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Nonlocal transformations of Kolmogorov equations into the backward heat equation

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Abstract

We extend and solve the classical Kolmogorov problem of finding general classes of Kolmogorov equations that can be transformed to the backward heat equation. These new classes include Kolmogorov equations with time-independent and time-dependent coefficients. Our main idea is to include nonlocal transformations. We describe a step-by-step algorithm for determining such transformations. We also show how all previously known results arise as particular cases in this wider framework.
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1. Introduction

In this paper, we extend previous work on finding one-dimensional Kolmogorov equations that can be mapped into the backward heat equation. Let $p(t,x)$ satisfy a Kolmogorov equation

$$\frac{\partial p}{\partial t} + a(t,x)\frac{\partial^2 p}{\partial x^2} + b(t,x)\frac{\partial p}{\partial x} = 0, \quad a(t,x) > 0. \tag{1}$$

The Kolmogorov equation (1) is the partial differential equation (PDE) satisfied by the Green's function for the Fokker–Planck equation

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$$\frac{\partial q}{\partial t} - \frac{\partial^2}{\partial x^2}\big(a(t,x)q\big) + \frac{\partial}{\partial x}\big(b(t,x)q\big) = 0.$$

We find new sets of coefficients $\{a(t,x), b(t,x)\}$ for which (1) can be transformed to the backward heat equation

$$\frac{\partial \bar{p}}{\partial \bar{t}} + \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} = 0. \tag{2}$$

The Kolmogorov equation (1) can be transformed to the backward canonical PDE

$$\frac{\partial P}{\partial T} + \frac{\partial^2 P}{\partial X^2} + G(T,X)P = 0 \tag{3}$$

through the transformation

$$\begin{aligned} &T = t, \\ &X = X(t,x) = \int^x \frac{dz}{\sqrt{a(t,z)}}, \\ &P(T,X) = e^{1/2 \int^X D(t',Z)\,dZ} p(t,x), \end{aligned} \tag{4}$$

where

$$D(T,X) = \frac{\partial X}{\partial t} + a(t,x)\frac{\partial^2 X}{\partial x^2} + b(t,x)\frac{\partial X}{\partial x} \tag{5}$$

and, in terms of $D(T,X)$, the coefficient $G(T,X)$ of (3) satisfies

$$\frac{\partial G}{\partial X} = -\frac{1}{2}\left[\frac{\partial D}{\partial T} + \frac{\partial^2 D}{\partial X^2} + D\frac{\partial D}{\partial X}\right]. \tag{6}$$

From (6), it follows that if we set

$$D(T,X) = 2\frac{\partial}{\partial X}\log\big|\theta(T,X)\big|, \tag{7}$$

then $\theta(T,X)$ satisfies the backward equation

$$\frac{\partial \theta}{\partial T} + \frac{\partial^2 \theta}{\partial X^2} + \big[G(T,X) + \Gamma(T)\big]\theta = 0 \tag{8}$$

for arbitrary $\Gamma(T)$.

Let $u_1 = u_1(T,X) = P(T,X)$ and let $G_1(T,X) = G(T,X)$. Then Eq. (3) becomes the canonical PDE

$$\frac{\partial u_1}{\partial T} + \frac{\partial^2 u_1}{\partial X^2} + G_1(T,X)u_1 = 0. \tag{9}$$

Consequently, the problem of mapping (1) into (2) reduces to mapping PDE (9) to the backward heat equation (2).

In his celebrated paper [9], Kolmogorov posed the problem of finding the most general equation of the form (1) that can be mapped into the backward heat equation (2) by a point transformation. Cherkasov ([8]; see also [2,11,12]) partially solved this problem by restricting himself to a special class of point transformations considered by Kolmogorov [9]. The complete solution of Kolmogorov's problem was given in [3].

The main idea of the present paper is as follows. Suppose the Kolmogorov equation (1) can be embedded in an auxiliary system of PDEs so that the set of all solutions of the auxiliary system yields all solutions of the Kolmogorov equation but there is not a one-to-one correspondence between solutions of the Kolmogorov equation and those of the related auxiliary system. Then a point transformation of the variables of the auxiliary system, which maps a component of the auxiliary system into the backward heat equation, could yield a nonlocal transformation which maps the given Kolmogorov equation to a backward heat equation. We exhibit such nonlocal transformations, yielding wider classes of Kolmogorov equations transformable to the backward heat equation than those previously obtained by point (local) transformations. This is accomplished by embedding a Kolmogorov equation in an auxiliary "potential" system obtained through replacement of the Kolmogorov equation by an equivalent conservation law [4].

In Section 2 we present the previously known results on mapping Kolmogorov equations to the backward heat equation by point transformations.

In Section 3 we present our basic framework for obtaining mappings by nonlocal transformations. We begin by observing that any solution of the adjoint equation of the canonical PDE (9) yields a factor for an equivalent conservation law. In turn this conservation law yields a potential system. We then find the most general canonical PDE for which a point transformation maps a corresponding potential system to a special potential system related to the backward heat equation. Each component of this special potential system satisfies the backward heat equation. Such a point transformation of a potential system is shown to yield a nonlocal transformation of a canonical PDE to the backward heat equation.

Using the basic framework, in Section 4 we give a step-by-step procedure to obtain new classes of Kolmogorov equations that are transformable to the backward heat equation. A theorem is proved which characterizes the richness of our extension of the known (local) results. We show that our new classes arise from *any* solution of the backward heat equation other than its fundamental solution (modulo translations) or its traveling wave solution.

In Section 5 we describe further generalizations emanating from our basic framework through a recycling procedure which yields chains of Kolmogorov equations transformable to the backward heat equation. In Section 6 this recycling procedure is shown to be of particular value when the coefficients of a Kolmogorov equation are time-independent. In Section 7, as examples we exhibit d-Bessel processes [10] which are only transformable to the backward heat equation by nonlocal transformations. As a consequence, a spherically symmetric $(2k+1)$-dimensional heat equation can be mapped into the one-dimensional heat equation only by a nonlocal transformation for $k = 2, 3, \ldots$.

In Section 8 we discuss connections with symmetry analysis.

The approach presented in this paper can be used in other mapping problems. For example, new classes of Schrödinger equations transformable to the free particle equation by nonlocal transformations were found in [7].

