Dispersion and Strichartz estimates for some finite rank perturbations of the Laplace operator

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Abstract

We consider the dispersion properties in L^p spaces of Schrödinger hamiltonians with a large number of obstacles modelled by rank one perturbations. We obtain both for the dispersion an Strichartz estimates non perturbative results with respect to the coupling constants.

1 Introduction

It is well known that the free Schrödinger group in $\mathbb{R}^d$ satisfies the dispersion estimate

$$\left\|e^{it\Delta}\right\|_{\mathcal{L}(L^p;L^{p'})} \leq C_p t^{-d(\frac{1}{p}-\frac{1}{2})} \quad \text{for } 1 \leq p \leq 2,\ \frac{1}{p}+\frac{1}{p'}=1.$$

The Strichartz estimates

$$\left\|e^{it\Delta}f\right\|_{L^q_t L^r_x} \leq C_q \left\|f\right\|_{L^2} \quad \text{for } 2 \leq q \leq \infty,\ \frac{d}{r}+\frac{2}{q}=\frac{d}{2},\ d \geq 3$$

can be viewed as a consequence of this (see [7] [9] and for the initial approach [14]). Note that a local in time Strichartz estimate can hold while the dispersion estimate fails as shows the analysis on riemannian manifold ([3][2], [4]). Motivated by nonlinear problems, many efforts have been made to extend the dispersion or the Strichartz estimates to the perturbed case $H = -\Delta + V(x)$. Two different approaches were developed to attack this problem which mixes harmonic analysis and spectral theory: 1) a time dependent approach developed by the second author with J.L. Journé and C.D. Sogge in [8] 2) a stationary one developed by K. Yajima in [16][17][18] which consists in showing the L^p-boundedness of the wave operators and reduces the perturbed case to the free one. Recently, I. Rodnianski and W. Schlag in [11] have obtained results in dimension 3 which improve the previous ones and also hold for time-dependent potentials, with a method which is close to the first one. In these two approaches the analysis is crucially dimension-dependent in two points: a) an obvious one which can be summarized as the dimension dependence of Sobolev embeddings; b) The analysis of low energies and especially the influence of zero resonances and eigenvalues which requires Jensen-Kato theory and the expression of the Green functions for $-\Delta - k^2$.

These two approaches also require essentially the same type of assumptions on the perturbation V: a) it is local i.e. $V = V(x)$ is a multiplication operator; b) it has to decay rather fast, as x goes to ∞.

The first assumption is used in some cancellation property for high frequencies which appears in different forms in the two approaches. There is no doubt that this should work also for some pseudo-differential perturbation but nothing is written on this subject.

The best results concerned with the second assumptions are the recent ones of I. Rodnianski and W. Schlag in [11]. An aim of this article is to show that whatever the improvement could be made in this direction, the theory would remain incomplete. Another motivation is concerned with the analysis of ballistic transport in random media.

The situation is the following: Consider $H = -\Delta + V(x-x_1) + V(x-x_2)$ where V is a fast decaying potential with all the necessary assumptions. The physical intuition about this hamiltonian is that as $|x_2 - x_1|$ goes to infinity the two potentials are decoupled and that the properties of the propagator e^{-itH} should be the same as for e^{-itH_k} with $H_k = -\Delta + V(x - x_k)$, $k = 1, 2$. Contrary to what would suggest any weak decay assumption the situation is better and better as $|x_2 - x_1|$ is larger and larger.

Notations:

- For $\delta \in \mathbb{R}$, $[\delta]$ and $\delta+0$ respectively denote the integer part of δ and any real number greater than δ.
- For $1 \leq p \leq \infty$, p' denotes the dual exponent given by $\frac{1}{p} + \frac{1}{p'} = 1$.
- For $y \in \mathbb{R}^d$, τ_y is the x-translation: $\tau_y\varphi(x) = \varphi(x-y)$ and $\langle y\rangle = (1 + |y|^2)^{1/2}$.
- We use the notation D or D_x for $\frac{1}{i}\partial_x$ on $\mathbb{R}^d$ and the Fourier transform is normalized as

$$\hat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^d} e^{-i\xi.x}\varphi(x)\ dx$$

$$\varphi(x) = \left(\mathcal{F}^{-1}\hat{\varphi}\right)(x) = \int_{\mathbb{R}^d} e^{ix.\xi}\hat{\varphi}(\xi)\ d\!\!\!{}^-\xi, \quad (d\!\!\!{}^-\xi = \frac{d\xi}{(2\pi)^d}).$$

We shall consider a situation where the spectral analysis can be carried over as explicitly as possible, namely the case of finite rank perturbations and more precisely the case where each obstacle is described by a rank one perturbation. Namely, we shall study the dispersion for hamiltonians of the form

$$H = H_0 + \sum_{j=0}^{N} \alpha_j \tau_{x_j} |\psi\rangle\langle\psi| \tau_{-x_j} = H_0 + \sum_{j=0}^{N} \alpha_j |\tau_{x_j}\psi\rangle\langle\tau_{x_j}\psi| \quad \text{with } H_0 = -\Delta,$$

where the function ψ and the distribution of obstacles satisfy the following assumptions.
Hypotheses:

0) Dimension: $d \geq 3$.

1) Decay and smoothness: The function ψ is a normalized $L^2\left(\mathbb{R}^d\right)$ function such that $\langle x\rangle^s \langle D\rangle^\sigma \psi \in L^2\left(\mathbb{R}^d\right)$ where $s > 1/2$ and $\sigma \geq 0$ will be specified for every intermediate result.

2) Absence of pure point spectrum: The coefficients α_j are all positive and the Fourier transform $\hat{\psi}$ satisfies:

$$\forall \lambda \geq 0, \left(\int_{S^{d-1}} \left| \hat{\psi}(\sqrt{\lambda}\omega) \right|^2 d\omega = 0 \right) \Rightarrow \left(\int_{\mathbb{R}^d} (|\xi|^2 - \lambda)^{-1} \left| \hat{\psi}(\xi) \right|^2 d\xi \geq 0 \right).$$

We will write $\alpha = \max_{j \in \{0,\ldots,N\}} \alpha_j$.

3) Spreading of obstacles: There exists $\varepsilon > 0$ so that

$$\forall i, j \in \{0, \ldots, N\}, i \neq j, |x_j - x_i| \geq \frac{1}{\varepsilon}.$$

Here are our results.

