right] - \left(1 + \frac{\lambda a}{2(1-a)}\right)\left[\left(k - 2 + \frac{(2-\lambda)a}{2(1-a)}\right)zP_{k,z} + zP_{k-1,z} + \left(k + \frac{a}{1-a}\right)zP_{k,z}\right] = \left[\sum_{i=0}^{k} \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i} P_{k-i,z}\right]_z, \tag{63}$$

where $z = xt^{-1-a\lambda/(2-2a)}$, $P_i = t^{-i-a/(1-a)}u_i$.

Case 2. $a = 1$.

By $X_1 + X_5$, Eq. (16) with $F(u) = u^\lambda$ can be reduced to

$$P_{k,zz} + P_{k-1,z} = \frac{\lambda+2}{\lambda}\left[\sum_{i=0}^{k} \frac{2(\lambda-1)! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i}\right], \tag{64}$$

where $z = t$, $P_i = x^{-2/\lambda}u_i$.

In particular, $\lambda = -2$, after integrating once, Eq. (64) becomes

$$P_{k,z} + P_{k-1} = c_1, \tag{65}$$

which has the general solutions in the form

$$P_k = \frac{(-1)^k c_1}{(k+1)!} z^{k+1} + \sum_{i=0}^{k} (-1)^i \frac{1}{i!}(c_{k+2-i} - c_1) z^i + c_1, \tag{66}$$

then Eq. (1) for $F(u) = u^{-2}$ has solution in the form

$$u = \frac{1}{x} \sum_{k=0}^{\infty} \epsilon^k \left[\frac{(-1)^k c_1}{(k+1)!} t^{k+1} + \sum_{i=0}^{k} (-1)^i \frac{1}{i!}(c_{k+2-i} - c_1) t^i + c_1\right]. \tag{67}$$

4.2.8. Reduction by $X_1 + X_5 + aX_3$

Here, we only consider the case $a \neq 0$, since with $a = 0$, $X_1 + X_5 + aX_3$ becomes $X_1 + X_5$, which is discussed in Section 4.2.7. By $X_1 + X_5 + aX_3$, Eq. (16) with $F(u) = u^\lambda$ can be reduced to

$$\frac{1}{a^2}(P_k + aP_{k-1}) - \frac{\lambda}{4a^2}[(4-\lambda)zP_{k,z} - z^2 P_{k,zz} + 2azP_{k-1,z}] = \left[\sum_{i=0}^{k} \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i} P_{k-i,z}\right]_z, \tag{68}$$

where $z = xe^{-\lambda t/(2a)}$, $P_i = e^{-t/a}u_i$.

4.2.9. Reduction by $-(\lambda - 2)X_1/2 + X_5 + aX_2$

The case $a = 0$ is not considered here because it is included in Section 4.2.7. By $-(\lambda - 2)X_1/2 + X_5 + aX_2$, Eq. (16) with $F(u) = u^\lambda$ can be reduced to

$$\left(k - 1 - \frac{2}{\lambda}\right)\left[\left(k - \frac{2}{\lambda}\right)P_k + \frac{2}{\lambda}P_{k,z} + P_{k-1}\right] + \frac{2}{\lambda^2}[(k\lambda - 2)P_{k,z} + 2P_{k,zz} + \lambda P_{k-1,z}] = \frac{1}{a^2}\left[\sum_{i=0}^{k} \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i} P_{k-i,z}\right]_z, \tag{69}$$

where $z = x/a + 2\ln t/\lambda$, $P_i = t^{2/\lambda - i}u_i$.

4.2.10. Reduction by X_6

By X_6, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$P_{k,zz} + P_{k-1,z} = 0, \tag{70}$$

where $P_{-1} = 0, z = t, P_i = x^3 u_i$.

Eq. (1) for $F(u) = u^{-4/3}$ has solutions in the form

$$u = \frac{1}{x^3}\sum_{k=0}^{\infty}\epsilon^k\left[\frac{(-1)^k c_1}{(k+1)!}t^{k+1} + \sum_{i=0}^{k}(-1)^i\frac{1}{i!}(c_{k+2-i} - c_1)t^i + c_1\right]. \tag{71}$$

4.2.11. Reduction by $aX_3 + X_6$ ($a \neq 0$)

By $aX_3 + X_6$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{a^2}P_{k,zz} + \frac{1}{a}P_{k-1,z} = 9\left[\sum_{i=0}^{k}\frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z, \tag{72}$$

where $\lambda = -4/3$ and $z = 2/x - t/a$, $P_i = x^3 u_i$.

4.2.12. Reduction by $5X_1 + 3X_5 + bX_6$

In this case, we consider the following two different cases.

Case 1. $b \neq 0$.