2. Mappings by point transformations

One can show that a point transformation

$$\tau = \tau(T, X, u_1), \quad y = y(T, X, u_1), \quad \tilde{u}_1 = \tilde{u}_1(T, X, u_1), \tag{10}$$

maps any PDE (9) into a homologous PDE, namely

$$\frac{\partial \tilde{u}_1}{\partial \tau} + \frac{\partial^2 \tilde{u}_1}{\partial y^2} + \tilde{G}_1(\tau, y)\tilde{u}_1 = 0 \tag{11}$$

for some $\tilde{G}_1(\tau, y)$ if and only if (10) is of the form

$$\begin{aligned} &y = \sigma(T)X + \rho(T), \qquad \tau = \int^T \sigma^2(\mu)\, d\mu, \\ &\tilde{u}_1 = \exp\left[\frac{\dot{\sigma}}{4\sigma}X^2 + \frac{\dot{\rho}}{2\sigma}X + \lambda\right]u_1, \end{aligned} \tag{12}$$

with

$$\begin{aligned} \tilde{G}_1(\tau, y) = \frac{1}{\sigma^2}\bigg[&G_1(T, X) + \frac{2\dot{\sigma}^2 - \sigma\ddot{\sigma}}{4\sigma^2}X^2 + \frac{2\dot{\sigma}\dot{\rho} - \sigma\ddot{\rho}}{2\sigma^2}X \\ &+ \left(\frac{\dot{\rho}^2}{4\sigma^2} - \frac{\dot{\sigma}}{2\sigma} - \dot{\lambda}\right)\bigg], \end{aligned} \tag{13}$$

where $\sigma(T)$, $\rho(T)$, $\lambda(T)$ are arbitrary functions of T and $\dot{\sigma} = d\sigma/dT$, etc.

Consequently, with respect to a point transformation, PDE (9) can be mapped into the backward heat equation if and only if $G_1(T, X)$ is of the form

$$G_1(T, X) = \alpha(T)X^2 + \beta(T)X + \gamma(T) \tag{14}$$

for arbitrary $\alpha(T)$, $\beta(T)$, $\gamma(T)$ (see [3]). The corresponding coefficients $\{\alpha(t, x), b(t, x)\}$ were given in [3] and will be exhibited in Section 4.

From Eq. (13), we see that if $G_1(T, X)$ is of the form (14), the mapping (12) which transforms the corresponding PDE (9) to the backward heat equation has $\sigma(T)$, $\rho(T)$, $\lambda(T)$ satisfying the system of ODEs

$$\begin{aligned} &\frac{\sigma\ddot{\sigma} - 2\dot{\sigma}^2}{4\sigma^2} = \alpha(T), \\ &\frac{\sigma\ddot{\rho} - 2\dot{\sigma}\dot{\rho}}{2\sigma^2} = \beta(T), \\ &\dot{\lambda} = \frac{\dot{\rho}^2}{4\sigma^2} - \frac{\dot{\sigma}}{2\sigma} + \gamma(T). \end{aligned} \tag{15}$$

The solution of system (15) is given in Appendix A.

3. The basic framework for nonlocal transformations

In $\mathbf{R}^n$, for any given linear operator L, its adjoint operator L^* is defined by

$$\phi Lu - uL^*\phi = \sum_{i=1}^{n} D_i f^i, \tag{16}$$

where $x = (x_1, \dots, x_n)$, the total derivative operator $D_i = \partial/\partial x_i$, f^i is a bilinear expression in u, ϕ and their derivatives, $i = 1, 2, \dots, n$. Consequently, if

$$L^*\phi = 0, \tag{17}$$

then $Lu = 0$ if and only if $\sum_{i=1}^{n} D_i f^i = 0$, i.e., *a given linear PDE*

$$Lu = 0 \tag{18}$$

is equivalent to a conservation law

$$\sum_{i=1}^{n} D_i f^i = 0 \tag{19}$$

for any ϕ satisfying its adjoint equation (17). Using (19), one can introduce an auxiliary potential system whose compatibility conditions yield (18). The set of all solutions of such a potential system yields the set of all solutions of (18) but the connection between these solution sets is not one-to-one (see [4]). When $n \geqslant 3$, one is confronted with the problem of a natural gauge arbitrariness. In this case, as shown in [1], the associated potential system must be augmented by gauge constraints.

Now we specialize to the situation when PDE (18) is the canonical PDE (9). Here the linear operator L is given by

$$L = \frac{\partial}{\partial T} + \frac{\partial^2}{\partial X^2} + G_1(T, X), \tag{20}$$

its adjoint L^* is given by

$$L^* = -\frac{\partial}{\partial T} + \frac{\partial^2}{\partial X^2} + G_1(T, X), \tag{21}$$

and the conservation law (19) is given by

$$\frac{\partial}{\partial T}(\phi u_1) + \frac{\partial}{\partial X}\left(\phi \frac{\partial u_1}{\partial X} - \frac{\partial \phi}{\partial X} u_1\right) = 0. \tag{22}$$

The potential system corresponding to (22), with auxiliary dependent variable $v_1(T, X)$, is given by

$$\frac{\partial v_1}{\partial X} = \phi u_1, \qquad \frac{\partial v_1}{\partial T} = \frac{\partial \phi}{\partial X} u_1 - \phi \frac{\partial u_1}{\partial X}, \tag{23}$$

where $\phi(T, X)$ is *any* solution of the adjoint PDE

$$L^*\phi = -\frac{\partial \phi}{\partial T} - \frac{\partial^2 \phi}{\partial X^2} + G_1(T, X)\phi = 0. \tag{24}$$

Note that if $(u_1(T,X), v_1(T,X), \phi(T,X))$ solves the potential/adjoint system (23), (24), then $u_1(T,X)$ solves the canonical PDE (9) and $v_1(T,X)$ solves the backward equation

$$\frac{\partial v_1}{\partial T} + \frac{\partial^2 v_1}{\partial X^2} - \frac{2}{\phi}\frac{\partial \phi}{\partial X}\frac{\partial v_1}{\partial X} = 0. \tag{25}$$

Moreover, if $u_1 = U_1(T,X)$ solves the canonical PDE (9) and $\phi = \Phi(T,X)$ solves (24), then from the integrability conditions of (23) it follows that there exist solutions $(u_1(T,X), v_1(T,X)) = (U_1(T,X), V_1(T,X))$ of potential system (23) with $V_1(T,X)$ unique to within an arbitrary constant. Hence for any given $\phi = \Phi(T,X)$ which solves (24), the relationship between the solution sets of PDE (9) and system (23) is not one-to-one.

