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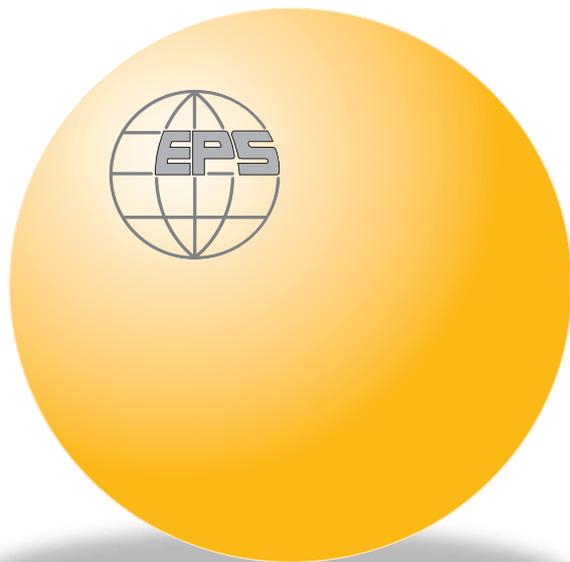
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Nonlinear effects in quantum tunnelling escape

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Abstract. – We consider the problem of tunnelling escape of particles from a multiparticle system confined within a potential trap. The process is nonlinear due to the interparticle interaction. Using the hydrodynamic representation for the quantum equations of the multiparticle system we find the tunnelling rate and time evolutions of the number of trapped particles for different nonlinearity values.

The recent advances in the experiments on real Bose-Einstein condensates (BEC) [1–3] and nonlinear optical waves [4] have generated a lot of theoretical works. The standard approach was based on the nonlinear Schrödinger equation with a potential depending on the particle density (also known as the time-dependent Gross-Pitaevskii (GP) equation [5–9]). The phenomena of coherence [10], macroscopic Josephson [11–14] and Landau-Zener [15] tunneling, vortex formation [16–19], instabilities, focusing and blowup are all new concepts, which are related to the *nonlinear* nature of the systems. Most of the analysis of these hard and fundamental analytic problems is so far being dealt with by a combination of numerical schemes (*e.g.*, refs. [12,14]) and finite-dimensional phenomenological models (*e.g.*, ref. [11]). The quest for a theory that can adequately give the relevant nonlinear effects in the time-dependent regime is therefore of major current interest.

We analyze the problem of tunneling in N -body quantum systems by solving the corresponding nonlinear problem in the leading relevant approximation. Our approach combines ideas from many-body theory, nonlinear partial differential equations and resonance theory in quantum mechanics to offer a unified approach to finding tunneling times for both nonlinear systems. We focus on the problem of a BEC droplet in a potential well that tunnels through a finite barrier and find the leading nonlinear corrections to the tunneling rate.

A droplet of N atoms with boson statistics is confined by an external potential $V_{ext}(\mathbf{R})$. The multiparticle wave function of such a system satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Psi(\{\mathbf{R}_i\})}{\partial t} = \left(-\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial \mathbf{R}_i^2} + \sum_i V_{ext}(\mathbf{R}_i) + \frac{1}{2} \sum_{i \neq j} V_{int}(\mathbf{R}_i - \mathbf{R}_j) \right) \Psi(\{\mathbf{R}_i\}). \quad (1)$$

We may now derive equations of motion for moments of the single-particle Wigner function $\rho_W(\mathbf{p}, \mathbf{R}, t)$. (see, *e.g.*, [20]). We concentrate on the condensate behavior, separated from the dissipative excitations (a detailed discussion of this separation is discussed, *e.g.*, in ref. [21]). The derivation can be made gauge invariant (see [20] and references therein). The most important of the moments are the density distribution of the particles

$$\rho(\mathbf{R}, t) = \frac{1}{(2\pi\hbar)^4} \int d^3p d\varepsilon \rho_W(\mathbf{p}, \varepsilon, \mathbf{R}, t)$$

