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On Shtelen's Solution of the Free Linear Schrödinger Equation

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Abstract

The solution of the three-dimensional free Schrödinger equation due to W.M. Shtelen based on the invariance of this equation under the Lorentz Lie algebra so(1,3) of nonlocal transformations is considered. Various properties of this solution are examined, including its extension to $n \geq 3$ spatial dimensions and its time decay; which is shown to be slower than that of the usual solution of this equation. These new solutions are then used to define certain mappings, F_n, on $L^2(\mathbb{R}^n)$ and a number of their properties are studied; in particular, their global smoothing properties are considered. The differences between the behavior of F_n and that of analogous mappings constructed from usual solutions of the free Schrödinger equation are discussed.

1. Introduction

It is well known [9, 6, 5] that the maximal Lie invariance algebra of the free-particle Schrödinger equation in three spatial dimensions,

$$(i\partial_t + \Delta)\psi = 0, \tag{1}$$

is the Schrödinger algebra sch(1,3), a Lie algebra which contains the Galilei and dilation algebras as well as some special conformal transformations. It was quite surprising therefore, when Fushchych and Segeda [4] showed that eq.(1) is also invariant under an algebra of nonlocal operators which satisfy the commutation relations of the three-dimensional Lorentz algebra so(1,3). Using this result, Shtelen [11] (see also [5]) constructed a new solution of eq.(1): ($c = \text{const.}$)

$$\psi_s\left(\underset{\sim}{x}, t\right) = c \int\limits_{\mathbb{R}^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \left(\underset{\sim}{k}^2\right)^{-1/4} \exp\left(-it\underset{\sim}{k}^2\right) dk, \qquad t > 0. \tag{2}$$

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This solution may be compared to the usual solution of eq.(1):

$$\begin{aligned}\psi\left(\underset{\sim}{x},t\right) &= c\int\limits_{\mathbb{R}^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\exp\left(-it\underset{\sim}{k}^2\right)dk\\ &= ct^{-3/2}\exp\left(\frac{i}{4t}\mid\underset{\sim}{x}\mid^2\right).\end{aligned}\tag{3}$$

It is well known that L^2 solutions of (1) can be generated by using (3) as the kernel of the following mapping:

$$\begin{aligned}\chi_t\left(\underset{\sim}{x}\right) &= (2\pi)^{-3/2}\int\limits_{\mathbb{R}^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\exp\left(-it\underset{\sim}{k}^2\right)\widetilde{\chi}(k)dk\\ &= (4\pi it)^{-3/2}\int\limits_{\mathbb{R}^3}\exp\left(\frac{i}{4t}\mid\underset{\sim}{x}-\underset{\sim}{y}\mid^2\right)\chi(y)dy,\end{aligned}\tag{4}$$

where $\widetilde{\chi}$ denotes the Fourier transform of χ. Then, with the choice of the multiplicative constant in (4), $\chi\mapsto\chi_t$ is unitary on $L^2(\mathbb{R}^3)$ and, when applied to nonrelativistic quantum mechanics, this result is usually given the physical interpretation that χ_t represents the wave function of a free quantum particle at time $t>0$ if χ represents the wave function of that particle at $t=0$. The unitary character of (4) plays an important role in this quantum theoretic interpretation and, more generally, in discussions of the conservation of probability in nonrelativistic quantum mechanics as a function of time (cf. [1]). We note for later reference that the unitarity of the mapping $\chi\mapsto\chi_t$ defined by (4) depends crucially on the $t^{-3/2}$ falloff of solution (3).

By analogy with (4), one may attempt to use Shtelen's solution (2) to construct an analogous mapping:

$$(Ff)\left(\underset{\sim}{x},\underset{\sim}{t}\right) = c\int\limits_{\mathbb{R}^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\left(\underset{\sim}{k}^2\right)^{-1/4}\exp\left(-it\underset{\sim}{k}^2\right)\tilde{f}(k)dk,\tag{5}$$

with $f\in L^2(\mathbb{R}^3)$. The purpose of this paper is to show that a mapping, F_n, analogous to (5) is well-defined on $L^2(\mathbb{R}^n)$ for $n\geq 3$ and to point out that it has quite different properties than the corresponding n-dimensional generalization of (4). For example, F_n is not unitary on $L^2(\mathbb{R}^n)$ for any choice of the undetermined multiplicative constant c, but instead maps $L^2(\mathbb{R}^n)$ onto the domains of $(-\Delta)^{1/4}$ densely contained in $L^2(\mathbb{R}^n)$. Similarly, it can be shown that F_n is not unitary on any weighted Hilbert space in which the weight is a power of $\underset{\sim}{k}^2$.