For any $\phi(T,X)$ that satisfies (24), the point transformation

$$w = \frac{v_1}{\phi} \tag{26}$$

maps (25) to the canonical PDE

$$\frac{\partial w}{\partial T} + \frac{\partial^2 w}{\partial X^2} + G_2(T,X)w = 0, \tag{27}$$

with $G_2(T,X)$ given by

$$G_2(T,X) = G_1(T,X) + 2\frac{\partial^2}{\partial X^2}\log|\phi|. \tag{28}$$

From Eqs. (23) and (26), it immediately follows that if $u_1(T,X)$ solves the canonical PDE (9) then

$$w = u_2(T,X) = \frac{1}{\phi(T,X)}\left[\int_k^X u_1(T,\xi)\phi(T,\xi)\,d\xi + B_2(T)\right], \tag{29}$$

with $B_2(T)$ satisfying the condition

$$\frac{dB_2}{dT} = \frac{\partial \phi}{\partial X}(T,k)u_1(T,k) - \phi(T,k)\frac{\partial u_1}{\partial X}(T,k) \tag{30}$$

for any constant k, solves the homologous equation (27) with $G_2(T,X)$ given by (28). Conversely, if $w(T,X)$ solves the canonical PDE (27) and $\phi(T,X)$ is any particular solution of

$$\frac{\partial \phi}{\partial T} - \frac{\partial^2 \phi}{\partial X^2} + 2\phi\frac{\partial^2}{\partial X^2}\log|\phi| - G_2(T,X)\phi = 0,$$

then solving the forward PDE (24) in terms of $G_1(T,X)$, i.e., setting

$$G_1(T,X) = \frac{1}{\phi}\left[\frac{\partial \phi}{\partial T} - \frac{\partial^2 \phi}{\partial X^2}\right], \tag{31}$$

it follows that

$$u_1(T,X) = \frac{\partial w}{\partial X} + \frac{w}{\phi}\frac{\partial \phi}{\partial X}$$

solves the homologous canonical equation (9) with $G_1(T,X)$ given by (31).

By direct calculation one can prove the following theorem.

Theorem 1. *Consider the canonical PDE* (9). *A point transformation maps the corresponding potential system* (23) *into the special backward heat equation potential system*

$$\frac{\partial \hat{v}}{\partial y} = \hat{u}, \qquad \frac{\partial \hat{v}}{\partial \tau} = -\frac{\partial \hat{u}}{\partial y}, \tag{32}$$

for which each component satisfies the backward heat equation, i.e., $\partial\hat{v}/\partial\tau + \partial^2\hat{v}/\partial y^2 = 0$, $\partial\hat{u}/\partial\tau + \partial^2\hat{u}/\partial y^2 = 0$, *if and on if* $G_1(T, X)$ *is of the form*

$$G_1(T, X) = 2\frac{\partial^2}{\partial X^2}\log|\psi| + \alpha(T)X^2 + \beta(T)X + \gamma(T), \tag{33}$$

where $\psi(T, X)$ *is any solution of*

$$\frac{\partial \psi}{\partial T} + \frac{\partial^2 \psi}{\partial X^2} + \big[\alpha(T)X^2 + \beta(T)X + \gamma(T)\big]\psi = 0 \tag{34}$$

for arbitrary $\alpha(T)$, $\beta(T)$, $\gamma(T)$. (*Note that* $\psi(T, X)$ *satisfies* (34) *if* $\phi(T, X) = 1/\psi(T, X)$ *satisfies the adjoint equation* (24) *with* $G_1(T, X)$ *given by* (33).) *The corresponding mapping of* (23) *into* (32) *is given by*

$$\begin{aligned}
y &= \sigma(T)X + \rho(T),\\
\tau &= \int^{T} \sigma^2(\mu)\,d\mu,\\
\hat{u} &= \frac{1}{\sigma}e^{g(T,X)}\left\{u_1 + \left(\frac{\dot{\sigma}X + \dot{\rho}}{2\sigma} + \frac{\psi_X}{\psi}\right)\psi v_1\right\},\\
\hat{v} &= e^{g(T,X)}\psi v_1,
\end{aligned} \tag{35}$$

where

$$g(T, X) = \frac{\dot{\sigma}}{4\sigma}X^2 + \frac{\dot{\rho}}{2\sigma}X + \lambda,$$

and $(\sigma(T), \rho(T), \lambda(T))$ *are related to* $(\alpha(T), \beta(T), \gamma(T))$ *through the system of ODEs* (15) *which is solved in Appendix* A. *The mapping* (35) *defines a point transformation on* (X, T, u_1, v_1)*-space that projects into a nonlocal transformation on* (X, T, u_1)*-space if the coefficient of* v_1 *is nonzero in Eq.* (35c).

The previously known result [3] about the equivalence of the canonical PDE (9) and the backward heat equation under a point transformation when the coefficient $G_1(T, X)$ of (9) is quadratic in X (Eq. (14)) immediately follows as a special case of Theorem 1. Indeed, if $\psi(T, X)$ satisfies

$$\frac{1}{\psi}\frac{\partial \psi}{\partial X} + \frac{\dot{\sigma}X + \dot{\rho}}{2\sigma} = 0,$$

then Eq. (33) yields

$$G_1(T, X) = \alpha(T)X^2 + \beta(T)X + \gamma(T) - \frac{\dot{\sigma}}{\sigma}.$$

It is easy to check that the mapping (35) yields a nonlocal transformation of (9) to the backward heat equation if and only if $\psi(T, X)$ is a solution of (34) satisfying the condition

$$\frac{\partial^3}{\partial X^3}\log|\psi| \not\equiv 0. \tag{36}$$

Moreover, the resulting expression (33) for $G_1(T, X)$ is not of the quadratic form (14) if and only if $\psi(T, X)$ satisfies the condition

$$\frac{\partial^5}{\partial X^5}\log|\psi| \not\equiv 0. \tag{37}$$

Let $\hat{\psi}(\tau, y)$ be *any* solution of the backward heat equation $\partial\hat{\psi}/\partial\tau + \partial^2\hat{\psi}/\partial y^2 = 0$. Then from (12) it follows that

$$\psi(T, X) = \exp\left[-\left\{\frac{\dot{\sigma}}{4\sigma}X^2 + \frac{\dot{\rho}}{2\sigma}X + \lambda\right\}\right]\hat{\psi}(\tau, y)$$

is a solution of (34), and hence (33) becomes

$$G_1(T, X) = \alpha(T)X^2 + \beta(T)X + \gamma(T) - 2\sigma^2\left[\frac{\hat{\psi}_\tau}{\hat{\psi}} + \left(\frac{\hat{\psi}_y}{\hat{\psi}}\right)^2\right] - \frac{\dot{\sigma}}{\sigma}, \tag{38}$$

where $y = \sigma X + \rho$, $\tau = \int^T \sigma^2(\mu)\,d\mu$, with $\sigma(T)$, $\rho(T)$ related to $\alpha(T)$, $\beta(T)$ through (15a,b). Hence through (38) *every* solution of the backward heat equation yields a coefficient $G_1(T, X)$ for which the corresponding backward equation (9) can be mapped into the backward heat equation. This relationship will be considered in more detail in the next section.