and the velocity field

$$m\mathbf{v}(\mathbf{R}, t) = \frac{1}{(2\pi\hbar)^4} \frac{1}{\rho(\mathbf{R}, t)} \int d^3p d\varepsilon \mathbf{p} \rho_W(\mathbf{p}, \varepsilon, \mathbf{R}, t)$$

for which the continuity equation

$$\frac{\partial \rho(\mathbf{R}, t)}{\partial t} + \nabla(\mathbf{v}(\mathbf{R}, t)\rho(\mathbf{R}, t)) = 0 \quad (2)$$

and the Euler-type equation

$$\frac{\partial \mathbf{v}(\mathbf{R}, t)}{\partial t} + (\mathbf{v}(\mathbf{R}, t) \cdot \nabla)\mathbf{v}(\mathbf{R}, t) = -\frac{1}{m} \nabla(V_{ext}(\mathbf{R}) + V_{qu}(\mathbf{R}) + V_{eff}(\mathbf{R})) \quad (3)$$

are obtained. The latter is not necessarily potential and in three dimensions allows for vortices.

The ‘‘quantum potential’’ $V_{qu}(\mathbf{R}) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho(\mathbf{R}, t)}}{\sqrt{\rho(\mathbf{R}, t)}}$ accounts for the quantum character of the liquid in the droplet, whereas the potential $V_{eff}(\mathbf{R})$ is due to the inter-particle interaction. The latter in the mean-field approximation takes the form of a functional of the particle density $\rho(\mathbf{R}, t)$. In the case of the contact interaction, $V_{int}(\mathbf{R}_i - \mathbf{R}_j) = \lambda \delta(\mathbf{R}_i - \mathbf{R}_j)$ direct calculations result in the lowest order in λ in $V_{eff}(\mathbf{R}) = \lambda \rho(\mathbf{R}, t)$. Equations (2) and (3) become equivalent to the time-dependent GP equation. Its hydrodynamic representation was recently discussed in ref. [22]. Higher-order terms in λ produce corrections to the potential $V_{eff}(\mathbf{R})$ as well as a dissipative contribution due to the interaction with excitations above the condensate.

We limit ourselves to the discussion of tunneling evaporation of a droplet kept within a one-dimensional potential depicted in fig. 1. and consider the one-dimensional versions of

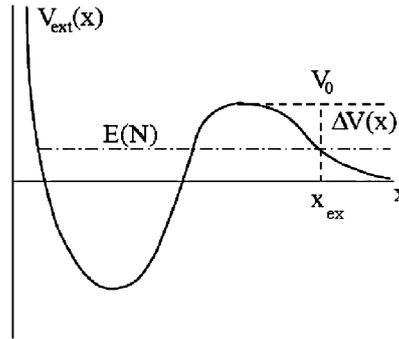


Fig. 1 – Potential well keeping the droplet. The dashed line shows the auxiliary potential $V_0(x)$ at larger x far from the barrier of the actual potential $V(x)_{ext}$. The dash-dotted line corresponds to the energy $E(N)$ for a given number of atoms N .

eqs. (2) and (3), which read now

$$\rho_t(x, t) + \frac{\partial}{\partial x}[\rho(x, t)v(x, t)] = 0, \quad (4)$$

$$v_t(x, t) + v(x, t)v_x(x, t) = \frac{1}{m} \frac{\partial}{\partial x} \left[V_{ext}(x) - \frac{1}{\sqrt{\rho(x, t)}} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \sqrt{\rho(x, t)} + U_{eff}(\rho(x, t)) \right]. \quad (5)$$

First, an auxiliary confining potential $V_0(x)$ is introduced, which coincides with the real potential $V_{ext}(x)$ within the potential well but differs from it (dashed line in fig. 1) outside the well. According to fig. 1, the asymptotic value V_0 of $V_0(x)$ at $x \rightarrow \infty$ is at the same time the barrier height of $V_{ext}(x)$. Stationary states are possible in the potential $V_0(x)$ whose density distribution follows from eq. (5) at $v(x, t) = 0$ with $V_0(x)$ substituted for $V_{ext}(x)$:

$$V_0(x)\sqrt{\rho(x, t)} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \sqrt{\rho(x, t)} + U_{eff}(\rho(x, t))\sqrt{\rho(x, t)} = E(N)\sqrt{\rho(x, t)}. \quad (6)$$