It will be shown that ψ_s has a slower time decay, $t^{-(\frac{n}{2}-\frac{1}{4})}$, than the characteristic falloff of the n-dimensional generalization of (4), $t^{-n/2}$. This property makes (5) and its generalizations F_n unacceptable for applications to quantum mechanics. On the other hand, for mathematical analyses of certain nonlinear evolution equations (e.g., nonlinear Schrödinger equations), it is very desirable that mappings determined by the linear part of such equations, such as (4) and (5), have smoothing properties; i.e., that elements of their range are smoother than elements of their domain. However, it is known that n-dimensional analogues of the mapping $\chi\mapsto\chi_t$ defined in (4) are unitary on all the Sobolev spaces $H^s(\mathbb{R}^n)(s \in \mathbb{R}$ as well as on $L^2(\mathbb{R}^n)$ and, therefore, do not have smoothing properties on these spaces. Many authors have tried to compensate for the lack of smoothing properties of these solutions on spatial domains by proving the existence of smoothing properties for them on space-time domains (cf. [3] and references cited therein).

We will show that n-dimensional versions, F_n, of the mapping F defined in (5) based upon Shtelen's solutions of the free Schrödinger equation have mild global smoothing properties on spatial domains in the sense that $F_n f$ has a "$\frac{1}{2}$ derivative" when $f \in L^2(\mathbb{R}^n)$. It seems to be a reasonable conjecture that such properties can be a first step in existence and uniqueness proofs for solutions of a class of nonlinear evolution equations which are different from presently known proofs. It is clear that such solutions, if they exist, will be different from the usual solutions based upon n-dimensional analogues of solutions (4) of the free Schrödinger equation. In the present however, we will concentrate on discussions of solutions of the free Schrödinger equation and of n-dimensional generalizations of (5).

In Section 2 we will discuss a number of properties of n-dimensional versions of Shtelen's solution (2), and Section 3 will be devoted to a discussion of some properties of n-dimensional generalizations F_n of the mapping F defined in (5); including a discussion of their global smoothing properties.

2. Properties of Shtelen's Solution

We first note that, although Shtelen only stated result (2) for the case $n = 3$, it is in fact valid for all $n \geq 3$. Indeed, the conditions [11,5]:

$$(x_a \partial_{x_b} - x_b \partial_{x_a})\psi = 0,\ a, b = 1, 2, ..., n,$$

$$\left[(-\Delta)^{1/2}\left(it\partial_{x_a} + \frac{1}{2}x_a\right) + \left(it\partial_{x_a} + \frac{1}{2}x_a\right)(-\Delta)^{1/2}\right]\psi = 0,$$

in which $(-\Delta)^{1/2}$ is interpreted as a pseudodifferential operator, lead to (2) in higher spatial dimensions in the same way as for $n = 3$. In the latter case, Shtelen evaluated the integrals in (2) to obtain:

$$\psi_s\left(\underset{\sim}{x}, t\right) = c\left|\underset{\sim}{x}\right|^{1/2} t^{-3/2} \exp\left(\frac{i\left|\underset{\sim}{x}\right|^2}{8t}\right)\left(J_{-\frac{1}{4}}\left(\frac{\left|\underset{\sim}{x}\right|^2}{8t}\right) + iJ_{\frac{3}{4}}\left(\frac{\left|\underset{\sim}{x}\right|^2}{8t}\right)\right), \quad t > 0, \tag{6}$$

where the J_ν are Bessel functions of the first kind. By using the well-known small-argument expansions for the Bessel functions, one obtains the following result for the asymptotic time dependence of ψ_s :

$$\psi_s\left(\underset{\sim}{x}, \underset{\sim}{t}\right) \sim c\, t^{-5/4}\left(1 + 0\left(\left|\underset{\sim}{x}\right|^2 t^{-1}\right)\right) \tag{7}$$

subject to the condition $\left|\underset{\sim}{x}\right|^2 \ll 8t$.

We will improve upon this result by deriving a representation for the n-dimensional version of (2) in terms of confluent hypergeometric functions. We write integrals (2) in terms of hyperspherical coordinates ([2], p.202):

$$\begin{aligned}
k_1 &= |k| \cos\theta_1 \\
k_2 &= |k| \sin\theta_1 \cos\theta_2 \\
&\vdots \\
k_{n-2} &= |k| \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2} \\
k_{n-1} &= |k| \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\phi, \\
k_n &= |k| \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\phi,
\end{aligned}$$

where $0 \leq \theta_i \leq \pi (i = 1, 2, ..., n-2)$ and $0 \leq \phi < 2\pi$. Then, choosing the direction of the vector x to lie along the k_1 axis, we obtain:

$$\begin{aligned}
\psi_s(x,t) &= c\, \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty |k|^{n-3/2} \exp\left(-it|k|^2\right) d|k| \times \int_0^\pi \exp\left(i|k|\,|x| \cos\theta_1\right) \sin^{n-2}\theta_1 d\theta_1 \\
&= c\, 2\pi^{\frac{n}{2}} \left(\frac{2}{|x|}\right)^{\frac{n}{2}-1} \int_0^\infty |k|^{\frac{n-1}{2}} \exp\left(-it|k|^2\right) J_{\frac{n}{2}-1}\left(|k||x|\right) d|k| \qquad (8) \\
&= c\, \pi^{\frac{n}{2}} (it)^{-\left(\frac{n}{2}-\frac{1}{4}\right)} \frac{\Gamma\left(\frac{n}{2}-\frac{1}{4}\right)}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}-\frac{1}{4}; \frac{n}{2}; \frac{i|x|^2}{4t}\right),
\end{aligned}$$

where the well-known integral representation for the Bessel functions has been used to obtain the second inequality ([8], p.79) and then the final expression (8) is obtained by integrating the power series for $J_{\frac{n}{2}-1}$ term by term.