4. New classes of Kolmogorov equations transformable to the backward heat equation

Now we apply the results of Section 3 to the Kolmogorov equation (1). Let $\psi(T, X)$ be any solution of (34) and let $G_1(T, X)$ be given by (33) for *arbitrary* $\alpha(T)$, $\beta(T)$, $\gamma(T)$. Then

$$\frac{\partial^3 G_1}{\partial X^3} = 2\frac{\partial^5}{\partial X^5}\log|\psi|.$$

Consequently, after using (6) with $G = G_1$, the work of the previous section can be restated as follows.

Through one of our potential systems, the Kolmogorov equation (1) can be mapped into the backward heat equation if and only if T, X, $D(T, X)$ defined by (4) and (5) satisfy

$$\frac{\partial^2}{\partial X^2}\left(\frac{\partial D}{\partial T} + D\frac{\partial D}{\partial X} + \frac{\partial^2 D}{\partial X^2}\right) = -4\frac{\partial^5}{\partial X^5}\log|\psi| \tag{39}$$

for any solution $\psi(T, X)$ of (34).

Now we show how the above result generalizes previous work presented in [3,8].

In [3] it was shown that the Kolmogorov equation (1) can be mapped into the backward heat equation through a point transformation if and only if $D(T, X)$ satisfies

$$\frac{\partial^2}{\partial X^2}\left(\frac{\partial D}{\partial T} + D\frac{\partial D}{\partial X} + \frac{\partial^2 D}{\partial X^2}\right) = 0, \tag{40}$$

and it was further shown that Cherkasov's special class of point transformations restricted $D(T, X)$ to the solutions of (40) that satisfy

$$\frac{\partial^2 D}{\partial X^2} = 0. \tag{41}$$

In terms of the original independent variables (x, t), note that

$$\frac{\partial}{\partial X} = \left(\frac{\partial X}{\partial x}\right)^{-1}\frac{\partial}{\partial x} = \sqrt{a(t,x)}\,\frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} - \frac{\partial X}{\partial t}\left(\frac{\partial X}{\partial x}\right)^{-1}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \sqrt{a(t,x)}\,X_t\frac{\partial}{\partial x}.$$

The previous discussion leads to the following step-by-step procedure to find new classes of Kolmogorov equations (1) that are transformable to the backward heat equation.

Let $\alpha(T)$, $\beta(T)$, $\gamma(T)$ be arbitrary functions of T.

(1) Let $\psi(T, X)$ be a solution of (34), found as follows.

For any given $\alpha(T)$, $\beta(T)$, $\gamma(T)$, let $\sigma = \sigma_1(T)$, $\rho = \rho_1(T)$, $\lambda = \lambda_1(T)$ be *any* solution of the system of ODEs (15), solved in Appendix A. Let

$$y_1 = \sigma_1(T)X + \rho_1(T), \qquad \tau_1 = \int^T \sigma_1^2(\mu)\,d\mu. \tag{42}$$

Let $\hat\psi(\tau_1, y_1)$ be *any* solution of the backward heat equation

$$\frac{\partial\hat\psi}{\partial\tau_1} + \frac{\partial^2\hat\psi}{\partial y_1^2} = 0 \tag{43}$$

and let

$$g_1(T, X) = \frac{\dot\sigma_1}{4\sigma_1}X^2 + \frac{\dot\rho_1}{2\sigma_1}X + \lambda_1.$$

Then

$$\psi(T, X) = e^{-g_1(T,X)}\hat\psi(\tau_1, y_1) \tag{44}$$

yields a solution of (34).

(2) Using *any* $\psi(T, X)$ given by (44), determine $G_1(T, X)$ from (33), i.e.,

$$G_1(T, X) = 2\frac{\partial^2}{\partial X^2}\log\left|\psi(T, X)\right| + \alpha(T)X^2 + \beta(T)X + \gamma(T).$$

(3) For any $\psi(T, X)$ given by (44), and corresponding $G_1(T, X)$ given by (33), use (7) and (8), with $G(T, X) = G_1(T, X)$, to determine $D(T, X)$, i.e.,

$$D(T, X) = 2\frac{\partial}{\partial X}\log\left|\theta(T, X)\right|,$$

where $\theta(T,X)$ is *any* solution of

$$\frac{\partial\theta}{\partial T}+\frac{\partial^2\theta}{\partial X^2}+\left[G_1(T,X)+\Gamma(T)\right]\theta=0 \tag{45}$$

for arbitrary $\Gamma(T)$. The procedure to solve PDE (45) now follows from the results of Theorem 1 as applied to Eq. (45).

Let $\sigma=\sigma_2(T)$, $\rho=\rho_2(T)$ be *any* solution of (15) and let $\lambda=\lambda_2(T)$ be *any* solution of

$$\dot\lambda=\frac{\dot\rho_2^2-2\sigma_2\dot\sigma_2}{4\sigma_2^2}+\gamma(T)+\Gamma(T).$$

Let

$$y_2=\sigma_2(T)X+\rho_2(T),\qquad \tau_2=\int^T\sigma_2^2(\mu)\,d\mu. \tag{46}$$

Let $\hat\theta(\tau_2,y_2)$ be *any* solution of the backward heat equation $\partial\hat\theta/\partial\tau_2+\partial^2\hat\theta/\partial y_2^2=0$, and let

$$g_2(T,X)=\frac{\dot\sigma_2}{4\sigma_2}X^2+\frac{\dot\rho_2}{2\sigma_2}X+\lambda_2.$$

Then

$$\theta(T,X)=e^{-g_2(T,X)}\left[\sigma_2\frac{\partial\hat\theta}{\partial y_2}-\left(\frac{\dot\sigma_2X+\dot\rho_2}{2\sigma_2}+\frac{\psi_X}{\psi}\right)\hat\theta\right] \tag{47}$$

yields a solution of (45) and hence leads to $D(T,X)=2(\partial/\partial X)\log|\theta(T,X)|$.

(4) For *any* $D(T,X)=2(\partial/\partial X)\log|\theta(T,X)|$, the corresponding coefficients $\{a(t,x), b(t,x)\}$ of the Kolmogorov equation (1) can be found as follows.

Let $D(T,X)=D(t,X(t,x))$, where $X(t,x)$ is given by (4b). Then from (5), the coefficients $\{a(t,x),b(t,x)\}$ are *any* solutions of

$$\frac{\partial X}{\partial t}+a(t,x)\frac{\partial^2X}{\partial x^2}+b(t,x)\frac{\partial X}{\partial x}=D\big(t,X(t,x)\big).$$

In particular, for *any* solution $\theta(T,X)$ of (4.7), and *arbitrary* $a(t,x)$, the coefficient

$$b(t,x)=2a(t,x)\frac{\partial}{\partial x}\log\big|\theta\big(t,X(t,x)\big)\big|-\sqrt{a(t,x)}\,X_t+\frac{1}{2}a_x(t,x) \tag{48}$$

with $X(t,x)=\int^x(1/\sqrt{a(t,z)})\,dz$.