Theorem 1.1. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma > \frac{d}{2}$, $\alpha = \max_{0 \leq j \leq N} \alpha_j$ *fixed. For* $1 < p \leq 2$, $1 < r < \min\left(p, \frac{2d}{d+2}\right)$ *and for* $\frac{(1/p-1/2)}{(1/r-1/2)}) < \theta \leq 1$, *there exist two constant* $C = C_{p,r,\theta,\alpha,\psi} > 0$ *and* $C' = C'_{p,r,\theta,\alpha,\psi} > 0$ *so that*

$$\left(N \leq \frac{1}{C\varepsilon^{d(r-1)}} \right) \Rightarrow \left(\forall t \in \mathbb{R} \setminus \{0\}, \left\| e^{-itH} \right\|_{\mathcal{L}\left(L^p, L^{p'}\right)} \leq C'(N+1)^\theta t^{-d(\frac{1}{p}-\frac{1}{2})} \right).$$

Theorem 1.2. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma > \frac{d}{2}$, $\alpha = \max_{0 \leq j \leq N} \alpha_j$ *fixed. For* $1 < p \leq 2$, $1 < r < \min\left(p, \frac{2d}{d+2}\right)$ *and for* $s' > \frac{d}{2}$, *there exist two constants* $C = C_{p,r,\alpha,\psi} > 0$ *and* $C' = C'_{s',p,r,\alpha,\psi} > 0$ *so that*

$$\forall t \in \mathbb{R} \setminus \{0\}, \forall u \in \langle x\rangle^{-s'} L^2(\mathbb{R}^d), \quad \left\| e^{-itH} u \right\|_{L^{p'}} \leq C' \min_{x_0 \in \mathbb{R}^d} \left\| \langle x - x_0\rangle^{s'} u \right\|_{L^2} t^{-d(\frac{1}{p}-\frac{1}{2})},$$

as soon as $N \leq \frac{1}{C\varepsilon^{d(r-1)}}$.

For the Strichartz estimate, we have

Theorem 1.3. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma > \max\left(\frac{d-3}{2}, \frac{d-2}{4}\right)$ *and* $\alpha = \max_{0 \leq j \leq N} \alpha_j$ *fixed. There exist a constant* $C = C_{\alpha,\psi} > 0$ *and for any* $q \in [2, \infty]$ *a constant* $C' = C'_{q,\alpha,\psi}$ *so that the Strichartz estimate*

$$\forall u \in L^2(\mathbb{R}^d), \quad \left\| e^{itH} u \right\|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}^d_x))} \leq C'(N+1)^{\frac{2}{q}} \|u\|_{L^2} \qquad \text{with} \quad \frac{d}{r} + \frac{2}{q} = \frac{d}{2},$$

holds as soon as $N \leq \frac{1}{C\varepsilon^{d\frac{d-2}{d+2}}}$.

Theorem 1.4. *We assume Hypotheses 0)1)2)3) with $s > \left[\frac{d}{2}\right] + 2$, $\sigma \geq \frac{d-2}{2}$ and $\alpha = \max_{0\leq j\leq N} \alpha_j$ fixed. There exist a constant $C = C_{\alpha,\psi} > 0$ and for any $s' > \frac{d}{2}$ a constant $C' = C'_{s',\alpha,\psi}$ so that the estimate*

$$\left(N \leq \frac{1}{C\varepsilon^{d\frac{d-2}{d+2}}}\right) \Rightarrow \left(\sup_{x_0\in\mathbb{R}^d} \left\| \langle x - x_0\rangle^{-s'} e^{-itH} \right\|_{\mathcal{L}\left(L^2(\mathbb{R}^d);L^2(\mathbb{R}^{d+1})\right)} \leq C'\right).$$

Those results are non perturbative in terms of the coupling constants α_j. They can be read in two ways: 1) For a fixed finite number N, uniform dispersion and Strichartz estimates hold as $\varepsilon \to 0$. 2) For $\varepsilon > 0$ it provides a sufficient conditions on N for the dispersion and Strichartz estimates. Notice also that the Strichartz estimate holds for larger N than what we are able to prove for the dispersion estimate. If these results are optimal (which is suspected), they cannot be derived directly with a stationary approach which would give the same condition on N for all the estimates.
The proof will be done in two steps: In Section 2 we will consider the case of one obstacle and show that in the rank one case the wave operators are bounded in L^p (stationary approach). The second one (Section 3) uses a bootstrap argument (time-dependent approach) and induction on N.

2 One obstacle

Rank one perturbations are known in spectral theory as basic perturbations for which everything can be computed explicitly. It is not only a toy model: First, trace class perturbations can be approximated by finite rank ones and the invariance principle for wave operators allows to reduce (very) short-range perturbations to this case. Secondly, any anti-Wick quantized operator is defined as a superposition of rank one perturbations. Surprisingly, nothing seems to have been written on the dispersive properties of rank one perturbed laplacian. One exception is the work of S.Albeverio, Z. Brzezniak and L. Dabrowski [1] where the kernel of the propagator was explicitly computed for point interaction potentials. In this special situation of local rank one potentials, the L^∞ norm decays like $\frac{1}{t^{1/2}}$ in dimension one (delta potentials) and like $\frac{1}{t^{1/2}}$ or $\frac{1}{t^{3/2}}$ in dimension three depending on the presence or not of zero resonances (see formula 16 of [1]).

We shall study the question of dispersion for regular rank one perturbation by proving that the wave operators are bounded in L^p spaces. We shall follow the techniques of K. Yajima in [16] [17] [18] and this paragraph can be viewed as a simple introduction to his very complete work. Here the case of local perturbations $V(x)$ will be an intermediate step (which seems necessary).

Let H_α denote the hamiltonian $H_0 + \alpha|\psi\rangle\langle\psi|$ on $\mathbb{R}^d$ with $H_0 = -\Delta$ and d, α, ψ according to Hypotheses 0) 1) 2).
We first recall the Aronszajn-Krein formulas which can be found in [12]: Let $F(z)$ denote the holomorphic function of $z \in \mathbb{C} \setminus \mathbb{R}_+$ given by

$$F(z) = \langle\psi|(H_0 - z)^{-1}|\psi\rangle. \tag{2.1}$$

With the decay assumptions on ψ, the boundary values

$$F_\pm(\lambda) = F(\lambda \pm i0) = \langle\psi|(H_0 - \lambda \mp i0)^{-1}|\psi\rangle, \quad \lambda \in \mathbb{R} \tag{2.2}$$

are everywhere defined functions and coincide on $(-\infty, 0)$. If $F_\alpha(z) = \langle\psi|(H_\alpha - z)^{-1}|\psi\rangle$, we deduce from the second resolvent formula, the relations for $z \in \mathbb{C} \setminus \mathbb{R}^+$:

$$F_\alpha(z) = \frac{F(z)}{1 + \alpha F(z)}, \tag{2.3}$$

$$(H_\alpha - z)^{-1} |\psi\rangle = \frac{1}{1 + \alpha F(z)} (H_0 - z)^{-1}|\psi\rangle \tag{2.4}$$

and $$(H_\alpha - z)^{-1} = (H_0 - z)^{-1} - \frac{\alpha}{1 + \alpha F(z)} (H_0 - z)^{-1} |\psi\rangle\langle\psi| (H_0 - z)^{-1}. \tag{2.5}$$

From the stationary expression of the wave operators $W_\pm = W_\pm(H_0 + V, H_0)$:

$$W_\pm = \frac{1}{2i\pi} \int_{\mathbb{R}} \left[\mathrm{Id} + (H_0 - \lambda \pm i0)^{-1} V\right]^{-1} \left[(H_0 - \lambda - i0)^{-1} - (H_0 - \lambda + i0)^{-1}\right] d\lambda$$

applied with $V = \alpha|\psi\rangle\langle\psi|$ and relation (2.5) we get the explicit expression for the wave operators $W_\pm(H_\alpha, H_0)$:

$$W_\pm(H_\alpha, H_0) = \mathrm{Id} - \frac{1}{2i\pi} \int_{\mathbb{R}^+} \frac{\alpha}{1 + \alpha F_\mp(\lambda)} (H_0 - \lambda \pm i0)^{-1} |\psi\rangle\langle\psi| \left[(H_0 - \lambda - i0)^{-1} - (H_0 - \lambda + i0)^{-1}\right] d\lambda. \tag{2.6}$$