Here $E(N)$ is the single-particle eigen-energy in the droplet with N atoms. One readily recognizes the stationary Schrödinger equation for the single-particle wave function $\psi(x, t) = \sqrt{\rho(x, t)}$ and with the effective potential $U_{eff}(\rho(x, t))$ accounting for the many-body effects. The solution of (6) at a given initial number of atoms N_0 is the initial state of the droplet in the external potential $V_{ext}(x)$. The density distribution $\rho(x, t)$ becomes time dependent and its dynamics is governed by eq. (5). We will follow the time evolution of the number of particles $N(t) = \int_0^{x_{ex}} dx \rho(x, t)$ within the droplet with x_{ex} being the exit point of the potential $V_{ext}(x)$ at a given energy $E(N)$. $N(t)$ at $t = 0$ coincides with the initial number of particles N_0 to within exponentially small corrections.

Equation (5) is solved assuming that the velocity field and the density distribution change adiabatically slowly with time so that the velocity time derivative, $v_t(x, t)$, in (5) is neglected. The applicability of this approximation will be discussed below. Since initially the density distribution $\rho(x, t)$ satisfies eq. (6), we find the velocity field $v(x, t) = \sqrt{\frac{2}{m} \Delta V(x)}$ where $\Delta V(x) = V_0 - V_{ext}(x)$. Below we will be interested only in the velocity near the exit point, defined by the energy distance between the exit point and the top of the potential.

Now we find the number of particles within the well by integrating the continuity equation (4) from 0 to the exit point x_{ex} . According to our approximation, both the density and the velocity fields vary slowly with time due to the variation of the number of particles $N(t)$. So does the exit point $x_{ex}(N)$ due to the drift of the eigen-energy $E(N)$. The velocity is now the function $v(x_{ex}(N), t) = \sqrt{\frac{2}{m}[V_0 - E(N(t))]}$ of the number of particles $N(t)$ defined by the equation

$$t = - \int_{N_0}^N d\tilde{N} \frac{1}{\rho(x_{ex}(\tilde{N})) \sqrt{\frac{2}{m}[V_0 - E(\tilde{N})]}}. \quad (7)$$

In order to estimate $\rho(x_{ex}(N))$ at the exit point the particle density for given N is written as $\rho(x; N) = N \varrho(x; N)$, with $\varrho(x; N)$ normalized to one. $\rho(x, N)$ satisfies eq. (6). Since we need to know its value only at the exit point where it is exponentially small, we may neglect the inter-particle interaction, *i.e.* $\sqrt{\varrho(x_{ex}(N); N)} \propto \exp[-\alpha(N)x_{ex}(N)]$ with $\alpha = \frac{1}{\hbar} \sqrt{2m(V_0 - E(N))}$ and the exit point position determined by the equation $V_{ext}(x_{ex}) = E(N)$.

If the dependence of the energy $E(N)$ on the number of particles in the droplet is known, eq. (7) can be solved and the time dependence $N(t)$ of the number of particles can be found. As an example we carry out the calculation for the contact inter-particle interaction when eqs. (4)

and (5) correspond to the GP equation. At not too large nonlinearity we may approximately assume a linear dependence of the energy on the number of particles, $E(N) = E(0) + \tilde{\lambda}N$, [23] and represent the integral (7) in the form

$$\frac{1}{2}\Gamma_0 t = - \int_{y_0}^y dy \frac{e^{\mp\beta, \alpha_0 x_0 y^2}}{y\sqrt{1 \mp y^2}}, \quad (8)$$

where $y^2 = \frac{N}{N_{GP}}$, with $N_{GP} = \frac{\hbar^2 \alpha_0^2}{2m|\tilde{\lambda}|} = \frac{V_0 - E(0)}{|\tilde{\lambda}|}$. The dimensionless factor

$$\beta = 2 \left[\tilde{\lambda} \left(\frac{m}{\hbar^2 \alpha_0^2} N_{GP} + \frac{N_{GP}}{x_0 V'_{ext}(x_0)} \right) \right] = \frac{x_0 V'_{ext}(x_0) + 2(V_0 - E(0))}{x_0 V'_{ext}(x_0)}$$

appearing in eq. (8) is close to one; $\Gamma_0 = \frac{\alpha_0 \hbar}{m} \varrho(x_0; 0)$ is the tunneling rate in the absence of the interaction. Here and below the sign + or - is chosen according to the sign of the interaction parameter $\tilde{\lambda}$.