(5) The procedure outlined in Theorem 1 relates the solution of the corresponding canonical backward equation (3) (or (9)) to *any* solution of the backward heat equation.

(6) In terms of $D(T,X)$, the corresponding solutions of the Kolmogorov equation (1) are found through transformation (4).

In the above step-by-step procedure we now show which solutions of the backward heat equation (43) yield Kolmogorov equations, i.e., coefficients $a(t,x)$, $b(t,x)$, which are transformable to the backward heat equation *only* by nonlocal transformations.

Theorem 2. *Let* $\hat{\psi}(\tau_1, y_1)$ *be a solution of the backward heat equation* (43). *By following the step-by-step procedure* (1)–(5), *such a solution yields the Kolmogorov equation* (1) *that is transformable to the backward heat equation* only *through a nonlocal transformation if and only if* $\hat{\psi}(\tau_1, y_1)$ *is* not *one of the forms*

$$\text{(I)} \quad \hat{\psi}(\tau_1, y_1) = e^{(Py_1 - P^2\tau_1)},$$
$$\text{(II)} \quad \hat{\psi}(\tau_1, \hat{y}_1) = \frac{1}{\sqrt{\tau_1 - \hat{\tau}_1}} \exp\left\{\frac{(y_1 - \hat{y}_1)^2}{4(\tau_1 - \hat{\tau}_1)}\right\}, \tag{49}$$

where P, y_1, $\hat{\tau}_1$ *are arbitrary constants.*

Proof. From condition (37), it follows that $\hat{\psi}(\tau_1, y_1)$ yields a Kolmogorov equation that is transformable to the backward heat equation only by a nonlocal transformation if and only if $\psi(T, X)$ given by (44) satisfies $(\partial^5/\partial X^5)\log|\psi(T, X)| \not\equiv 0$. Now suppose

$$\frac{\partial^5}{\partial X^5}\log\left|\psi(T, X)\right| \equiv 0.$$

Consequently, $(\partial^5/\partial X^5)\log|\hat{\psi}(\tau_1, y_1)| \equiv 0$, which, as it follows from (42) ($\sigma \neq 0$), is equivalent to

$$\frac{\partial^5}{\partial y_1^5}\log\left|\hat{\psi}(\tau_1, y_1)\right| \equiv 0.$$

Hence $\hat{\psi}(\tau_1, y_1)$ is of the form

$$\hat{\psi}(\tau_1, y_1) = \exp\left\{A(\tau_1)y_1^4 + B(\tau_1)y_1^3 + C(\tau_1)y_1^2 + D(\tau_1)y_1 + E(\tau_1)\right\} \tag{50}$$

for arbitrary $A(\tau_1)$, $B(\tau_1)$, $C(\tau_1)$, $D(\tau_1)$, $E(\tau_1)$. Substituting (50) into the backward heat equation (42), we see that

$$A(\tau_1) = B(\tau_1) \equiv 0, \qquad \frac{dC}{d\tau_1} + 4C^2 = 0,$$
$$\frac{dD}{d\tau_1} + 4CD = 0, \qquad \frac{dE}{d\tau_1} + 2C + D^2 = 0. \tag{51}$$

It is now straightforward to show that the solutions of (51) yield (48). □

For an arbitrary coefficient $a(t, x)$ in the Kolmogorov equation (1), we next show how the work presented in this paper generalizes previous works [3,8] on the types of coefficients $b(t, x)$ for which (1) can be transformed to the backward heat equation. Making the appropriate substitutions in (47), we obtain

$$\begin{aligned} b(t, x) = {} & -2\sqrt{a}\left(\frac{\dot{\sigma}_2}{2\sigma_2}X + \frac{\dot{\rho}_2}{2\sigma_2}\right) - \sqrt{a}\,X_t + \frac{1}{2}a_x + 2a\frac{\partial}{\partial x}\log\left|\hat{\theta}(\tau_2, y_2)\right| \\ & + 2a\frac{\partial}{\partial x}\log\left|\sigma_2\frac{\partial}{\partial y_2}\log\left|\hat{\theta}(\tau_2, y_2)\right| - \sigma_1\frac{\partial}{\partial y_1}\log\left|\hat{\psi}(\tau_1, y_1)\right|\right. \\ & \left. + \left(\frac{\dot{\sigma}_1}{2\sigma_1} - \frac{\dot{\sigma}_2}{2\sigma_2}\right)X + \left(\frac{\dot{\rho}_1}{2\sigma_1} - \frac{\dot{\rho}_2}{2\sigma_2}\right)\right|, \end{aligned} \tag{52}$$

with $X(t,x)$, $\sigma_i(t)$, $\rho_i(t)$, $\tau_i(t)$, $y_i(t, X(t,x))$, $i = 1, 2$, defined by Eqs. (4), (42) and (46); $\hat{\psi}(\tau_1, y_1)$, $\hat{\theta}(\tau_2, y_2)$ are *any* solutions of backward heat equations in terms of their respective arguments. Note that if $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, then $\tau_1 = \tau_2 = \tau$, $y_1 = y_2 = y$. One can show the following:

(I) Cherkasov's results [8] correspond to $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, $\hat{\theta}(\tau, y) = \text{const}$, $\hat{\psi}(\tau, y)$ is a solution of the backward heat equation which is one of the special forms (49).

(II) The results presented in [3] also correspond to $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, $\hat{\psi}(\tau, y)$ is a solution of the backward heat equation which is one of the special forms (49), but, unlike in Cherkasov's results, $\hat{\theta}(\tau, y)$ is allowed to be any solution of the backward heat equation.

(III) The results presented in this paper yield further new classes of $b(t,x)$, $a(t,x)$ if $\sigma_1 \neq \sigma_2$, $\rho_1 \neq \rho_2$. If $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, then further new classes of $b(t,x)$, $a(t,x)$ beyond those found in [3], are obtained for any pair of solutions $(\hat{\theta}(\tau, y), \hat{\psi}(\tau, y))$ of the backward heat equation except when

(1) $\hat{\psi}(\tau, y)$ is one of the special forms (49) or
(2) $\hat{\psi}(\tau, y) = C\hat{\theta}(\tau, y)$ for some constant C.