Theorem 2.1. *Under the Hypotheses 0) 1) and 2) with* $s > \left[\frac{d}{2}\right] + 2$ *and* $\sigma \geq \max\left(\frac{d-3}{2}, \frac{1}{2}\left[\frac{d}{2}\right]\right)$, *the hamiltonian* H_α *has only absolute continuous spectrum and the wave operators are bounded in* $L^p(\mathbb{R}^d)$, *for* $1 < p < \infty$:

$$\|W_\pm(H_\alpha, H_0)\|_{\mathcal{L}(L^p)} \leq C_{p,\psi,\alpha}.$$

Proof. We assumed $\alpha \geq 0$ so that $\sigma(H_\alpha) = \mathbb{R}_+$. Moreover regular ($\psi \in L^2(\mathbb{R})$) rank one perturbations exclude singular continuous spectrum and ensure the existence of the wave operators $W_\pm(H_\alpha, H_0)$. With hypothesis 2) we will show in Lemma 2.6 that the function $1 + \alpha F_\pm(\lambda)$ never vanishes so that H_α has no embedded eigenvalue.
For simplicity of notations, we focus on W_+ (the treatment of W_- is symmetric). We now write the stationary formula (2.6) in the form:

$$W_+(H_\alpha, H_0) - \mathrm{Id} = G_{|\psi\rangle\langle\psi|} \circ \frac{\alpha}{1 + \alpha F_-(H_0)},$$

where the operator G_V equals

$$G_V = \frac{-1}{2i\pi} \int_{\mathbb{R}^+} (H_0 - \lambda + i0)^{-1} V \left[(H_0 - \lambda - i0)^{-1} - (H_0 - \lambda + i0)^{-1}\right] d\lambda$$

and after changing the integration contour

$$G_V = \frac{-1}{2i\pi}\int_{\mathbb{R}} (H_0 - \lambda + i0)^{-1} V (H_0 - \lambda - i0)^{-1} d\lambda. \tag{2.7}$$

Hence the problem is reduced to: 1) the L^p-boundedness of the Fourier multiplier by $\frac{1}{1+\alpha F_-(|\xi|^2)}$ (Proposition 2.8); 2) the L^p-boundedness of $G_{|\psi\rangle\langle\psi|}$ (Proposition 2.4 and Proposition 2.5). □

The previous result and the intertwinning relation

$$e^{-itH_\alpha} = W_+(H_\alpha, H_0) e^{it\Delta} W_+(H_\alpha, H_0)^*,$$

yield the estimates for the perturbed hamiltonian. Note that the maximum value for r in the Strichartz estimates is $\frac{2d}{d-2} < \infty$ for $d \geq 3$.

Corollary 2.2. *The dispersion and Strichartz estimates hold for H_α:*

$$\left\| e^{-itH_\alpha} \right\|_{\mathcal{L}(L^p;L^{p'})} \leq C_{p,\alpha,\psi} t^{-d\left(\frac{1}{p}-\frac{1}{2}\right)} \quad \textit{for } 1 < p \leq 2, \tag{2.8}$$

$$\textit{and} \quad \left\| e^{-itH_\alpha} f \right\|_{L^q_t L^r_x} \leq C_{q,\alpha,\psi} \left\| f \right\|_{L^2} \quad \textit{for } 2 \leq q \leq \infty,\ \frac{d}{r} + \frac{2}{q} = \frac{d}{2}\ (d \geq 3). \tag{2.9}$$

We will also need the regular dispersion estimate which writes for H_0

$$\left\| e^{-itH_0} u \right\|_{L^{p'}} \leq C_{p,s} \left\| \langle D\rangle^\sigma \langle x\rangle^s u \right\|_{L^2} \langle t\rangle^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}, \quad 1 \leq p \leq 2,\ s, \sigma > \frac{d}{2}.$$

By noticing that $(1+H_\alpha)^{\pm\left[\frac{\sigma}{2}\right]\pm 1}(1+H_0)^{\mp\left[\frac{\sigma}{2}\right]\mp 1}$ is bounded on $L^2(\mathbb{R}^d)$ for $\langle D\rangle^\sigma \psi \in L^2(\mathbb{R}^d)$ we get the

Corollary 2.3. *We assume the Hypotheses 0)1)2) with $s > \left[\frac{d}{2}\right] + 2$ and $\sigma > \frac{d}{2}$. For $s', \sigma' > \frac{d}{2}$, there exist a constant $C_{s',\sigma',p,\alpha,\psi}$ so that*

$$\forall t \in \mathbb{R},\ \left\| e^{-itH_\alpha} u \right\|_{L^{p'}} \leq C_{s',\sigma',p,\alpha,\psi} \left\| \langle D\rangle^{\sigma'} \langle x\rangle^{s'} u \right\|_{L^2} \langle t\rangle^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}, \quad 1 < p \leq 2.$$

We next give the details of the proof of Theorem 2.1.

2.1 L^p-boundedness of $G_{|\psi\rangle\langle\psi|}$.

This part on the L^p-boundedness of the operator G_V essentially relies on results by Yajima in [16][17][18]. The case where $V = V(x)$ is treated in the Proposition 2.13 of [17] which we recall here:

Proposition 2.4. *For $V = V(x)$ with $\langle D\rangle^{\frac{d-3}{2}} \langle x\rangle^{1/2+0} V \in L^2(\mathbb{R}^d)$, we set for $t \in \mathbb{R}$ and $\omega \in \mathbb{S}^{d-1}$*

$$K_V(t,\omega) = \frac{i}{2(2\pi)^d} \int_0^\infty \hat{V}(r\omega) r^{d-2} e^{itr/2}\, dr$$

and $x_\omega = x - 2(x.\omega)\omega$ denotes the reflection along the ω-axis of $x \in \mathbb{R}^d$. Then:

1) *The operator G_V given by (2.7) can be expressed as follows:*

$$(G_V u)(x) = \int_{\mathbb{S}^{d-1}} \int_{2x.\omega}^{+\infty} K_V(t,\omega) u(t\omega + x_\omega)\, dt\, d\omega. \tag{2.10}$$

2) *For any p, $1 \le p \le \infty$, the operator G_V is bounded on $L^p(\mathbb{R}^d)$ and we have for $\sigma > 1/2$:*

$$\|G_V\|_{\mathcal{L}(L^p)} \le \|K_V\|_{L^1\left(\mathbb{R}\times\mathbb{S}^{d-1}\right)} \le \left\| \langle D \rangle^{\frac{d-3}{2}} \langle x \rangle^{\sigma} V \right\|_{L^2}.$$

The translation invariance of the Laplace operator, allows to reduce the case $V = |\psi\rangle\langle\psi|$ to the previous one and we have the

Proposition 2.5. *Under Hypotheses 0),1),2) with $s > \frac{d+1}{2}$ and $\sigma \ge \frac{d-3}{2}$, the operator $G_{|\psi\rangle\langle\psi|}$ is bounded in $L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$:*

$$\left\|G_{|\psi\rangle\langle\psi|}\right\|_{\mathcal{L}(L^p)} \le C_{p,s} \left\| \langle D \rangle^{\frac{d-3}{2}} \langle x \rangle^{s} \psi \right\|_{L^2}^2.$$