We use (8) to obtain a bound on $\psi_s(x,t)$ by utilizing the following integral representation for ${}_1F_1$ ([8], p.274): ($Re\ c > Re\ a > 0$)

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt.$$

It follows that the absolute value of the ${}_1F_1$ function in (8) is bounded by unity, so that:

$$|\ \psi_s(x,t)\ | \leq C(n) t^{-\left(\frac{n}{2}-\frac{1}{4}\right)}, \qquad t > 0,\ n \geq 3,$$

uniformly in $x \in \mathbb{R}^n$. When $n = 3$, the time dependence of the above bound agrees with the asymptotic result (7) obtained from Shtelen's representation (6).

3. n-dimensional generalizations of (5)

We now consider expressions of the form (5):

$$(F_n f)(x,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ik\cdot x} (k^2)^{-1/4} \exp(-itk^2) \widetilde{f}(k) dk \qquad (9)$$

with $finL^2(\mathbb{R}^n), n \geq 3$. We recall that -$\Delta$ is a positive semidefinite operator on $L^2(\mathbb{R}^n)$ with the domain $\{u \in H^1(\mathbb{R}^n) : -\Delta u \in L^2(\mathbb{R}^n)\}$, where Δu is defined in the sense of distributions. The fractional powers of $-\Delta$ can be consistently defined ([7], Section 1.4) and we obtain from (9)

$$\|(-\Delta)^{\frac{1}{4}} \, F_n f\|_{L^2(\mathbb{R}^n)} = \|f_t\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)},$$

since the mapping $f \mapsto f_t$ defined analogously to (4) is unitary on $L^2(\mathbb{R}^n)$. It follows that $F_n f \in D((-\Delta)^{\frac{1}{4}}) \subset L^2(\mathbb{R}^n)$ when $f \in L^2(\mathbb{R}^n)$. The domains of fractional powers of $-\Delta$ have been characterized ([7], Section 1.6) and we see that F_n are smoothing in the sense that, roughly speaking, Ff has a "$\frac{1}{2}$ derivative" if $f \in L^2(\mathbb{R}^n)$.

To conclude the paper, we restrict f to the smaller class of functions $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and show that F_n is well-defined as a map from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. This result will exhibit the time decay properties of F_n in an explicit manner. In addition, we will prove that, for f contained in the indicated class of functions, $F_n f$ is also given by an expression analogous to the second line of (4) with the Gaussian kernel replaced by Shtelen's solution (8).

Theorem. *Define the mappings F_n by (9) with $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then they are bounded from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ if $t > 0$. Moreover, if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $F_n f$ can be written as the following convolution integral:*

$$(F_n f)(x,t) = \int\limits_{\mathbb{R}^n} \psi_s(x-y,t) f(y) dy, \tag{10}$$

where $\psi_s(x,t)$ is given by (8) with $c = (2\pi)^{-n/2}$.

Proof. We begin by introducing the definition of the Fourier transform $\widetilde{f}$ into (9). In order to write the resulting expression in the form (10), we must justify the reversal of the order of integrating over y and k. Since $f \in L^1(\mathbb{R}^n)$, one has $\widetilde{f} \in L^\infty(\mathbb{R}^n)$ and from (9) and (8),

$$\begin{aligned} |(F_n f)(x,t)| &= \left| (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ik\cdot x} (k^2)^{-1/4} \exp(-itk^2) \left(\int\limits_{\mathbb{R}^n} e^{-ik\cdot y} f(y) dy \right) dk \right| \\ &\leq K(n) t^{-(\frac{n}{2}-\frac{1}{4})} \parallel f \parallel_{L^1(\mathbb{R}^n)} \text{ uniformly in } x \\ &\text{in } \mathbb{R}^n, \end{aligned} \tag{11}$$

showing that $F_n f \in L^\infty(\mathbb{R}^n)$ and that $F_n f \in L^1(\mathbb{R}^{2n}, dy \times dk)$ (y-integration performed first). It follows from the form of Fubini's theorem given in ([10], Section I.4) that the orders of performing the integrations in (11) may be reversed and that (9) and (10) are equal for all $x \in \mathbb{R}^n$ and all $t > 0$. This completes the proof.

We note from (11) the characteristic time decay of $F_n f$ is $t^{-(\frac{n}{2}-\frac{1}{4})}$ when $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which is seen to be slower than the corresponding time decay of the n-dimensional generalization of (4) considered as a map from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$:

$$\parallel \chi_t \parallel_{L^\infty(\mathbb{R}^n)} \leq c \, t^{-n/2} \parallel \chi \parallel_{L^1(\mathbb{R}^n)} .$$

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