By $5X_1 + 3X_5 + bX_6$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\begin{aligned} P_{k,zz} + P_{k-1,z} = {} & 3b\sum_{i=0}^{k}[-4z^2(AP_{k-i,z})_z + 2(2k-2i+3)z(AP_{k-i})_z \\ & + (2k-1)(2k-2i+3)AP_{k-i} - 2(2k-1)zAP_{k-i,z}], \end{aligned} \tag{73}$$

where $\lambda = -4/3$ and $z = te^{-6/(bx)}$, $P_i = x^3\exp[-3(2i+3)/(bx)]u_i$ and

$$\begin{aligned} & \sum_{i=0}^{k}\left[3(2k-1)AP_{k-i} + 6z(AP_{k-i})_z + (2k-2i+3)AP_{k-i} + 2zAP_{k-i,z}\right] \equiv 0, \\ & A = \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}, \end{aligned} \tag{74}$$

which is proved by mathematical induction in the Appendix.

Case 2. $b = 0$.

By $5X_1 + 3X_5$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\left(k+\frac{3}{2}\right)\left[\left(k+\frac{1}{2}\right)P_k + P_{k-1}\right] = \left[\sum_{i=0}^{k}\frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z, \tag{75}$$

where $\lambda = -4/3$ and $z = x$, $P_i = t^{-i-3/2}u_i$.

4.2.13. Reduction by $X_3 + X_6 - bX_2$

Here, due to $b > 0$, without loss of generality, we set $b = 3\mu^2$ in order to simplify the computations. By $X_3 + X_6 - bX_2$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{9}\left[P_{k,zz} + 3\mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{k-1,z}\right] = \sum_{i=0}^{k}\left[\left(\frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right)_z - \frac{27\mu^2\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i}\right], \tag{76}$$

where

$$\sum_{i=0}^{k}\left[\frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z} + 3\left(\frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right)_z\right] \equiv 0, \tag{77}$$

and $\lambda = -4/3$ and $z = t/3 + 1/(3\mu)\arctan[x/(3\mu)]$, $P_i = (9\mu^2 + x^2)^{3/2}u_i$. Eq. (77) can be proved with similar method as Eq. (74) in the Appendix.

4.2.14. Reduction by $X_5 + X_6 - bX_2$

In this case, the characteristic equations for $X_5 + X_6 - bX_2$ are

$$\frac{\mathrm{d}x}{-\frac{1}{3}x^2 - \frac{5}{3}x - b} = \frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}u_i}{(x-i+1)u_i}, \tag{78}$$

so the following three different cases arise based on the discriminant of quadratic equation $x^2/3 + 5/3x + b = 0$.

Case 1. $12b - 25 > 0$.

By $X_5 + X_6 - bX_2$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{9}\mathrm{e}^{6z}[(P_{k,zz} + 3P_{k,z}) - 3\mathrm{e}^{3z}P_{k-1,z}] = \sum_{i=0}^{k}\Big[(3k-4)AP_{k-i,z} - 9bAP_{k-i} + 3(k-i-1)[(4-3k)AP_{k-i} + (AP_{k-i})_z] + (AP_{k-i,z})_z\Big], \tag{79}$$

where $\lambda = -4/3$,

$$z = \frac{2}{\sqrt{12b-25}}\arctan\left(\frac{2x+5}{\sqrt{12b-25}}\right) - \frac{1}{3}\ln t,$$
$$P_i = (x^2+5x+3b)^{\frac{3}{2}}\exp\left(\frac{3(2i+3)}{\sqrt{12b-25}}\arctan\left(\frac{2x+5}{\sqrt{12b-25}}\right)\right)u_i, \tag{80}$$

and

$$A = \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i},$$
$$\sum_{i=0}^{k}[3(AP_{k-i})_z + 3(4k-i)AP_{k-i} + AP_{k-i,z}] \equiv 0, \tag{81}$$

which can be proved with similar method as Eq. (74) in the Appendix.

Case 2. $12b - 25 = 0$.

By $X_5 + X_6 - bX_2$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{144}\mathrm{e}^{24z}[P_{k,zz} + 12P_{k,z}] - 12\mathrm{e}^{12z}P_{k-1,z} = \sum_{i=0}^{k}\Big[4(3k-4)AP_{k-i,z} - 300AP_{k-i} - 48(i-1)(4-3k)AP_{k-i} + 12(i-1)(AP_{k-i,z})_z\Big], \tag{82}$$

where $\lambda = -4/3$ and $z = -1/(4x+10) - \ln t/12$, $P_i = (2x+5)^3\exp(6i+9)/(2x+5)u_i$ and

$$A = \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i},$$
$$\sum_{i=0}^{k}[AP_{k-i,z} + 40AP_{k-i} + 12(3k+i)AP_{k-i,z} + 3(AP_{k-i})_z] \equiv 0, \tag{83}$$

which can be proved with similar method as Eq. (74) in the Appendix.

Case 3. $12b - 25 < 0$.