Proof. We again follow Yajima in [18]-Lemma 4.4. For $u \in S(\mathbb{R}^d)$ and for $\lambda \in \mathbb{R}$, we have

$$\left(|\psi\rangle\langle\psi|(H_0 - \lambda - i0)^{-1}|u\rangle\right)(x) = \psi(x) \int_{\mathbb{R}^d} \overline{\psi(y)} \left[(H_0 - \lambda - i0)^{-1} u\right](y)\, dy.$$

The change of variable $y \to x - y$ in the integral and the translation invariance of $(H_0 - \lambda - i0)^{-1}$ gives

$$\left(|\psi\rangle\langle\psi|(H_0 - \lambda - i0)^{-1}|u\rangle\right)(x) = \int_{\mathbb{R}^d} V_y(x) \left[(H_0 - \lambda - i0)^{-1} \tau_y u\right](x)\, dy$$

where $V_y(x) = \psi(x)\overline{\psi}(x-y)$. Integrating with respect to $\lambda \in \mathbb{R}$ leads to

$$G_{|\psi\rangle\langle\psi|} u = \int_{\mathbb{R}^d} G_{V_y} \tau_y u\, dy \quad \text{in } S'(\mathbb{R}^d)$$

and to the estimate (the case $p = \infty$ follows by duality)

$$\|G_V\|_{\mathcal{L}(L^p)} \le \int_{\mathbb{R}^d} \left\|G_{V_y}\right\|_{\mathcal{L}(L^p)} dy \le \int_{\mathbb{R}^d} \left\| \langle D \rangle^{\frac{d-3}{2}} \langle x \rangle^{s_1} \psi(x) \overline{\psi}(x-y) \right\|_{L^2_x} dy, \quad (s_1 > 1/2).$$

With $\frac{\langle y \rangle^s}{\langle x \rangle^s} \le C_s \langle x - y \rangle^s$, $s \ge 0$, we have

$$\int_{\mathbb{R}^{2d}} \langle y \rangle^{d+2(s_2-s_1)} \langle x \rangle^{2s_1} |\psi(x)|^2 |\psi(x-y)|^2\, dxdy \le C \left\| \langle x \rangle^{\frac{d}{2}+s_2} \psi \right\|_{L^2}^4 \quad (s_2 > s_1 > 1/2).$$

With Cauchy-Schwarz, we finally get

$$\int_{\mathbb{R}^d} \left\| \langle x \rangle^{s_1} \psi(x) \overline{\psi}(x-y) \right\|_{L^2_x} dy, \le C_{s_2} \left\| \langle x \rangle^{\frac{d}{2}+s_2} \psi \right\|_{L^2}^2 \quad (s_2 > s_1 > 1/2).$$

The treatment of integer derivatives is similar and the final result is obtained via bilinear complex interpolation (see [15] [5]). □

2.2 The Fourier multiplier $\frac{1}{1+\alpha F_-(|\xi|^2)}$.

We first check what we announced in the proof of Theorem 2.1, namely the absolute continuity of the spectrum of H_α.

Lemma 2.6. *Under Hypotheses 0)1)2) with $s > 3/2$ and $\sigma \geq 0$, the function $1 + \alpha F_-(\lambda)$ is continuous on $\mathbb{R} \cup \{\infty\}$ and never vanishes. As a consequence, the spectrum of H_α is absolutely continuous.*

Proof. We first note that F_- is the Fourier transform of the time-dependent function $-i.1_{\mathbb{R}^+}(t)\langle\psi|e^{itH_0}|\psi\rangle$,

$$F_-(\lambda) = \langle\psi|(H_0 - \lambda + i0)^{-1}|\psi\rangle = -i\int_0^{+\infty} e^{-it\lambda}\langle\psi|e^{itH_0}|\psi\rangle\ dt.$$

If $\langle x\rangle^{1+0}\psi$ belongs to $L^2(\mathbb{R}^d)$, then ψ belongs to $L^p(\mathbb{R}^d)$ with $\frac{1}{p} > \frac{1}{d} + \frac{1}{2}$. As a consequence of the dispersion estimate for H_0, the function $1_{\mathbb{R}^+}(t)\langle\psi|e^{itH_0}|\psi\rangle$ belongs to $L^1(\mathbb{R}_t)$ and its Fourier transform F_- is continuous on $\mathbb{R}$ and vanishes at infinity.

It remains to check that $1 + \alpha F_-(\lambda)$ never vanishes.

a) $\lambda \leq 0$: For $\lambda < 0$, the real part of $F_-(\lambda)$ equals

$$\mathbb{R}\mathrm{e}(F_-(\lambda)) = \int_{\mathbb{R}^d} \frac{1}{|\xi|^2 - \lambda}\left|\hat{\psi}(\xi)\right|^2\ d\!\!\!{}^-\xi > 0.$$

Thus we have

$$\forall\lambda \in (-\infty, 0],\ \mathbb{R}\mathrm{e}(1 + \alpha F_-(\lambda)) \geq 1.$$

b) $\lambda > 0$: The trace theorem with $\langle x\rangle^{3/2+0}\psi \in L^2(\mathbb{R}^d)$ ensures that $\hat{\psi}(\sqrt{\lambda}.)$ is a $L^2(\mathbb{S}^{d-1})$-valued $\mathcal{C}^1$ function of $\lambda \in (0, +\infty)$. Hence, whenever the imaginary part of $F_-(\lambda)$ vanishes,

$$\mathcal{I}\mathrm{m}(F_-(\lambda)) = \frac{\lambda^{\frac{d-1}{2}}}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\left|\hat{\psi}(\sqrt{\lambda}\omega)\right|^2\ d\omega = 0,$$

then its real part equals

$$\mathbb{R}\mathrm{e}(F_-(\lambda)) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\frac{(|\xi|^2 - \lambda)}{(|\xi|^2 - \lambda)^2 + \varepsilon^2}\left|\hat{\psi}(\xi)\right|^2\ d\!\!\!{}^-\xi = \int_{\mathbb{R}^d}\frac{1}{(|\xi|^2 - \lambda)}\left|\hat{\psi}(\xi)\right|^2\ d\!\!\!{}^-\xi.$$

The right-hand side is positive in such a case by Hypothesis 2) and therefore $1 + \alpha F_-(\lambda)$ never vanishes on $(0, +\infty)$.