Now we integrate eq. (8) by parts, take the slow logarithmic function out of the integral, and arrive at the equation $\nu_{\pm}(y) = \nu_{\pm}(y_0) \exp[-\Gamma_0 t e^{\pm\beta \alpha_0 x_0 y^2}]$ in which $\nu_{\pm}(y) = \mp \frac{\sqrt{\mp y^2 + 1} - 1}{1 + \sqrt{\mp y^2 + 1}}$. Assuming in the first iteration that $y = y_0$ in the exponential, the time dependence of the number of particles in the well is

$$N(t) = N_{GP} \frac{4\nu_{1,\pm}}{(1 \pm \nu_{1,\pm})^2} \quad (9)$$

with $\nu_{1,\pm} = \nu_{\pm}(y_0) \exp[-\Gamma_0 t \exp[\mp\beta \alpha_0 x_0 y_0^2]]$. At $N \ll N_{GP}$, eq. (9) results in an exponential decay $N(t) = N_0 e^{-\Gamma_0 t}$ with the rate Γ_0 being the tunneling rate of non-interacting particles. At N comparable to N_{GP} , the initial decay may strongly deviate from the exponential behavior. However, N diminishes in the time course and at large time its asymptotic behavior becomes exponential.

Before discussing the result we estimate the validity of the above adiabatic approximation by comparing the time derivative of the velocity, v_t , near the exit point x_{ex} , neglected in the above calculations, with the right-hand side of eq. (5). The time dependence of the velocity results mainly from the time drift of the exit point, so that

$$v_t = \frac{2\dot{x}_0}{mv(x_{ex})} V'_{ext}(x_{ex}) \approx \frac{1}{mv(x_{ex})} \frac{V_0}{l} \dot{x}_{ex} \quad (10)$$

with l being the typical scale of the potential $V_{ext}(x)$. Since x_0 is the solution of equation $V_{ext}(x_{ex}) = E(0) - \tilde{\lambda}N(t)$, we get $\dot{x} \approx \frac{\dot{N}}{N} \frac{\tilde{\lambda}N}{V_0} l$. The r.h.s. of eq. (5) is approximately estimated as V_0/ml so that comparing it with the time derivative of the velocity field (10) we need to check the inequality

$$\frac{v_t ml}{V_0} \approx \frac{\dot{x}_{ex}}{v(x_{ex})} \approx \tau_{tr} \frac{\dot{N}}{N} \frac{\tilde{\lambda}N}{V_0} \ll 1. \quad (11)$$

Here $\tau_{tr} = l/v$ is roughly the time needed for a tunneling particle to traverse the classically forbidden underbarrier region. At any reasonably high potential barrier the tunneling rate is small enough so that $\tau_{tr} \dot{N}/N \ll 1$. The ratio $\tilde{\lambda}N/V_0$ cannot be large and in many realistic cases it is small. We conclude that the inequality (11) is robust and holds at all reasonable parameters of the system, which justifies the adiabatic approximation applied in this paper.