By $X_5 + X_6 - bX_2$, Eq. (16) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{9}\mathrm{e}^{6z}[(P_{k,zz} + 3P_{k,z}) - 3\mathrm{e}^{3z}P_{k-1,z}] = \sum_{i=0}^{k}[(3k-4)AP_{k-i,z} - 9bAP_{k-i} - 3(i-1)(4-3k)AP_{k-i} + (AP_{k-i,z})_z], \tag{84}$$

where $\lambda = -4/3$,

$$z = -\frac{2}{\sqrt{25-12b}}\mathrm{arctanh}\left(\frac{2x+5}{\sqrt{25-12b}}\right) - \frac{1}{3}\ln t,$$
$$P_i = (x^2+5x+3b)^{\frac{3}{2}}\exp\left(\frac{3(2i+3)}{\sqrt{25-12b}}\mathrm{arctanh}\left(\frac{2x+5}{\sqrt{25-12b}}\right)\right)u_i, \tag{85}$$

and

$$A = \frac{\lambda! P_0^{\lambda-j}}{(\lambda-j)! j_0! \cdots j_i!} P_{l_0}^{j_0} \cdots P_{l_i}^{j_i},$$

$$\sum_{i=0}^{k} [3(AP_{k-i})_z + 3(3k+i)AP_{k-i} + AP_{k-i,z}] \equiv 0, \quad (86)$$

which can be proved with similar method as Eq. (74) in the Appendix.

The reduced equations by the operators $X_1 + X_6 - bX_2$ and $X_1 + X_2 - bX_6$ are similar with the one by $X_5 + X_6 - bX_2$, so we omit them.

5. Connections between ASM and AHSM for Eq. (1)

In this section, we first concentrate on the complete approximate homotopy symmetry classification of Eq. (1), and then give a comparative study of ASM and AHSM for Eq. (1).

5.1. Approximate homotopy symmetry classification

For approximate homotopy symmetry of Eq. (1), we consider the following simple homotopy model

$$H(u, q) = (1-q)[u_{tt} - [F(u)u_x]_x] + q\omega[u_{tt} - [F(u)u_x]_x + \mu u_t] = 0 \quad (87)$$

with $\mu = \epsilon$. The above homotopy model has the property

$$H(u, 0) = u_{tt} - [F(u)u_x]_x, \qquad H(u, 1) = u_{tt} - [F(u)u_x]_x + \mu u_t. \quad (88)$$

Assuming $\omega = 1 - \theta$, one has

$$(1-\theta q)[u_{tt} - [F(u)u_x]_x] + q\mu(1-\theta)u_t = 0. \quad (89)$$

Expanding the dependent variable with respect to q as (10), then one can expand $F(u)$ in a series in homotopy parameter q

$$F(u) = F\left(\sum_{k=0}^{\infty} q^k u_k\right) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \left[\left. \frac{\partial^k F(u)}{\partial q^k} \right|_{q=0} \right]. \quad (90)$$

Substituting the expansions into Eq. (1) and separating at each order of homotopy parameter, one has

$$u_{k,tt} - \left[\sum_{i=0}^{k} \frac{F^{(j)}(u_0)}{j_0! j_1! \cdots j_i!} u_{l_0}^{j_0} u_{l_1}^{j_1} \cdots u_{l_i}^{j_i} u_{k-i,x} \right]_x + \mu(1-\theta) \sum_{i=0}^{k-1} \theta^{k-1-i} u_{i,t} = 0. \quad (91)$$

The kth order approximate homotopy symmetries of Eq. (1) correspond to exact symmetries of the first $k+1$ equations in Eq. (91).

Obviously, when $\theta = 0$, Eq. (91) is identical to Eq. (16) after using the scaling transformation $u_k = \mu^k \widetilde{u}_k$ $(k = 0, 1, 2, \ldots)$. For $\theta \neq 0$, the relationship between the coupled system from perturbation with first-order precision derived by two methods for Eq. (1) is easily obtained and given in Theorem 1.

Theorem 1. *For Eq. (1), the first-order coupled system obtained from two methods are equivalent under the scaling transformation* $u_0 = \widetilde{u}_0$, $u_1 = \mu(1-\theta)\widetilde{u}_1$.

The proof of Theorem 1 is directly obtained by expanding Eqs. (16) and (91) with $k = 1$, so we omit it.

Based on Theorem 1 and Lie algorithm, first-order approximate homotopy symmetries are the same with first-order approximate symmetries. However, for the higher-order cases with $\theta \neq 0$, the relationship between Eqs. (16) and (91) cannot be determined easily. Hence, we perform approximate homotopy symmetry reductions to search for the links.

Assuming that operator (19) leaves the first $k+1$ equations in Eq. (91) invariant, with similar method as approximate symmetry, we get the following results in Table 5.

By means of the method adopted in Section 4.1, we obtain an optimal system of one-dimensional subalgebras of the approximate homotopy Lie algebra admitted by Eq. (1) in Proposition 3.