The absolute continuity of the spectrum of H_α now follows from the fact that the boundary values of the resolvent are locally uniformly bounded in weighted L^2 spaces which excludes the presence of embedded eigenvalues (see [6][10]). □

Remark 2.7. *The condition of Hypothesis 2) allows low energy cut-off but not high-energy cut-off. As an example, if* $\psi = \chi(H_0)\psi$ *for some compactly supported function* χ*, then for* E *larger than any* $\lambda \in \mathrm{supp}(\chi)$*, one can find* $u \in L^2(\mathbb{R}^d)$ *so that* $(H_0 - E)|u\rangle = |\psi\rangle$*. Then we have*

$$H_\alpha |u\rangle = E|u\rangle + (1 + \alpha\langle\psi|u\rangle)\,|\psi\rangle$$
$$\text{with} \qquad \langle\psi|u\rangle = \int_{\mathbb{R}^d} (|\xi|^2 - E)^{-1} \left|\hat{\psi}(\xi)\right|^2 \, d\!\!{}^-\xi < 0$$

and H_α *has the embedded eigenvalue* E *for* $\alpha = \frac{-1}{\langle\psi|u\rangle}$*.*

We recall the Marcinkiewicz Fourier multiplier theorem (see [13][15]) which says that $m(D)$ is bounded in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, provided that the function m is $\left[\frac{d}{2}\right] + 1$-times continuously differentiable on $\mathbb{R}^d \setminus \{0\}$ and that the derivatives satisfy the uniform estimates

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}\,, \ \left|\partial_\xi^\beta m(\xi)\right| \le C_\beta \,|\xi|^{-|\beta|}\,, \qquad \text{for } 0 \le |\beta| \le \left[\frac{d}{2}\right] + 1.$$

For a function $m(\xi) = g(|\xi|^2)$, it becomes $g \in \mathcal{C}^{\left[\frac{d}{2}\right]+1}((0, +\infty))$ and

$$\forall \lambda \in (0, +\infty), \ \left|(\lambda\partial_\lambda)^k g(\lambda)\right| \le C_k, \qquad \text{for } 0 \le k \le \left[\frac{d}{2}\right] + 1.$$

We will prove the

Proposition 2.8. *Under Hypotheses 0)1)2) with* $s > \left[\frac{d}{2}\right] + 2$ *and* $\sigma \ge \frac{1}{2}\left[\frac{d}{2}\right]$*, the function* F_- *is* $\left[\frac{d}{2}\right] + 1$ *times continuously differentiable on* $\mathbb{R}^*$ *with the estimate*

$$\forall \lambda \in \mathbb{R}^*, \ \left|(\lambda\partial_\lambda)^k F_-(\lambda)\right| \le C_{s,k} \left\|\langle D\rangle^{\frac{1}{2}\left[\frac{d}{2}\right]} \langle x\rangle^s \psi\right\|_{L^2}^2, \quad \textit{for } 0 \le k \le \left[\frac{d}{2}\right] + 1.$$

Hence, the operator $\frac{1}{1+\alpha F_-(H_0)}$ *is bounded on* $L^p(\mathbb{R}^d)$ *for* $1 < p < \infty$*.*

Proof. The last assertion is a direct consequence of the first one with the non vanishing of $1 + \alpha F_-$ on $\mathbb{R} \cup \{\infty\}$. After taking the inverse Fourier transform, it suffices to check for $0 \le k \le \left[\frac{d}{2}\right] + 1$ the estimates

$$\left\|(\partial_t t)^k \langle\psi|e^{itH_0}|\psi\rangle\right\|_{L^1(\mathbb{R})} \le C_{s,k} \left\|\langle D\rangle^{\frac{1}{2}\left[\frac{d}{2}\right]} \langle x\rangle^s \psi\right\|_{L^2}^2, \quad (s > \left[\frac{d}{2}\right] + 2).$$

with $\psi \in S(\mathbb{R}^d)$.

Let $1 = \chi_0^3(\lambda) + \sum_{j=1}^\infty \chi^3(2^{-j}\lambda)$ be a dyadic partition of unity with $\chi_0 \in \mathcal{C}_0^\infty([0, 2))$ and $\chi \in \mathcal{C}_0^\infty((1/2, 3/2))$. We write

$$\langle\psi|e^{itH_0}|\psi\rangle = \langle\psi|e^{itH_0}\chi_0^3(|D|)|\psi\rangle + \sum_{j=1}^{+\infty} \langle\psi|e^{itH_0}\chi^3(2^{-j}\,|D|)|\psi\rangle$$

Thus, we have to consider two kinds of terms

$$\begin{aligned} & I_0(t) = \langle \psi_0 | e^{itH_0} \chi_0(|D|) | \psi_0 \rangle \\ \text{and for } j \geq 1 \quad & I_j(t) = \langle \psi_j | e^{itH_0} \chi(2^{-j} |D|) | \psi_j \rangle, \end{aligned}$$

where ψ_j equals $\chi_0(|D|)\psi$ and $\chi(2^{-j}|D|)\psi$ respectively for $j = 0$ and for $j \geq 1$.

We notice that for $\varphi \in S(\mathbb{R}^d)$ we have

$$\begin{aligned} (\partial_t t) \int_{\mathbb{R}^d} e^{-it|\xi|^2} \varphi(\xi)\, đ\xi &= \int_{\mathbb{R}^d} (-\frac{i}{2}\xi.\partial_\xi + 1) e^{-it|\xi|^2} \varphi(\xi)\, đ\xi \\ &= \int_{\mathbb{R}^d} e^{-it|\xi|^2} (\frac{i}{2}\xi.\partial_\xi + \frac{di}{2} + 1)\varphi(\xi)\, đ\xi. \end{aligned}$$

$I_j(t),\ j \geq 1$: For $0 \leq k \leq \left[\frac{d}{2}\right] + 1$, we write

$$(\partial_t t)^k I_j(t) = \int_{\mathbb{R}^d} e^{-it|\xi|^2} \left[\sum_{|\beta| \leq k} 2^{|\beta| j} \theta_{\beta,k}(2^{-j}\xi) \partial_\xi^\beta \left| \hat{\psi}_j(\xi) \right|^2 \right] đ\xi$$

where the functions $\theta_{\beta,k}$ belong to $\mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$. After using non stationary phase (integration by part with $\frac{\xi}{t|\xi|^2}\partial_\xi$) and interpolation we get

$$\forall t \in \mathbb{R}^*,\ \left|(\partial_t t)^k I_j(t)\right| \leq C_{k,s_1} 2^{jk} \min\left(1, \frac{1}{(2^j t)^{s_1}}\right) \|\langle x\rangle^{s_1} \psi_j\|_{L^2}^2, \quad \text{for } s_1 \geq 0.$$

After time integration, we deduce the estimate:

$$\left\|(\partial_t t)^k I_j(t)\right\|_{L^1(\mathbb{R})} \leq C_{k,s_2} 2^{j(k-1)} \left\|\langle x\rangle^{k+s_2} \psi_j\right\|_{L^2}^2 \quad \text{for } s_2 > 1. \tag{2.11}$$

$I_0(t)$: For $0 \leq k \leq \left[\frac{d}{2}\right] + 1$, we write

$$(\partial_t t)^k I_0(t) = \int_{\mathbb{R}^d} e^{-it|\xi|^2} \left[\sum_{|\beta| \leq k} \chi_{\beta,k}(\xi) \partial_\xi^\beta \left| \hat{\psi}(\xi) \right|^2 \right] đ\xi$$

where the functions $\chi_{\beta,k}$ belong to $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Since the operators $\chi_{\beta,k}(D)$ are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, we deduce like in the proof of Lemma 2.6 the estimates

$$\left\|(\partial_t t)^k I_0(t)\right\|_{L^1(\mathbb{R})} \leq C_{k,s} \left\|\langle x\rangle^{k+s} \psi\right\|_{L^2}^2 \quad \text{for } s > 1. \tag{2.12}$$

We obtain the result for $\psi \in S(\mathbb{R}^d)$ after taking the sum of all the terms (2.11)(2.12) and finally for all ψ by density. □

2.3 Two estimates for $\langle \tau_y \psi | e^{-itH_\alpha} | \psi \rangle$.

The results of this section will be used further. The next one can be viewed as an extension of Lemma 2.6 and Proposition 2.8.