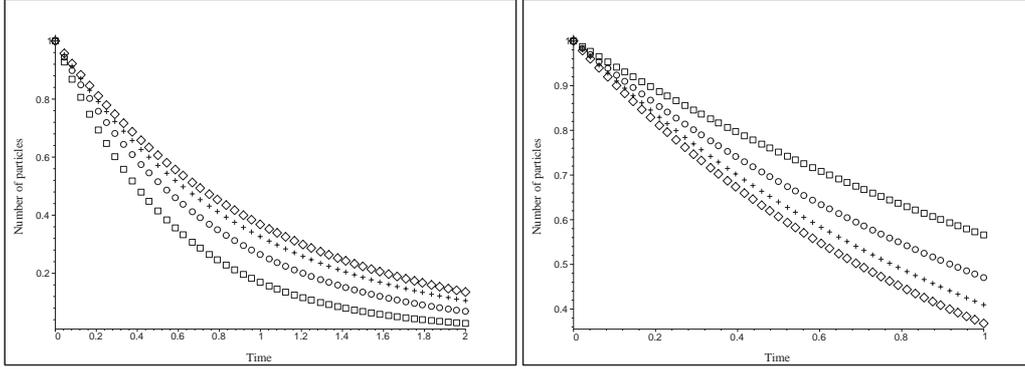


Fig. 2 – Relative number of atoms $(N(t)/N_0)$ as a function of time (in units $1/\Gamma$) at $\tilde{\lambda} > 0$ (left figure) and $\tilde{\lambda} < 0$ (right figure) for different values of the initial number of atoms N_0/N_{GP} and $\beta\alpha_0x_0 = 6$: $N_0/N_{GP} = 0$ (diamonds); $N_0/N_{GP} = 0.02$ (crosses); $N_0/N_{GP} = 0.05$ (circles); $N_0/N_{GP} = 0.1$ (squares).

Figures 2 show the time dependence of the number of atoms $N(t)$ in droplets with different initial numbers N_0 for repelling ($\tilde{\lambda} > 0$) and attracting $\tilde{\lambda} < 0$ interactions. The number of particles is scaled with the quantity N_{GP} defined in eq. (8), which can be thought of as a number of particles, at which the nonlinear contribution to the single-particle energy $E(N)$ becomes comparable with the linear part $E(0)$ of the energy. The time decay of the droplet is purely exponential, if the initial number N_0 is small. At larger values of N_0 it starts deviating from this exponential behavior and at large N_0 in the escape rate becomes essentially more rapid for the repelling interaction (fig. 2 left) and slower for the attracting interaction (fig. 2 right). The larger N_0 the stronger the deviation, but in the long time limit the behavior of the decay curve tends to the exponential one in both cases.

The initial rate of the decay as a function of N_0 ,

$$-\frac{1}{N\Gamma_0} \left. \frac{dN}{dt} \right|_{t=0} = \sqrt{1 + \frac{N}{N_{GP}}} \exp \left[\pm \beta\alpha_0x_0 \frac{N}{N_{GP}} \right]. \quad (12)$$

is also plotted for various values of the parameter $\beta\alpha_0x_0$. The nonlinearity causes an increase of the escape rate with the increasing initial number of particles N_0 in the droplet for the repelling interaction (fig. 3 left), or a decrease of the rate for the attractive interaction (fig. 3 right). This trend is stronger for higher and wider barriers (large values of $\beta\alpha_0x_0$). The trend is mainly determined by the exponential function in eq. (12), corresponding to the variation of the density of particles at the exit point x_{ex} . However, the square-root factor, reflecting the variation of the particle velocity with the shift of the exit point, counters this trend and can even reverse it, as is shown in fig. 3 right. The latter may, however, happen only for rather low and narrow barriers when $\beta\alpha_0x_0 < 1$.

We have demonstrated here how the hydrodynamic approach to the description of a multiparticle quantum system leads to a solution of the nonlinear problem of particle tunneling escape from a trap. Using the adiabatic approximation (neglect of v_t), justified for nonlinearities, which are not necessarily small, we obtained an analytical solution in a one-dimensional case for the time dependence of the number of trapped particles and for the tunneling rates at different nonlinearity values and signs. The technique, we use, goes beyond the WKB approximation. In particular, the divergence at the turning point of the trap potential, typical of WKB, does not appear here. That is why applying the same technique directly to