Proposition 3. *For each case of the approximate homotopy symmetry classification results of Eq. (1), the corresponding optimal system of one-dimensional subalgebras is given in Table 6.*

In general, after obtaining the optimal system, one should perform approximate homotopy symmetry reductions for Eq. (1) via the Lie method. However, there is no need to construct the reduced equations by $b_1Y_1 + b_iY_i$ in Table 6 by the standard Lie method, since we can achieve the goal by acting a scaling transformation on the reductions by the operators $b_1X_1 + b_iX_i$ in Table 2, where $b_1 \neq 0$, b_i $(i = 2, \ldots, 6)$ are arbitrary constants in what follows.

Table 5
Approximate homotopy symmetry classification of Eq. (1).

$F(u)$	Approximate homotopy symmetry operators
Arbitrary	$Y_1 = x\partial_x + t\partial_t + \sum_{i=0}^{\infty}[iu_i - (i-1)\theta u_{i-1}]\partial_{u_i}$, $Y_2 = \partial_x$, $Y_3 = \partial_t$
$e^{\lambda u}$	$Y_1, Y_2, Y_3, Y_4 = \frac{\lambda}{2}x\partial_x + \partial_{u_0}$
$u^{\lambda}\left(\lambda \neq -\frac{4}{3}\right)$	$Y_1, Y_2, Y_3, Y_5 = \frac{1}{2}(\lambda-2)x\partial_x - t\partial_t + \sum_{i=0}^{\infty}(i-1)[\theta u_{i-1} - u_i]\partial_{u_i}$
$u^{-4/3}$	$Y_1, Y_2, Y_3, Y_5, Y_6 = -\frac{1}{3}x^2\partial_x + x\sum_{i=0}^{\infty} u_i\partial_{u_i}$

Table 6
Optimal system of one-dimensional subalgebras of Eq. (1) by AHSM.

$F(u)$	The operators of the optimal system
Arbitrary	$Y_1, Y_3 + aY_2, a \in R$
$e^{\lambda u}$	$Y_4, Y_1 + aX_4, Y_2 + cY_3, aY_3 + Y_4, -\frac{\lambda}{2}Y_1 + Y_4 + aX_2, a \in R$
$u^{\lambda}\left(u \neq -\frac{4}{3}\right)$	$Y_5, Y_1 + aY_5, Y_2 + aY_3, Y_1 + Y_5 + aY_3, -\frac{1}{2}(\lambda-2)Y_1 + Y_5 + aY_2, a \in R$
$u^{-\frac{4}{3}}$	In addition to the operators of $F(u) = u^{\lambda}$ in this table: $aY_3+Y_6, 5Y_1+3Y_5+aY_6, Y_3+Y_6-bY_2, Y_2+bY_3-Y_6, Y_5+Y_6-bY_2, Y_2+Y_5-bY_6, Y_1+Y_6-bY_2, Y_1+Y_2-bY_6, a \in R, b \in R^+$

5.2. Connections between ASM and AHSM for Eq. (1)

In this subsection, we first give approximate homotopy symmetry reductions based on the results with ASM, and then perform a comparison of three-order series solutions.

5.2.1. Reductions by $b_1Y_1 + b_iY_i$

According to the reduction results by ASM for Eq. (1), we construct the higher-order reduced equations derived by $b_1Y_1 + b_iY_i$ through a scaling transformation in Theorem 2.

Theorem 2. *The reduced equations constructed by the operators $b_1Y_1+b_iY_i$ are equivalent to the corresponding one by $b_1X_1+b_iX_i$ under the scaling transformation*

$$P_0 = \widetilde{P}_0, \qquad P_1 = \mu(1-\theta)\widetilde{P}_1, \qquad P_2 = [\mu(1-\theta)]^2\widetilde{P}_2, \quad \ldots, \quad P_k = [\mu(1-\theta)]^k\widetilde{P}_k. \tag{92}$$

Proof. We take operator Y_1 as an example to show it, the other cases can be done with similar method. The reduced equations by Y_1 for each order are listed as follows

$$\begin{aligned}
&k = 0 : z^2P_{0,zz} + 2zP_{0,z} - [F(P_0)P_{0,z}]_z = 0,\\
&k = 1 : z^2P_{1,zz} - [P_{1,z}F(P_0) + F'(P_0)P_1P_{0,z}]_z + \mu(1-\theta)zP_{0,z} = 0,\\
&k = 2 : z^2P_{2,zz} - 2zP_{2,z} - [P_{2,z}F(P_0) + F'(P_0)P_1P_{1,z} + \frac{1}{2}P_{0,z}P_1^2F''(P_0) + P_2F'(P_0)]_z\\
&\qquad + 2P_2 + \mu(1-\theta)[P_1 - zP_{1,z}] = 0,\\
&\ldots\\
&\text{The } k\text{th} : z^2P_{k,zz} + (k-1)[kP_k - 2zP_{k,z}] - \left[\sum_{i=0}^{k}\frac{F^{(j)}(P_0)}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}\ldots P_{l_i}^{j_i}P_{k-i,z}\right]_z + \mu(1-\theta)[P_{k-1} - zP_{k-1,z}] = 0.
\end{aligned} \tag{93}$$

Obviously, the first equation in (93) is the unperturbed equations which is identical to the first equation in Eq. (50) with $k = 0$.