Proposition 2.9. *We assume Hypotheses 0)1)2) with* $s > \left[\frac{d}{2}\right] + 2$ *and* $\sigma \geq \max\left(\frac{d-3}{2}, \frac{d}{4}\right)$. *For* $1 < r \leq 2$, *for* $0 \leq \theta \leq 1$ *and* $s' > \frac{d}{2}$, *there exists a constant* $C_{s',r,\theta,\alpha,\psi}$ *so that*

$$\forall y \in \mathbb{R}^d, \forall t \in \mathbb{R}, \ \left|\langle \tau_y u | e^{-itH_\alpha} | u \rangle\right| \leq C_{s',r,\theta,\alpha,\psi} \left\| \langle x \rangle^{s'} \langle D \rangle^{\frac{d}{4}} u \right\|_{L^2}^2 \left(\langle t \rangle^\theta \langle y \rangle^{1-\theta} \right)^{-d(\frac{1}{r}-\frac{1}{2})}.$$

Proof. For $s' > \frac{d}{2}$ fixed, we take u in $\langle x \rangle^{-s'} \langle D \rangle^{-\frac{d}{4}} L^2(\mathbb{R}^d)$. The Duhamel formula for H_α gives

$$\langle \tau_y u | e^{-itH_\alpha} | u \rangle = \langle \tau_y u | e^{-itH_0} | u \rangle - i\alpha \int_0^t \langle \tau_y u | e^{-i(t-t')H_0} | \psi \rangle \langle \psi | e^{-it'H_\alpha} | u \rangle \ dt'$$

Corollary 2.2 says that the dispersion estimate holds for H_α. Since our assumptions ensure that both ψ and u belong to $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we have

$$\left| \langle \psi | e^{-it'H_\alpha} | u \rangle \right| \leq C_{s',r,\alpha,\psi} \langle t \rangle^{-d(\frac{1}{r}-\frac{1}{2})} \left\| \langle x \rangle^{s'} u \right\|_{L^2} \quad \text{for } 1 < r \leq 2. \tag{2.13}$$

The next Lemma 2.10 states that the estimates hold with H_0 instead of H_α. We now take r close enough to 1 and the integrability of $\langle t' \rangle^{-d(\frac{1}{r}-\frac{1}{2})}$ provides the estimate for $\theta = 0$. Meanwhile the estimate $\int_0^t \langle t - t' \rangle^{-s_1} \langle t' \rangle^{-s_1} \ dt' \leq C_{s_1} \langle t \rangle^{-s_1}$, for $s_1 > 1$, gives the result for $\theta = 1$. The case of general $\theta \in [0, 1]$ and $r \in (1, 2]$ follows by interpolation. □

Lemma 2.10. *For* $s > d/2$, *for* $1 \leq r \leq 2$ *and for* $0 \leq \theta \leq 1$, *there exists a constant* $C_{r,\theta,s} > 0$ *so that*

$$\forall y \in \mathbb{R}^d, \forall t \in \mathbb{R} \ \left| \langle \tau_y u | e^{-itH_0} | u \rangle \right| \leq C_{r,\theta,s} \left\| \langle x \rangle^s \langle D \rangle^{\frac{d}{4}} u \right\|_{L^2}^2 \left(\langle t \rangle^\theta \langle y \rangle^{1-\theta} \right)^{-d(\frac{1}{r}-\frac{1}{2})}.$$

Proof. It relies on a combination of propagation estimates (given here by non stationary phase) and dispersion estimates. The result for bounded $y \in \mathbb{R}^d$ is a consequence of the dispersion estimate and we can assume $|y| \geq 1$. We introduce like in the proof of Proposition 2.8 the dyadic partition of unity on $\mathbb{R}$: $1 = \chi_0(\lambda)^3 + \sum_{j=1}^\infty \chi(2^{-j}\lambda)^3$ with $\chi_0 \in \mathcal{C}_0^\infty([0, 2))$ and $\chi \in \mathcal{C}_0^\infty((1/2, 3/2))$. Here the terms with the factors $\chi_0(\lambda)$ and $\chi(2^{-j}\lambda)$ are treated in the same way and we set for $j \geq 1$ $\chi_j(\lambda) = \chi(2^{-j}\lambda)$. We write for $t \in \mathbb{R}$

$$\langle \tau_y u | e^{-itH_0} | u \rangle = \sum_{j=0}^\infty a_j(t)$$

$$\text{with} \quad a_j(t) = \langle \tau_y u | e^{-itH_0} \chi_j(|D|)^3 | u >= \int_{\mathbb{R}^d} e^{it|\xi|^2} e^{-iy.\xi} \chi_j(|\xi|) \left| \hat{u_j}(\xi) \right|^2 \ d\!\!\!{}^-\xi$$

where $u_j = \chi_j(|D|)u$. We split the analysis of $a_j(t)$ in two regimes

$|t| \leq \frac{1}{10}2^{-j}|y|$**:** In this case the phase $\varphi(x,\xi,t) = -y.\xi + t|\xi|^2$ is not stationary and we use integration by part with

$$\frac{1}{1+|\partial_\xi \varphi|^2}\partial_\xi \varphi.\partial_\xi, \quad \partial_\xi \varphi = -y + 2t\xi.$$

On the support of χ_j we have

$$|\partial_\xi \varphi(x,\xi,t)| = |-y+2t\xi| \geq |y| - 2\,|t|\,|\xi| \geq |y|\left(1 - \frac{3}{10}\right) \geq \frac{|y|}{2}$$

and therefore

$$|\partial_\xi \left(\partial_\xi \varphi\right)| = |2t| \leq \frac{1}{5}2^{-j}|y| \leq |\partial_\xi \varphi|\,.$$

For any $k \in \mathbb{N}$ ($u \in S(\mathbb{R}^d)$), we get the estimate

$$\forall j \in \mathbb{N},\ |a_j(t)| \leq C_k \left\|\langle x\rangle^k u_j\right\|_{L^2}^2 |y|^{-k} \quad \text{for } |t| \leq \frac{1}{10}2^{-j}|y|$$

and after interpolation it holds for any $k \in \mathbb{R}^+$ and $\langle x\rangle^k u \in L^2(\mathbb{R}^d)$.

