Conceptual Problems Multivariable Calculus

Please submit your solutions to the highlighted problems.

1 Vectors and Geometry

1. Two lines on the xy plane either intersect at a point or are parallel. a) When you have two lines in 3d space (so xyz space), what are the possibilities? b) What about two planes in 3d space? c) What about three planes in 3d space? d) Or four planes in 3d space? e) What about a line and a plane in 3d space?

2. Two lines on the xy plane determine a unique line passing through these points. a) Is this still true in 3d space? b) How many points do you need to specify a plane in 3d space?

3. Suppose $\mathbf{v}$ is a vector with certain physical units. For example, if $\mathbf{v}$ were a velocity vector then its units would be length/time. How are the units of $|\mathbf{v}|$ related to the units of $\mathbf{v}$? What are the units of $\frac{\mathbf{v}}{|\mathbf{v}|}$?

4. What is the line of intersection between the xz and yz planes?

5. What does the equation $x = 3$ correspond to if a) x is the only variable being considered, b) x, y are the only variables being considered, c) x, y, z are the only variables being considered.

6. What does the equation $x^2 + y^2 = 4$ correspond to if a) x, y are the only variables being considered, b) x, y, z are the only variables being considered.

7. Suppose $\mathbf{v}$ is a vector on the xy plane different from $\mathbf{0}$. Then $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ is known as the *polar decomposition* of the vector. Can you think of a reason why this name is given to the previous equation? Hint: think of complex numbers.

8. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ are two vectors and $\mathbf{u}$ another vector such that $\text{proj}_{\mathbf{u}}\mathbf{v}_1 = 3\mathbf{i} - 5\mathbf{k}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}_2 = 2\mathbf{j} + 7\mathbf{k}$. Find a) $\text{proj}_{\mathbf{u}}(\mathbf{v}_1 + 2\mathbf{v}_2)$, b) $\text{proj}_{\mathbf{u}}3\mathbf{v}_1$ and c) $\text{proj}_{3\mathbf{u}}\mathbf{v}_2$.

9. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three different vectors. Does the expression $\mathbf{v}_1 \cdot \mathbf{v}_2 \cdot \mathbf{v}_3$ make sense? What about $\mathbf{v}_1 \times \mathbf{v}_2 \times \mathbf{v}_3$? What about $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$? Is the last expression the same as $\mathbf{v}_1 \times (\mathbf{v}_2 \cdot \mathbf{v}_3)$, or the same as $(\mathbf{v}_1 \cdot \mathbf{v}_2) \times \mathbf{v}_3$?

10. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$?

11. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$

12. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

13. T/F If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then so is $\mathbf{a} \times \mathbf{b}$.

14. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$.

15. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$.

16. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{a}\|$.

2 Curves

1. What curve do you get by intersecting the sphere $x^2 + y^2 + z^2 = 4$ with the plane $x = y$?

2. Suppose a particle is moving with position vector $\mathbf{r}(t)$ and velocity vector $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$. a) if $\mathbf{v}(t)$ is constant, must $|\mathbf{v}(t)|$ be constant? b) if $|\mathbf{v}(t)|$ is constant, must $\mathbf{v}(t)$ be constant? c) if $\mathbf{v}(t)$ and $\mathbf{r}(t)$ are orthogonal vectors, must $\mathbf{a}(t)$ and $\mathbf{v}(t)$ be orthogonal?

3. Find the intersection between the sphere $3x^2 + y^2 + z^2 = 4$ and the cylinder $y^2 + z^2 = 1$.

4. T/F Suppose that a GPS satellite is orbiting the Earth in such a way that its distance from the planet remains constant. Then the velocity vector of the satellite is always perpendicular to its position vector.

5. Suppose that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two curves which intersect at a point P. If $P = \mathbf{r}_1(t_1)$ and $P = \mathbf{r}_2(t_2)$, does this mean that $t_1 = t_2$?

6. Suppose that $\mathbf{r}_1(t)$ is the equation of a curve and one defines $\mathbf{r}_2(t) = \mathbf{r}_1(\lambda t)$ for λ a positive constant (basically $\mathbf{r}_2$ is using a different time scale than the one used for $\mathbf{r}_1$). How is the velocity vector of $\mathbf{r}_2$ related to the one of $\mathbf{r}_1$? What about the acceleration vectors? If one defines $\mathbf{r}_3(t) = \mathbf{r}_1(t - t_0)$, how is the velocity vector of $\mathbf{r}_3$ related to the one of $\mathbf{r}_1$? What about the acceleration vectors?

7. Suppose $\mathbf{r}(t) = R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j}$ gives the equation of a circle. What are the units of ω if t is given in units of seconds, or more generally, time? What values t must take if we want $\mathbf{r}(t)$ to give a parametrization of the circle? The period of $\mathbf{r}(t)$ is the smallest positive number T such that $\mathbf{r}(t+T) = \mathbf{r}(t)$. What is T in this situation?