The second equation is linear about P_1 and $P_{1,z}, P_{1,zz}$, so setting $P_1 = \mu(1-\theta)\widetilde{P}_1$, then we convert it to the form

$$z^2\widetilde{P}_{1,zz} - [\widetilde{P}_{1,z}F(\widetilde{P}_0) + F'(\widetilde{P}_0)\widetilde{P}_1\widetilde{P}_{0,z}]_z + z\widetilde{P}_{0,z} = 0, \tag{94}$$

which has the same form as the second one in Eq. (50).

For $k = 2$, assuming $P_1 = \mu(1-\theta)\widetilde{P}_1$, $P_2 = [\mu(1-\theta)]^2\widetilde{P}_2$, we have

$$z^2\widetilde{P}_{2,zz} - 2z\widetilde{P}_{2,z} - [\widetilde{P}_{2,z}F(\widetilde{P}_0) + F'(\widetilde{P}_0)\widetilde{P}_1\widetilde{P}_{1,z} + \frac{1}{2}\widetilde{P}_{0,z}\widetilde{P}_1^2F''(\widetilde{P}_0) + \widetilde{P}_2F'(\widetilde{P}_0)]_z + 2\widetilde{P}_2 + [\widetilde{P}_1 - z\widetilde{P}_{1,z}] = 0, \tag{95}$$

which has the same form as the third one in Eq. (50).

Deriving by inductive reasoning, using the transformation

$$P_0 = \widetilde{P}_0, \qquad P_1 = \mu(1-\theta)\widetilde{P}_1, \qquad P_2 = [\mu(1-\theta)]^2\widetilde{P}_2, \quad \ldots, \quad P_k = [\mu(1-\theta)]^k\widetilde{P}_k, \tag{96}$$

we convert Eq. (93) to Eq. (50). That is to say, the two reduced equations, Eqs. (50) and (93), are equivalent under the scaling transformation (92). This establishes the theorem. □

Using Theorem 2, one can obtain approximate homotopy series solutions through acting transformation (92) on approximate solutions with ASM for the same equations.

5.2.2. Other reductions by AHSM

The operators which do not contain parameter θ are the same, so do the corresponding similarity variables, but the reduced equations are different. Below, we perform symmetry reductions using these operators.

A1. *Reduction by* $cY_2 + Y_3$

In terms of the similarity variables $z = x - ct$, $u_i = P_i$, Eq. (91) is converted to

$$c^2 P_{k,zz} - c\mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z} = \left[\sum_{i=0}^{k}\frac{F^{(j)}(P_0)}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z. \tag{97}$$

A2. *Reduction by* $aY_3 + Y_4$

To proceed, two different cases are distinguished.

(I) $a \neq 0$

By $aY_3 + Y_4$, Eq. (91) with $F(u) = e^{\lambda u}$ can be reduced to

$$\begin{aligned}
&\frac{1}{a^2}P_{0,zz} = -\frac{2}{\lambda}e^{\lambda P_0}(P_{0,z}-1) + \frac{4}{\lambda^2}[e^{\lambda P_0}(P_{0,z}-1)]_z,\\
&\frac{1}{a^2}P_{1,zz} + \frac{1}{a}\mu(1-\theta)P_{0,z} = -\frac{2}{\lambda}e^{\lambda P_0}(P_{1,z}+\lambda P_1P_{0,z}-\lambda P_1) + \frac{4}{\lambda^2}[e^{\lambda P_0}(P_{1,z}+\lambda P_1P_{0,z}-\lambda P_1)]_z,\\
&\frac{1}{a^2}P_{k,zz} + \frac{1}{a}\mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z} = -2e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-1}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\cdots P_{l_i}^{j_i}P_{k-i,z}\\
&\qquad + 4\left[e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-2}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\dots P_{l_i}^{j_i}P_{k-i,z}\right]_z, \quad k>1,
\end{aligned} \tag{98}$$

where $z = t/a - 2/\lambda \ln x$, $P_0 = u_0 - 2/\lambda \ln x$, $P_i = u_i$, $(i = 1, 2, \dots, k)$.