$|t| \geq \frac{1}{10}2^{-j}|y|$**:** We combine the dispersion estimate with the uniform boundedness of $\chi_j(|D|)$ on $L^r(\mathbb{R}^d)$, $1 \leq r \leq \infty$, and the inclusion $\langle x\rangle^{-s'}L^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ($s' > d/2$). We thus obtain for $1 \leq r \leq 2$ the estimate

$$\forall j \in \mathbb{N},\ |a_j(t)| \leq C_{r,s'} \left\|\langle x\rangle^{s'} u_j\right\|_{L^2}^2 |t|^{-d\left(\frac{1}{r}-\frac{1}{2}\right)}, \quad \text{for } |t| \geq \frac{1}{10}2^{-j}|y|.$$

This estimate implies for $0 \leq \theta \leq 1$:

$$\forall j \in \mathbb{N},\ |a_j(t)| \leq C_{r,\theta,s'}2^{jd/2} \left\|\langle x\rangle^{s'} u_j\right\|_{L^2}^2 \left(|t|^\theta\,|y|^{1-\theta}\right)^{-d\left(\frac{1}{r}-\frac{1}{2}\right)}, \quad \text{for } |t| \geq \frac{1}{10}2^{-j}|y|.$$

For $1 \leq r \leq 2$, $s' > d/2$ and $0 \leq \theta \leq 1$, we have found a constant $C_{r,\theta,s'}$ so that

$$\forall j \in \mathbb{N}, \forall t \in \mathbb{R},\ |a_j(t)| \leq C_{r,\theta,s'}2^{jd/2} \left\|\langle x\rangle^{s'} u_j\right\|_{L^2}^2 \left(\langle t\rangle^\theta\,|y|^{1-\theta}\right)^{-d\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Taking the sum with respect to $j \in \mathbb{N}$ yields the result. □

The next result will be used in the analysis of the Strichartz estimate.

Proposition 2.11. *We assume Hypotheses 0)1)2) with* $s > \left[\frac{d}{2}\right]+2$ *and* $\sigma \geq \max\left(\frac{d-3}{2}, \frac{d-2}{4}\right)$. *For* $s' > \frac{d}{2}$, *there exists a constant* $C_{s',\alpha,\psi}$ *so that*

$$\forall y \in \mathbb{R}^d,\ \left\|\langle \tau_y u| e^{-itH_\alpha} |u\rangle\right\|_{L^1_t(\mathbb{R})} \leq C_{s',\alpha,\psi} \left\|\langle x\rangle^{s'} \langle D\rangle^{\frac{d-2}{4}} u\right\|_{L^2}^2 |y|^{-\left(\frac{d}{2}-1\right)}.$$

Proof. Let u belong to $\langle x\rangle^{-s'}\langle D\rangle^{-\frac{d-2}{4}}L^2(\mathbb{R})$ with $s'>\frac{d}{2}$. Like in the proof of Proposition 2.9, our assumptions yield the estimate (2.13) and therefore

$$\left\|\langle\psi|e^{-itH_\alpha}|u\rangle\right\|_{L^1_t(\mathbb{R})}\leq C_{s',\alpha,\psi}\left\|\langle x\rangle^{s'}u\right\|_{L^2}.$$

Using again Duhamel formula the problem is reduced to the case $\alpha=0$ (bilinear version with (u,u) and (u,ψ)). Like in Lemma 2.10 we write

$$\langle\tau_y u|e^{-itH_\alpha}|u\rangle=\sum_{j=0}^{\infty}a_j(t)$$

where the terms $a_j(t)$ satisfy

$$|a_j(t)|\leq C_{s'}\left\|\langle x\rangle^{s'}u_j\right\|_{L^2}^2|y|^{-s'},\qquad \text{for } |t|\leq\frac{1}{10}2^{-j}|y|$$
$$\text{and}\qquad |a_j(t)|\leq C_{s'}\left\|\langle x\rangle^{s'}u_j\right\|_{L^2}^2 t^{-d/2}\qquad \text{for } |t|\geq\frac{1}{10}2^{-j}|y|.$$

After integration with respect to $t\in\mathbb{R}$ we get

$$\int_{\mathbb{R}}|a_j(t)|\leq C_{s'}2^{j(d/2-1)}|y|^{-(d/2-1)}\left\|\langle x\rangle^{s'}u_j\right\|_{L^2}^2.$$

We conclude by summing with respect to $j\in\mathbb{N}$. □

Remark 2.12. a) *Note that with the L^q norm in time, one can get the decay $\langle y\rangle^{-(\frac{d}{2}-\frac{1}{q})}$ with q-dependent regularity assumptions.*

b) *The results of Proposition 2.9, Lemma 2.10 and Proposition 2.11 are optimal: An explicit integration in the case $\alpha=0$ with the gaussian wave function $u=\frac{1}{\pi^{d/4}}e^{-\frac{x^2}{2}}$ gives*

$$\left|\langle\tau_y u|e^{-itH_0}u\rangle\right|=\frac{1}{\langle t\rangle^{d/2}}e^{-\frac{|y|^2}{\langle t\rangle^2}}.$$

3 N obstacles.

For the final analysis, it is convenient to change the numbering of obstacles. For a subset $\mathcal{K}$ of $\mathbb{Z}^d$ with $\#\mathcal{K}=N+1$ and a bijection $j:\mathcal{K}\to\{0,\dots,N\}$ we write

$$H=H_0+\sum_{k\in\mathcal{K}}|\psi_k><\psi_k|$$

with $\psi_k=\alpha_{j(k)}^{1/2}\tau_{x_{j(k)}}\psi$. Moreover with Hypothesis 3), this set $\mathcal{K}\subset\mathbb{Z}^d$ and the bijection $j:\mathcal{K}\to\{0,\dots,N\}$ can be chosen so that

$$\mathcal{K}\subset\mathbb{Z}^d\cap B(0,C_dN^{1/d})$$
$$\text{and}\qquad \forall k,k'\in\mathcal{K},\ \left|x_{j(k)}-x_{j(k')}\right|\geq\frac{1}{C_d\varepsilon}|k-k'|,$$

where the constant $C_d > 1$ only depends on the dimension d. For any subset $\mathcal{K}'$ of $\mathcal{K}$, the hamiltonian $H_{\mathcal{K}'}$ will be given by

$$H_{\mathcal{K}'} = H_0 + \sum_{k\in\mathcal{K}'} |\psi_k >< \psi_k|.$$

For $1 < p < \frac{2d}{d+2}$, we set $C_{d,p} = 2^{d\left(\frac{1}{p}-\frac{1}{2}\right)+1} \int_0^\infty \langle t\rangle^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}\, dt$.

3.1 Dispersion estimates

The bootstrap argument is performed in the next two Lemmas. The final proofs of Theorem 1.1 and Theorem 1.2 simply gathers all the estimates.

For $n \in \{0, \ldots, N-1\}$ and $t \in \mathbb{R}_+$ we introduce the quantity

$$S_n(t) = \sup_{\substack{\#\mathcal{K}' = n+1 \\ k_0 \in \mathcal{K}'}} \sum_{k\in\mathcal{K}\setminus\mathcal{K}'} \left|\langle\psi_k|e^{-itH_{\mathcal{K}'}}|\psi_{k_0}\rangle\right|. \tag{3.1}$$

Lemma 3.1. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma \geq \max\left(\frac{d-3}{2}, \frac{d}{4}\right)$ *and* $\alpha = \max_{k\in\mathcal{K}} \alpha_{j(k)}$ *fixed. Then for* $1 < r < p < \frac{2d}{d+2}$, *there exists a constant* $C = C_{p,r,\alpha,\psi} > 0$ *so that the estimate*

$$S_n(t) \leq 2C\varepsilon^{\frac{d}{r'}} N^{\frac{1}{r}} \langle t\rangle^{-d\left(\frac{1}{p}-\frac{1}{2}\right)},$$

holds uniformly in $t \in \mathbb{R}^+$, $n \in \{0, \ldots, N-1\}$, *as soon as*

$$N \leq \frac{1}{(4C_{d,p}C)^r \varepsilon^{d(r-1)}}.$$

Proof. Let p and r satisfy $1 < r < p < \frac{2d}{d+2}$. We study by induction on $n \in \{0, \ldots, N-1\}$ the boundedness of

$$C_n = \left\|\langle t\rangle^{d(\frac{1}{p}-\frac{1}{2})} S_n(t)\right\|_{L^\infty}.$$