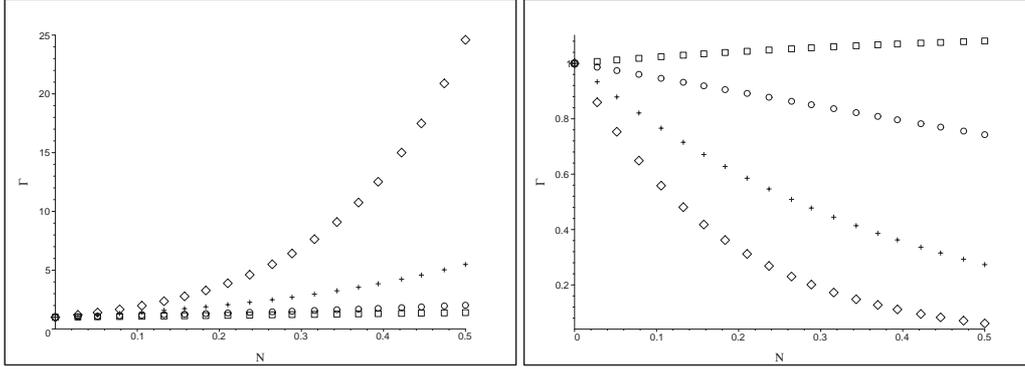


Fig. 3 – Initial rate Γ of the tunneling escape for the positive ($\tilde{\lambda} > 0$, left figure) and negative ($\tilde{\lambda} < 0$, right figure) nonlinearity in units of Γ_0 as a function of the initial number of particles N_0/N_{GP} for several choices of $\beta\alpha_0x_0$: $\beta\alpha_0x_0 = 6$ (diamonds); $\beta\alpha_0x_0 = 3$ (crosses); $\beta\alpha_0x_0 = 1$ (circles); $\beta\alpha_0x_0 = 0.25$ (squares).

three-dimensional systems seems to be straightforward. Contrary to the standard WKB approach, velocity is not necessarily a potential field (see, *e.g.*, the discussion in ref. [20]) and in three-dimensional cases vortices can be considered. Currently the influence of nonlinearity on the tunneling rate and time evolution of the system is being studied by solving the Gross-Pitaevskii equation numerically and a good correspondence to the results obtained in this paper is found [24].

These results can be verified experimentally by measuring tunneling escape rate at different values of the nonlinearity parameter $\tilde{\lambda}$. The Feshbach resonance [25, 26] provides us with a tool to tune the scattering length of ultracold atoms [5], and hence the interaction parameter $\tilde{\lambda}$, by an external magnetic field. At the fields near the resonance the interaction parameter $\tilde{\lambda}$ may become very large and change its sign so that the dynamics of the tunneling escape will be strongly influenced by the magnetic field.

Traps are typically characterized by the energy of a few μK and width of a few μm . The energy of a single atom bound by such a trap can be estimated as roughly $1 \mu\text{K}$ or 10^{-9} eV , meaning that the zero-point oscillation frequency is 10^3 s^{-1} (see, *e.g.* refs. [11, 14]). The tunneling rate in the absence of interaction will be, say, from several times to one or two orders of magnitude smaller, *i.e.* $\Gamma_0 \sim 10$ to 300 s^{-1} . The nonlinear GP corrections to the particle energies are about $0.1 \mu\text{K}$ or less for typical densities of ultracold gases $n \sim 10^{18} \text{ m}^{-3}$ for which the interaction parameter $na^3 \sim 10^{-6}$, however, even now the Feshbach resonance allows one to increase this correction by close to an order of magnitude [27].

The important dimensionless parameter in our one-dimensional model is the ratio N/N_{GP} . In a real three-dimensional experimental situation the relevant parameter will be the ratio of the actual density of particles n to the density n_{GP} at which the GP correction becomes comparable to the binding energy of an atom in the trap. Far from the Feshbach resonance n_{GP} exceeds the typical densities $n \sim 10^{18} \text{ m}^{-3}$ by one or two orders of magnitude which can make the nonlinear corrections rather small. However, approaching the Feshbach resonance by varying magnetic field we can make the ratio n/n_{GP} approaching unity. Then an observation of a dependence of the tunneling rate on the density of atoms within the trap or of a non-exponential time dependence of the density of atoms will be a direct indication of nonlinear effects in tunneling.

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