(II) $a = 0$

By Y_4, Eq. (91) with $F(u) = e^{\lambda u}$ can be reduced to

$$P_{k,zz} + \mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z} = -2e^{\lambda P_0}\sum_{i=0}^{k}\frac{\lambda^{j-1}}{j_0!j_1!\cdots j_i!}P_{l_0}^{j_0}P_{l_1}^{j_1}\cdots P_{l_i}^{j_i}, \tag{99}$$

where $z = t$, $P_0 = u_0 - 2/\lambda \ln x$, $P_i = u_i$, $(i = 1, 2, \dots)$.

A3. *Reduction by* $aY_3 + Y_6$

According to parameter a, we consider two cases.

(I) When $a \neq 0$, by $aY_3 + Y_6$, Eq. (91) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{a^2}P_{k,zz} + \frac{1}{a}\mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z} = 9\left[\sum_{i=0}^{k}\frac{\lambda(\lambda-1)\dots(\lambda-j+1)P_0^{\lambda-j}}{j_0!\cdots j_i!}P_{l_0}^{j_0}\dots P_{l_i}^{j_i}P_{k-i,z}\right]_z, \tag{100}$$

where $\lambda = -4/3$ and $z = 2/x - t/a$, $P_i = x^3u_i$.

(II) For $a = 0$, by Y_6, Eq. (91) with $F(u) = u^{-4/3}$ can be reduced to

$$P_{k,zz} + \mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z} = 0, \tag{101}$$

where $P_{-1} = 0$, $z = t$, $P_i = x^3u_i$.

Integrate Eq. (101) once, one has

$$P_{k,z} + \mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_i = c_1, \tag{102}$$

where c_1 is an integrated constant.

Eq. (1) for $F(u) = u^{-4/3}$ has solution in the form

$$u = \frac{1}{x^3}\sum_{k=0}^{\infty}q^k\int\left[c_1 - \mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_i(t)\right]dt. \tag{103}$$

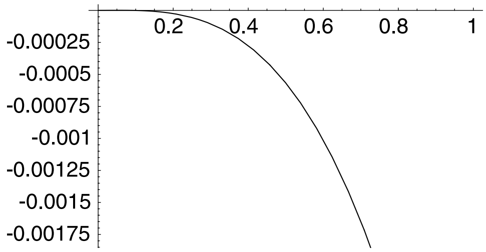

Fig. 1. Horizontal axis denote θ, vertical axis denote the error.

A4. *Reduction by* $Y_3 + Y_6 - bY_2$

Here, due to $b > 0$, without loss of generality, we set $b = 3\mu^2$ in order to simplify the computations. By $Y_3 + Y_6 - bY_2$, Eq. (91) with $F(u) = u^{-4/3}$ can be reduced to

$$\frac{1}{9}\left[P_{k,zz} + 3\mu(1-\theta)\sum_{i=0}^{k-1}\theta^{k-1-i}P_{i,z}\right] = \sum_{i=0}^{k}\left[\left(\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right)_z - \frac{27\mu^2\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i}\right], \tag{104}$$

where

$$\sum_{i=0}^{k}\left[\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z} + 3\left[\frac{\lambda!P_0^{\lambda-j}}{(\lambda-j)!j_0!\cdots j_i!}P_{l_0}^{j_0}\cdots P_{l_i}^{j_i}P_{k-i,z}\right]_z\right] \equiv 0, \tag{105}$$

and $\lambda = -4/3$ and $z = t/3 + 1/(3\mu)\arctan[x/(3\mu)]$, $P_i = (9\mu^2 + x^2)^{3/2}u_i$. Eq. (105) can be proved with similar method as Eq. (74) in the Appendix.

5.2.3. *A comparison of three-order series solutions*

In this subsection, we give a comparison of three-order series solutions for the case $F(u) = u^2$ with ASM and AHSM.

For $F(u) = u^2$, we get particular solutions with three-order precision of Eq. (50) by ASM

$$\widetilde{P}_0 = y, \qquad \widetilde{P}_1 = -\frac{y}{6} + \frac{c_1}{\sqrt{y}}, \qquad \widetilde{P}_2 = -\frac{y}{36} - \frac{7c_1}{12\sqrt{y}} + c_2,$$
$$\widetilde{P}_3 = -\frac{1}{64}y^{-\frac{3}{2}}(5y^4 + 2c_1) - \frac{y}{54} + c_2\sqrt{y} + \frac{1}{96}\left(\frac{19}{\sqrt{y}} - 64c_1\right). \tag{106}$$

Then, substituting the solution

$$u = P_0 + \epsilon t\widetilde{P}_1 + \epsilon^2 t^2\widetilde{P}_2 + \epsilon^3 t^3\widetilde{P}_3 \tag{107}$$

into Eq. (1) and setting $c_1 = c_2 = 1$, $x = t = 2$, $\epsilon = 0.01$, we get

$$u_{tt} + \epsilon u_t - (u^2u_x)_x \approx 7.82945 \times 10^{-8}. \tag{108}$$

On the other hand, using transformation (92) in Theorem 3, we obtain

$$P_0 = \widetilde{P}_0, \qquad P_1 = \mu(1-\theta)\widetilde{P}_1, \qquad P_2 = [\mu(1-\theta)]^2\widetilde{P}_2, \qquad P_3 = [\mu(1-\theta)]^3\widetilde{P}_3, \tag{109}$$

then a three-order approximate homotopy series solution of Eq. (1) is given as

$$u = P_0 + ptP_1 + p^2(t^2P_2 + t\theta P_1) + p^3(t^3P_3 + 2t^2\theta P_2 + t\theta^2P_1). \tag{110}$$