5. A recycling procedure further extending classes of Kolmogorov equations transformable to the backward heat equation

In this section we enhance the results presented in Sections 3 and 4 by describing a recycling procedure which generates sequential chains of Kolmogorov equations transformable to the backward heat equation. Such chains will emanate from any Kolmogorov equation with $G_1(T,X) = \Gamma_1(T,X)$ of the form

$$\Gamma_1(T,X) = 2\frac{\partial^2}{\partial X^2}\log\bigl|\psi(T,X)\bigr| + \alpha(T)X^2 + \beta(T)X + \gamma(T), \tag{53}$$

where $\psi(T,X)$ is any solution of

$$\frac{\partial \psi}{\partial T} + \frac{\partial^2 \psi}{\partial X^2} + \bigl[\alpha(T)X^2 + \beta(T)X + \gamma(T)\bigr]\psi = 0. \tag{54}$$

This recycling procedure generalizes previous work presented in [6] which dealt with a T-independent $\Gamma_1(T,X)$. To establish such chains we proceed as follows.

Suppose $u_1(T,X)$ is any solution of (9) and $\phi(T,X) = \phi_1(T,X) \neq 1/\psi(T,X)$ is a particular solution of (24) with $G_1(T,X) = \Gamma_1(T,X)$ given by (53) and (54). Then from Eq. (29) it follows that

$$u_2(T,X) = \frac{1}{\phi_1(T,X)}\left[\int_k^X u_1(T,\xi)\phi_1(T,\xi)\,d\xi + B_2(T)\right],$$

with $B_2(T)$ satisfying condition (30), solves

$$\frac{\partial u_2}{\partial T} + \frac{\partial^2 u_2}{\partial X^2} + \Gamma_2(T,X)u_2 = 0, \tag{55}$$

where

$$\Gamma_2(T,X)=\Gamma_1(T,X)+2\frac{\partial^2}{\partial X^2}\log\left|\phi_1(T,X)\right|.$$

Consequently, the canonical equation (55) is transformable to the backward heat equation. (Since (9), with $G_1(T,X)=\Gamma_1(T,X)$, can be mapped into the backward heat equation, in this manner one may obtain all solutions $u_1(T,X)$ by using step (3) of the step-by-step procedure outlined in Section 4 with $\Gamma_1(T)\equiv 0$ (see formula (47)). However, there is no general procedure to obtain particular solutions of the adjoint equation (24) when $\Gamma_1(T,X)$ is T-dependent. In Section 6 we will show how to obtain the general solution of (24) when $\Gamma_1(T,X)$ and $\phi(T,X)$ are T-independent.)

In general, suppose $u_n(T,X)$ is any solution of

$$\frac{\partial u_n}{\partial T}+\frac{\partial^2 u_n}{\partial X^2}+\Gamma_n(T,X)u_n=0, \tag{56}$$

and $\phi_n(T,X)$ is a particular solution of

$$-\frac{\partial \phi_n}{\partial T}+\frac{\partial^2 \phi_n}{\partial X^2}+\Gamma_n(T,X)\phi_n=0.$$

Then the system

$$\frac{\partial}{\partial X}(\phi_n u_{n+1})=\phi_n u_n,\qquad \frac{\partial}{\partial T}(\phi_n u_{n+1})=\frac{\partial \phi_n}{\partial X}u_n-\phi_n\frac{\partial u_n}{\partial X}$$

yields a nonlocal transformation relating the canonical PDE (56) to the homologous canonical PDE

$$\frac{\partial u_{n+1}}{\partial T}+\frac{\partial^2 u_{n+1}}{\partial X^2}+\Gamma_{n+1}(T,X)u_{n+1}=0, \tag{57}$$

where

$$\Gamma_{n+1}(T,X)=\Gamma_n(T,X)+2\frac{\partial^2}{\partial X^2}\log\left|\phi_n(T,X)\right|.$$

The corresponding nonlocal transformations connecting solutions are given by

$$u_{n+1}(T,X)=\frac{1}{\phi_n(T,X)}\left[\int_k^X u_n(T,\xi)\phi_n(T,\xi)\,d\xi+B_{n+1}(T)\right],$$

with $B_{n+1}(T)$ satisfying the condition

$$\frac{dB_{n+1}}{dT}=\frac{\partial \phi_n}{\partial X}(T,k)u_n(T,k)-\phi_n(T,k)\frac{\partial u_n}{\partial X}(T,k),$$

$n=1,2,\ldots$. Consequently, sequential chains of canonical PDEs (57) are transformable to the backward heat equation.

The coefficients $a=a_n(t,x)$, $b=b_n(t,x)$ of the corresponding chains of Kolmogorov equations follow from step (4) of the step-by-step procedure presented in Section 4. In particular, let

$$D_n(T,X)=2\frac{\partial}{\partial X}\log\left|u_n(T,X)\right|$$

for any solution $u_n(T, X)$ of (55). Without loss of generality, $a(t, x)$ is arbitrary. Then

$$b_n(t,x) = 2\sqrt{a(t,x)}\,\frac{\partial}{\partial X}\log\left|u_n(t,X)\right| - \sqrt{a(t,x)}\,X_t + \frac{1}{2}a_x(t,x),$$

with $X = X(t, x)$ given by (4b).

6. The recycling procedure for Kolmogorov equations with time-independent coefficients

In the important special case when the coefficients $a(t, x)$, $b(t, x)$ of the Kolmogorov equation (1) are time-independent, i.e, $a(t, x) \equiv a(x)$, $b(t, x) \equiv b(x)$, all of the equations are completely solvable in the recycling procedure outlined in Section 5.

Here $X(t, x) \equiv X(x)$, $D(t, x) \equiv D(x)$, $G(t, x) \equiv G(x)$. From (4)–(6), one obtains

$$X(x) = \int^{x} \frac{dz}{\sqrt{a(z)}}, \tag{58}$$

$$D(X) = a(x)\frac{d^2X}{dx^2} + b(x)\frac{dX}{dx} = \frac{1}{\sqrt{a(x)}}\left[b(x) - \frac{1}{2}a'(x)\right], \tag{59}$$

$$G(X) = -\frac{1}{2}\left[D'(X) + \frac{1}{2}D^2(X)\right]. \tag{60}$$

In the recycling procedure, the equations are solved as follows.