8. Is the velocity vector $\mathbf{v}(t)$ of a curve $\mathbf{r}(t)$ always perpendicular to the acceleration vector $\mathbf{a}(t)$?

3 Functions of Several Variables

1. Suppose that the level curves of a function $z = f(x, y)$ consists of straight lines. Must the graph of f be a plane?

2. Suppose that $T(x, y, z)$ represents the temperature at the point (x, y, z), measured in Kelvin. What are the units of $\frac{\partial T}{\partial x}$? What about the units of $\frac{\partial^2 T}{\partial x^2}$?

3. Suppose $f(x, y)$ is a function defined on the xy plane. Is it possible for $f(x, y)$ to have only local maxima but no local minima? What about the converse: can a function have only local minima but no local maxima? If a function has a critical point which is a saddle point, must it have critical points which are either local maxima or local minima?

4. If (a, b) is a critical point of f (in the sense that $f_x(a, b) = f_y(a, b) = 0$), is it true that (a, b) is a critical point of f^2? How about the converse? Namely, if (a, b) is a critical point of f^2, does that mean that (a, b) is a critical point of f?

5. True/False: if $\nabla f(a, b) = (0, 0)$, $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, then f has a local minimum at (a, b).

6. The Coulomb potential generated by a charge q located at the origin of the xy plane is $V(x, y) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2}}$, where ϵ_0 is a constant. a) What is the domain of V? b) What is the domain of V if it is considered as a function of x, y, z , namely, $V(x, y, z) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2}}$?

7. Suppose $T(x, y)$ represents the temperature of the floor at the point (x, y). If we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we can think of T as a function of r, θ. Find $\frac{\partial T}{\partial r}$ and $\frac{\partial T}{\partial \theta}$ in terms of $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$. We say that the temperature is *isotropic* if $\frac{\partial T}{\partial \theta} = 0$. Is the function $T(x, y) = \frac{y}{x}$ isotropic? What about $T(x, y) = x^2 + y^2$?

8. Suppose the domain of $f(x, y)$ consists of the rectangle $-1 \le x \le 1$, $0 \le y \le 1$, while the domain of $g(x, y)$ consists of the rectangle $0 \le x \le 1$, $-1 \le y \le 1$. What is the domain of $h(x, y) = f(x, y)g(x, y)$?

9. Suppose two level surfaces $f_1(x, y, z) = c_1$ and $f_2(x, y, z) = c_2$ intersect on a curve. What is an easy way to find the tangent vector $\mathbf{v}$ to any point on this curve from ∇f_1 and ∇f_2?

10. T/F The function $f(x, y) = (1 - x^2 - y^2)^{1/2}\ln(x^2 + y^2 - 1)$ has empty domain.

11. True/ False: If $\nabla f(x, y) = (0, 0)$, then (x, y) is a local minimum or local maximum of f

12. True/False: For any unit vector $\mathbf{u}$, $Df_{-\mathbf{u}}(\mathbf{r}) = -Df_{\mathbf{u}}(\mathbf{r})$

13. True/False: if $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$

14. True/False: if f is differentiable at (a, b) and $\nabla f(a, b) = (0, 0)$, then f has a local maximum or minimum at (a, b)

15. True/False: if $f(x, y)$ has two local maxima then f must have a local minimum

4 Double and Triple Integrals

1. Suppose $f(x, y)$ and $g(x, y)$ are continuous functions on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Is it true that $\int_0^1 \int_0^1 f(x, y)g(x, y)dydx = \left(\int_0^1 \int_0^1 f(x, y)dydx\right)\left(\int_0^1 \int_0^1 g(x, y)dydx\right)$

2. Suppose $\rho(x, y)$ has units of electric charge per unit area. What are the units of $\int\int_R \rho(x, y)dA$? More generally, how are the units of $\int\int_R f(x, y)dA$ related to those of $f(x, y)$?

3. If $\int\int_R f(x, y)dA \geq 0$, does that mean that $f(x, y) \geq 0$ at every point of the region of integration R?

4. True/False: the integral $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\theta d\rho d\theta d\phi$ gives the volume of 1/4 of a sphere

5. True/False: For any $a, b \in \mathbb{R}$ and a continuous function f, $\int_0^a \int_0^b f(x, y)dydx = \int_0^b \int_0^a f(x, y)dxdy$

6. True/False: If $f(x, y) = g(x)h(y)$ then $\int\int_D f(x, y)dA = \left(\int\int_D g(x)dA\right)\left(\int\int_D h(y)dA\right)$

7. True/False: $\int_0^1 \int_0^x \sqrt{x + y^2}dydx = \int_0^x \int_0^1 \sqrt{x + y^2}dxdy$

5 Vector Fields

1. True/False: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

2. True/False: If $\mathbf{F}, \mathbf{G}$ are vector fields and $\nabla \times \mathbf{F} = \nabla \times \mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

3. True/False: If $\mathbf{F}$ is conservative then $\nabla \cdot \mathbf{F} = 0$

4. True/False: curl(div$\mathbf{F}$)) is not a meaningful expression