Inserting (110) into Eq. (1) and also choosing $p = c_1 = c_2 = 1$, $x = t = 2$, $\mu = \epsilon = 0.01$, we get Fig. 1 which demonstrates the error of solution (110) with respect to the auxiliary parameter θ.

From Fig. 1, in particular, assuming convergence-control parameter $\theta = 0.0894$, one has

$$u_t + [F(u)u_x]_x + \epsilon u_{xxxx} \approx -5.7284 \times 10^{-11}. \tag{111}$$

This example shows that appropriate convergence-control parameter θ can adjust the precision of series solutions.

6. Summary and discussion

In the framework of the approximate symmetry method originated by Fushchich and Shtelen, we investigate a class of perturbed wave equations and summarize the formulas for different order similarity reductions based on an optimal system of one-dimensional subalgebras of Eq. (1). For arbitrary $F(u)$ and three types of special equations of Eq. (1), zero-order similarity reduction equations are nonlinear ordinary differential equations while kth order ($k = 1, 2, \ldots$) similarity reduction equations are variable coefficient linear ordinary differential equations of $P_k(z)$ which depend on particular solutions of the first k similarity reduction equations.

In addition, we study approximate homotopy symmetry of Eq. (1) and show that first-order coupled equations and the higher-order reduced equations derived by the operators containing convergence-control parameter are equivalent under two scaling transformations, respectively. The comparison of three-order series solutions demonstrates that AHSM is superior to ASM for Eq. (1) because the convergence of series solutions by AHSM can be controlled by adjusting the parameter θ.

Note that the problem of classification of all possible potential symmetries for Eq. (1) still remains open and can form a subject of future investigation of properties of Eq. (1). Furthermore, whether the present work about connections between ASM and AHSM hold for any perturbed nonlinear PDEs is worthy of further consideration. These topics are in preparation and will be reported in our future works.

Appendix

By means of mathematical induction, we prove the following identical equation

$$\sum_{i=0}^{k}[3(2k-1)AP_{k-i} + 6z(AP_{k-i})_z + (2k-2i+3)AP_{k-i} + 2zAP_{k-i,z}] \equiv 0 \tag{A.1}$$

where $A = \lambda! P_0^{\lambda-j} P_{l_0}^{j_0} \ldots P_{l_i}^{j_i} / ((\lambda - j)! j_0! \cdots j_i!)$ with $\lambda = -4/3$.

Proof. 1. For $k = 0$, then $i = 0$. Hence, $A = P_0^{-4/3}$, Eq. (A.1) becomes

$$-3P_0^{-\frac{4}{3}}P_0 + 6z(P_0^{-\frac{4}{3}}P_0)_z + 3P_0^{-\frac{4}{3}}P_0 + 2zP_0^{-\frac{4}{3}}P_{0,z} \equiv 0. \tag{A.2}$$

2. Assuming Eq. (A.1) holds for the integer k, we prove it holds for $k + 1$, i.e.

$$\sum_{i=0}^{k+1}[3(2k+1)AP_{k+1-i} + 6z(AP_{k+1-i})_z + (2k-2i+5)AP_{k+1-i} + 2zAP_{k+1-i,z}] \equiv 0. \quad \square \tag{A.3}$$

Next, we search for all terms which contain P_{k+1} in Eq. (A.3). After direct computations, we find P_{k+1} only appear when $i = 0$ or $k + 1$, hence, we list all terms as follows:

$$i = 0, \quad A = P_0^{\lambda}; \qquad i = k+1, \quad A = \lambda P_{k+1} P_0^{\lambda-1}. \tag{A.4}$$

Calculating all terms which contain P_{k+1} and $P_{k+1,z}$ and setting $\lambda = -4/3$, we get

$$\begin{aligned} &3(2k+1)(\lambda+1)P_{k+1}P_0^{\lambda} + 6z(\lambda+1)(P_{k+1}P_0^{\lambda})_z + (2k+5)P_{k+1}P_0^{\lambda} \\ &\quad + 3\lambda P_{k+1}P_0^{\lambda} + 2z[P_{k+1,z}P_0^{\lambda} + \lambda P_{k+1}P_0^{\lambda-1}P_{k+1}P_{0,z}] \equiv 0. \end{aligned} \tag{A.5}$$

Hence, with Eqs. (A.1) and (A.5), Eq. (A.3) is an identity, i.e., Eq. (A.1) holds for $k + 1$. The desired identical equation (A.1) follows.