Let $\psi(X)$ be any solution of the ODE

$$\frac{d^2\psi}{dX^2} + [\alpha X^2 + \beta X + \gamma]\psi = 0 \tag{61}$$

for some constants α, β, γ, and let

$$\Gamma_1(X) = 2\frac{d^2}{dX^2}\log|\psi| + \alpha X^2 + \beta X + \gamma. \tag{62}$$

Suppose $u_1(T, X)$ is any solution of

$$\frac{\partial u_1}{\partial T} + \frac{\partial^2 u_1}{\partial X^2} + \Gamma_1(X)u_1 = 0.$$

All such solutions $u_1(T, X)$ are given by following step (3) of the step-by-step procedure of Section 4 with $\Gamma(T) \equiv 0$ (see formula (47)). Now let $\phi(X) = \phi_1(X) \neq 1/\psi(X)$ be a particular solution of the ODE

$$\frac{d^2\phi}{dX^2} + \Gamma_1(X)\phi = 0. \tag{63}$$

Unlike the situation in the time-dependent case, one is able to find the *general solution* of (63) since $\phi(X) = 1/\psi(X)$ is a particular solution of (63). Specifically,

$$\phi(X) = \phi_1(X; K_1) = \frac{1}{\psi(X)}\left[K_1 + \int^{X} \psi^2(z)\,dz\right] \tag{64}$$

for arbitrary constant K_1. Then

$$u_2(T,X) = \frac{1}{\phi_1(X;K_1)}\left[\int_k^X u_1(T,\xi)\phi_1(\xi;K_1)\,d\xi + B_2(T)\right],$$

with $B_2(T)$ satisfying the condition

$$\frac{dB_2}{dT} = \frac{\partial^2\phi_1}{\partial X}(k;K_1)u_1(T,k) - \phi_1(k;K_1)\frac{\partial u_1}{\partial X}(T,k)$$

for arbitrary constant k, solves

$$\frac{\partial u_2}{\partial T} + \frac{\partial^2 u_2}{\partial X^2} + \Gamma_2(X;K_1)u_2 = 0,$$

where

$$\Gamma_2(X;K_1) = \Gamma_1(X) + 2\frac{d^2}{dX^2}\log\left|\phi_1(X;K_1)\right|.$$

In general, suppose $u_n(T,X)$ is any solution of

$$\frac{\partial u_n}{\partial T} + \frac{\partial^2 u_n}{\partial X^2} + \Gamma_n(X;K_1,\ldots,K_{n-1})u_n = 0.$$

The general solution of the ODE

$$\frac{\partial^2\phi_n}{\partial X^2} + \Gamma(X;K_1,\ldots,K_{n-1})\phi_n = 0$$

can be obtained in the same manner as that used to obtain expression (64) and it has the form

$$\phi_n(X;K_1,\ldots,K_n) = \frac{1}{\phi_{n-1}(X;K_1,\ldots,K_{n-1})} \times\left[K_n + \int^X \left[\phi_{n-1}(z;K_1,\ldots,K_{n-1})\right]^2 dz\right]. \tag{65}$$

Then the function

$$u_{n+1}(T,X) = \frac{1}{\phi_n(X;K_1,\ldots,K_n)} \times\left[\int_k^X u_n(T,\xi)\phi_n(\xi;K_1,\ldots,K_n)\,d\xi + B_{n+1}(T)\right],$$

with $B_{n+1}(T)$ satisfying the condition

$$\frac{dB_{n+1}}{dT} = \frac{\partial\phi_n}{\partial X}(k;K_1,\ldots,K_n)u_n(T,k) - \phi_n(k;K_1,\ldots,K_n)\frac{\partial u_n}{\partial X}(T,k),$$

solves

$$\frac{\partial u_{n+1}}{\partial T} + \frac{\partial^2 u_{n+1}}{\partial X^2} + \Gamma_{n+1}(X; K_1, \ldots, K_n) u_{n+1} = 0, \tag{66}$$

where

$$\Gamma_{n+1}(X; K_1, \ldots, K_n) = \Gamma_n(X; K_1, \ldots, K_{n-1}) + 2\frac{d^2}{dX^2}\log\left|\phi_n(X; K_1, \ldots, K_n)\right|,$$

$n = 1, 2, \ldots,$ and $\phi_0(X) = \psi(X)$ is any solution of (61). Consequently, sequential chains of canonical equations (66) are transformable to the backward heat equation. For each member of such a sequential chain,

$$D_n(X; K_1, \ldots, K_n) = 2\frac{d}{dX}\log\left|\phi_n(X; K_1, \ldots, K_n)\right|.$$

Hence, with $a(x) = a_n(x)$ arbitrary, the coefficients $b(x)$ of the corresponding Kolmogorov equations (1), which are transformable to the backward heat equation, are given by

$$b(x) = b_n(x; K_1, \ldots, K_n) = 2\sqrt{a(x)}\,\frac{d}{dX}\log\left|\phi_n(X; K_1, \ldots, K_n)\right| + \frac{1}{2}a'(x), \tag{67}$$

$n = 0, 1, 2, \ldots,$ with $\phi_0(X) = \psi(X)$. In particular, when $a(x) \equiv \text{const} = a$, formula (67) becomes

$$b(x) = b_n(x; K_1, \ldots, K_n) = 2\sqrt{a}\,\frac{d}{dX}\log\left|\phi_n(X; K_1, \ldots, K_n)\right| \tag{68}$$

with $X = x/\sqrt{a}$.

The term $n = 0$ of a sequential chain corresponds to the local case which was completely considered in [3], the term $n = 1$ corresponds to the nonlocal extension which was completely considered in Section 4 of this paper, and further nonlocal extensions resulting from recycling correspond to terms $n = 2, 3, \ldots$. In effect there are $4 + n$ arbitrary fitting constants in term n.

7. An example: a *d*-Bessel process

For a d-Bessel process (see, for example, [10]) the Kolmogorov equation (1) has coefficients

$$a(x) = \frac{1}{2}, \qquad b(x) = \frac{\varepsilon}{x}, \tag{69}$$

with $\varepsilon = (d-1)/2$. Using (58)–(60), one finds that

$$X = \sqrt{2}\,x, \quad D(X) = \sqrt{2}\,\frac{\varepsilon}{x} = \frac{2\varepsilon}{X}, \quad G(X) = \frac{\varepsilon(1-\varepsilon)}{X^2}. \tag{70}$$

If $\varepsilon = 0$ $(d = 1)$ or $\varepsilon = 1$ $(d = 3)$, then the coefficient $G(X) \equiv 0$; if $\varepsilon = 2$ $(d = 5)$, then the coefficient $G(X)$ from (70) coincides with $\Gamma_1(X)$ given by (62) with $\psi = X$, $\alpha = \beta = \gamma = 0$. Hence the Kolmogorov equation (1), with its coefficients given by (69), for $\varepsilon = 2$ can be mapped by a *nonlocal* transformation (but not by any local one) into the backward heat equation.

Further application of the recycling procedure of Section 6 leads to the following result. Any Kolmogorov equation (1) with coefficients

$$a(x)=\frac{1}{2}, \qquad b(x)=b_n(x)=\frac{n+1}{x}, \quad n=0,1,2,\ldots \tag{71}$$

(which corresponds to $d=3,5,7,\ldots$), can be mapped into the backward heat equation. This follows immediately from (65) and (68) with $\phi_{n-1}(X)=X^n$ and $K_n=0$, $n=2,3,\ldots$.