As for the identities (77), (81), (83) and (86), one can prove them with mathematical induction as above procedures, so we omit them.

References

[1] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1993.
[2] L.V. Ovsyannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.
[3] G.W. Bluman, S.C. Anco, Symmetry and Integration Methods for Differential Equations, Springer-Verlag, New York, 2004.
[4] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.
[5] G.W. Bluman, G.J. Reid, S. Kumei, New classes of symmetries for partial differential equations, J. Math. Phys. 29 (1988) 806–811.
[6] N.H. Ibragimov, R.L. Anderson, Lie-Bäcklund tangent transformations, J. Math. Anal. Appl. 59 (1977) 145–162.
[7] N.M. Ivanova, C. Sophocleous, On the group classification of variable coefficient nonlinear diffusion–convection equations, J. Comput. Appl. Math. 197 (2006) 322–344.
[8] C.Z. Qu, L.N. Ji, L.Z Wang, Conditional Lie-Bäcklund symmetries and sign-invariants to quasi-linear diffusion equations, Stud. Appl. Math. 119 (2007) 355–391.

[9] O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, Acta Appl. Math. 106 (2009) 1–46.
[10] N.H. Ibragimov, Handbook of Lie Group Analysis of Differential Equations, vol. 1, CRC Press, Boca Raton, 1994; vol. 2, 1995; vol. 3, 1996.
[11] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, Perturbation methods in group analysis, J. Math. Sci. 55 (1991) 1450–1490.
[12] W.I. Fushchich, W.H. Shtelen, On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter, J. Phys. A: Math. Gen. 22 (1989) L887–L890.
[13] N. Euler, M.W. Shulga, W.H. Steeb, Approximate symmetries and approximate solutions for a multi-dimensional Landau–Ginzburg equation, J. Phys. A: Math. Gen. 25 (1992) L1095–L1103.
[14] M. Euler, N. Euler, A. KHohler, On the construction of approximate solutions for a multi-dimensional nonlinear heat equation, J. Phys. A: Math. Gen. 27 (1994) 2083–2092.
[15] N. Euler, M. Euler, Symmetry properties of the approximations of multidimensional generalized Van der Pol equations, J. Nonlinear Math. Phys. 1 (1994) 41–59.
[16] M. Pakdemirli, M. Yurusoy, T. Dolapc, Comparison of approximate symmetry methods for differential equations, Acta Appl. Math. 80 (2004) 243–271.
[17] R. Wiltshire, Two approaches to the calculation of approximate symmetry exemplified using a system of advection–diffusion equations, J. Comput. Appl. Math. 197 (2006) 287–301.
[18] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
[19] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Doctor Dissertation, Shanghai Jiao Tong University, Shanghai, 1992.
[20] X.Y. Jiao, Y. Gao, S.Y. Lou, Approximate homotopy symmetry method: homotopy series solutions to the sixth-order Boussinesq equation, Sci. China Ser. G-Phys. Mech. Astron. 52 (2009) 1169–1178.
[21] X.Y. Jiao, R.X. Yao, S.Y. Lou, Approximate similarity reduction for singularly perturbed Boussinesq equation via symmetry perturbation and direct method, J. Math. Phys. 49 (2008) 093505.
[22] I.A. Charnyi, Unstable motion of a real fluid in pipes, Nedra, Moscow, 1975 (in Russian).
[23] Z.Y. Zhang, Y.F. Chen, X.L. Yong, Classification and approximate solutions to a class of perturbed nonlinear wave equations, Commun. Theor. Phys. 52 (2009) 769–772.
[24] Z.Y. Zhang, B. Gao, Y.F. Chen, Second-order approximate symmetry classification and optimal system of a class of perturbed nonlinear wave equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 2709–2719.
[25] W.F. Ames, R.J. Lohner, E. Adams, Group properties of $u_{tt} = [f(u)u_x]_x$, Internat. J. Non-Linear Mech. 16 (1981) 439–447.
[26] N.J. Zabusky, Exact solution for vibrations of a nonlinear model string, J. Math. Phys. 3 (1962) 1028–1039.
[27] D. Chand, D.T. Davy, W.F. Ames, On the similarity solutions of wave propagation for a general class of non-linear dissipative materials, Internat. J. Non-Linear Mech. 11 (1976) 191–205.
[28] F.M. Mahomed, C.Z. Qu, Approximate conditional symmetries for partial differential equations, J. Phys. A: Math. Gen. 33 (2000) 343–356.
[29] Temuerchaolu, X.S. Gao, Nearly characteristic set of differential polynomial system, Acta Math. Sin. 45 (2002) 1041–1050.
[30] Temuerchaolu, An algorithmic theory of reduction of differential polynomials system, Adv. Math. 32 (2003) 208–220.