A d-Bessel process corresponds to the spherically symmetric heat equation in $\mathbf{R}^d$, i.e.,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial R^2}-\frac{(d-1)}{R}\frac{\partial u}{\partial R}=0. \tag{72}$$

From the above results it follows that (72) can be mapped into the heat equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}=0$$

(I) by a point transformation if and only if $d=3$ (a well-known result);
(II) by a nonlocal transformation if and only if $d=2k+1$, $k=2,3,\ldots$.

8. Remarks on connections with symmetry analysis

It is well known that the scalar Kolmogorov equation (1) can be mapped into the backward heat equation by a point transformation if and only if (1) admits a six-parameter Lie group of point transformations (see [3,5]). One can show that the backward heat equation potential system (32) admits a six-parameter Lie group of point transformations. Hence it is necessary that the potential system (23) admit a six-parameter Lie group of point transformations in order that (23) can be mapped into (32) by a point transformation. Consequently, when such a mapping (35) defines a nonlocal transformation acting on (X,T,u_1)-space, it follows that (23) must admit a six-parameter Lie group of point transformations whereas the corresponding canonical equation (9) may not admit a six-parameter Lie group of point transformations.

As an example, consider the d-Bessel process for $d=5$. Here the canonical equation (9) becomes

$$\frac{\partial u_1}{\partial T}+\frac{\partial^2 u_1}{\partial X^2}-\frac{2}{X^2}u_1=0,$$

and only admits a four-parameter Lie group of point transformations with its infinitesimal generators given by

$$\mathbf{X}_1=\frac{\partial}{\partial T}, \qquad \mathbf{X}_2=2T\frac{\partial}{\partial T}+X\frac{\partial}{\partial X},$$
$$\mathbf{X}_3=T^2\frac{\partial}{\partial T}+TX\frac{\partial}{\partial X}+\frac{1}{2}(X^2-2T)u_1\frac{\partial}{\partial u_1}, \qquad \mathbf{X}_4=u_1\frac{\partial}{\partial u_1}.$$

Since $\phi=X^{-1}$ (see Section 7), the corresponding potential system (23) takes the form

$$\frac{\partial v_1}{\partial X} = \frac{1}{X} u_1, \qquad \frac{\partial v_1}{\partial T} = -\frac{1}{X^2} u_1 - \frac{1}{X} \frac{\partial u_1}{\partial X}. \tag{73}$$

System (73) admits a six-parameter Lie group of point transformations with its infinitesimal generators given by

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial T}, \qquad \mathbf{X}_2 = 2T \frac{\partial}{\partial T} + X \frac{\partial}{\partial X}, \\
\mathbf{X}_3 &= T^2 \frac{\partial}{\partial T} + TX \frac{\partial}{\partial X} + \left[\left(\frac{1}{4} X^2 - \frac{3}{2} T\right) u_1 + \frac{1}{2} X^2 v_1\right] \frac{\partial}{\partial u_1} \\
&\quad + \left(\frac{1}{4} X^2 - \frac{3}{2} T\right) v_1 \frac{\partial}{\partial v_1}, \\
\mathbf{X}_4 &= u_1 \frac{\partial}{\partial u_1} + v_1 \frac{\partial}{\partial v_1}, \qquad \mathbf{X}_5 = \frac{\partial}{\partial X} + \frac{1}{X} v_1 \frac{\partial}{\partial u_1} - \frac{1}{X} v_1 \frac{\partial}{\partial v_1}, \\
\mathbf{X}_6 &= T \frac{\partial}{\partial X} + \left[\frac{1}{2} X u_1 + \left(\frac{1}{2} X + \frac{T}{X}\right) v_1\right] \frac{\partial}{\partial u_1} + \left(\frac{1}{2} X - \frac{T}{X}\right) v_1 \frac{\partial}{\partial v_1}.
\end{aligned}$$

From the mapping (35), it follows that the point transformation

$$X = y, \quad T = \tau, \quad v_1 = \frac{\hat{v}}{y}, \quad u_1 = \hat{u} - \frac{\hat{v}}{y}$$

transforms (73) into the backward heat equation potential system (32).

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Appendix A. Solution of system (15)

In system (15), suppose $\alpha(T)$, $\beta(T)$, $\gamma(T)$ are given functions of T. Then $\sigma(T)$, $\rho(T)$, $\lambda(T)$ are found as follows.

To find $\sigma(T)$, first let

$$\sigma(T) = \frac{1}{s(T)}.$$

Then Eq. (15a) becomes a linear ODE in terms of $s(T)$, namely

$$\frac{d^2 s}{dT^2} + 4\alpha(T) s = 0. \tag{A.1}$$

Let $s = S(T)$ be any solution of (A.1). Then the general solution of Eqs. (15b,c) is given by

$$\rho(T) = 2 \int^T \frac{1}{S^2(t_2)}\, dt_2 \int^{t_2} \beta(t_1) S(t_1)\, dt_1,$$

$$\lambda(T) = \frac{1}{2}\log\left|S(T)\right| + \int^{T} \left\{ \frac{1}{S^2(t_2)} \left[\int^{t_2} \beta(t_1) S(t_1)\, dt_1 \right]^2 + \gamma(t_2) \right\} dt_2.$$

References

[1] S.C. Anco, G. Bluman, Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations, J. Math. Phys. 38 (1997) 3508–3532.

[2] A.T. Bharucha-Reid, Elements of the Theory of Markov Processes and Their Applications, McGraw–Hill, New York, 1960.

[3] G. Bluman, On the transformation of diffusion processes into the Wiener process, SIAM J. Appl. Math. 39 (1980) 238–247.

[4] G. Bluman, P. Doran-Wu, The use of factors to discover potential systems or linearizations, Acta Appl. Math. 41 (1995) 21–43.

[5] G. Bluman, S. Kumei, Symmetries and Differential Equations, in: Applied Mathematical Sciences, vol. 81, Springer, New York, 1989.

[6] G. Bluman, G. Reid, Sequences of related linear PDEs, J. Math. Anal. Appl. 144 (1989) 565–585.

[7] G. Bluman, V. Shtelen, New classes of Schrödinger equations equivalent to the free particle equation through non-local transformations, J. Phys. A 29 (1996) 4473–4480.

[8] I.D. Cherkasov, On the transformation of the diffusion process to a Wiener process, Theory Probab. Appl. 2 (1957) 373–377.

[9] A. Kolmogoroff, Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104 (1931) 415–458.

[10] M. Nagasawa, Schrödinger Equations and Diffusion Theory, Birkhäuser, Basel, 1993.

[11] L.M. Ricciardi, On the transformation of diffusion processes into the Wiener process, J. Math. Anal. Appl. 54 (1976) 185–199.

[12] L.M. Ricciardi, Diffusion Processes and Related Topics in Biology, in: Lecture Notes in Biomathematics, vol. 14, Springer, Berlin, 1977.