$n = 0$ **:** In this case, $H_{\mathcal{K}'} = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$ and the result follows from Proposition 2.9. By replacing $1 < r < p$ by $1 < r_1 < p$ and taking $\theta = \frac{1/p-1/2}{1/r_1-1/2}$, it gives the estimate

$$\begin{aligned}\left\|\langle t\rangle^{d(\frac{1}{p}-\frac{1}{2})}\langle\psi_k|e^{-itH_{\mathcal{K}'}}|\psi_{k_0}\rangle\right\|_{L^\infty} &\leq C_{p,r_1,\alpha,\psi}\left|x_{j(k)} - x_{j(k_0)}\right|^{-d\left(\frac{1}{r_1}-\frac{1}{p}\right)} \\ &\leq C_{p,r_1,\alpha,\psi}\varepsilon^{d\left(\frac{1}{r_1}-\frac{1}{p}\right)}|k-k_0|^{-d\left(\frac{1}{r_1}-\frac{1}{p}\right)}.\end{aligned}$$

Then the sum with respect to $k \in \mathcal{K}\setminus\{k_0\}$ is estimated by

$$\left\|\sum_{k\neq k_0}\left|\langle\psi_k|e^{-itH_{\mathcal{K}'}}|\psi_{k_0}\rangle\right|\right\|_{L^\infty} \leq C_{p,r_1,\alpha,\psi}\varepsilon^{d\left(\frac{1}{r_1}-\frac{1}{p}\right)} N^{1-\left(\frac{1}{r_1}-\frac{1}{p}\right)}.$$

We take r so that $1/r = 1 + 1/p - 1/r_1$ (symmetry on the interval $(1/p, 1)$) and we obtain

$$C_0 = C_{p,r_1,\alpha,\psi}\varepsilon^{d/r'} N^{1/r}.$$

$n \geq 1$ **:** We assume that the constant C_m are known for $m < n$ and we take $\mathcal{K}' \subset \mathcal{K}$ with $\#\mathcal{K}' = n+1$ and $k_0 \in \mathcal{K}'$. The identity (A.3) of Lemma A.2 applied with $A_0 = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$ and $A_{\mathcal{K}'\setminus\{k_0\}} = H_{\mathcal{K}'}$ and Lemma A.1 yields

$$\langle t\rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)} \left\| \sum_{k\in\mathcal{K}\setminus\mathcal{K}'} \left|\langle\psi_k|e^{-itH_{\mathcal{K}'}}|\psi_{k_0}\rangle\right| \right\|_{L^\infty} \leq \sum_{m=0}^{n} C_{d,p}^m C_0 C_1 \dots C_m.$$

We take the maximum with respect to $(\mathcal{K}', k_0)$ and multiply the relation by $C_{d,p}$. Then by setting $C'_m = C_{d,p}C_m$, we get the relation

$$C'_n = \sum_{m=0}^{n} C'_0 \dots C'_m.$$

It is now a simple exercise to check the implication

$$(C'_0 \leq 1/4) \Rightarrow (\forall n < N,\ C'_n \leq 2C'_0) \Leftrightarrow (\forall n < N,\ C_n \leq 2C_0)\,.$$

The hypothesis gives the condition on N while the conclusion gives the estimate for all $n < N$. □

Before completing the proof of Theorem 1.1 we give a variant of the previous result for the quantities $\tilde{S}_{n,p}(t)$ defined for $1 < p < \frac{2d}{d+2}$, $N \in \mathbb{N}$ and $t \in \mathbb{R}$:

$$\tilde{S}_N(t) = \sup_{\substack{\#\mathcal{K} = N+1 \\ k_0 \in \mathcal{K}}} \left\| e^{-itH_{\mathcal{K}}}\psi_{k_0} \right\|_{L^{p'}}. \tag{3.2}$$

Lemma 3.2. *We assume Hypotheses 0)1)2)3) with $s > \left[\frac{d}{2}\right] + 2$, $\sigma > \frac{d}{2}$, $\alpha = \max_{k\in\mathcal{K}} \alpha_{j(k)}$ fixed, and we take $1 < r < p < \frac{2d}{d+2}$. If $C = C_{p,r,\alpha,\psi}$ denotes the constant of Lemma 3.1, there exists a constant $C' = C'_{p,\alpha,\psi} > 0$ so that*

$$\left(N \leq \frac{1}{(8C_{d,p}C)^r \varepsilon^{d(r-1)}} \right) \Rightarrow \left(\forall t \in \mathbb{R}, \tilde{S}_{N,p}(t) \leq C'\langle t\rangle^{-d\left(\frac{1}{p}-\frac{1}{2}\right)}. \right)$$

Proof. Let r, p be fixed so that $1 < r < p < \frac{2d}{d+2}$. For $N \in \mathbb{N}$, $N \leq \frac{1}{(8C_{d,p}C)^r\varepsilon^{d(r-1)}}$ we have according to Lemma 3.1

$$\forall n < N,\ C_{d,p} \left\| \langle t\rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)} S_n(t) \right\|_{L^\infty} \leq \frac{1}{4}.$$

We set

$$E_N = \left\| \langle t\rangle^{d(\frac{1}{p}-\frac{1}{2})} \tilde{S}_{N,p}(t) \right\|_{L^\infty}.$$

$N = 0$**:** The Corollary 2.3 states $E_0 = C_{p,\psi,\alpha} < +\infty$.

$N \geq 1$**:** We first fix $k_0 \in \mathcal{K}$ and we apply again the identity (A.3) of Lemma A.2 with $A_0 = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$ and $A_{\mathcal{K}\setminus\{k_0\}} = H_{\mathcal{K}}$. With Lemma A.1, it leads to

$$E_N \leq \sum_{n=0}^{N} E_n C_{d,p}^n \prod_{m=0}^{n-1} \left\| \langle t\rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)} S_m(t) \right\|_{L^\infty} \leq \sum_{n=0}^{N} E_n \left(\frac{1}{4}\right)^n$$

after taking the maximum value with respect to $k_0 \in \mathcal{K}$, $\#\mathcal{K} = N+1$.

In two steps, one easily deduces from the previous recurrence relation the estimate $E_N \leq E_0 \left(\frac{4}{3}\right)^N$ and consequently $E_N \leq \frac{3}{2}E_0$. ☐

End of the proof of Theorem 1.1. Let us fix p, r, p_1 so that $1 < p \leq 2$, $1 < r < p_1 < \min\left(p, \frac{2d}{d+2}\right)$. We set $\theta = \frac{(1/p-1/2)}{(1/p_1-1/2)}$ and we notice $\frac{(1/p-1/2)}{(1/r-1/2)} < \theta < 1$. According to Lemma 3.1 and Lemma 3.2 there exist two constants $C = C_{p_1,r,\alpha,\psi} > 0$ and $C' = C'_{p_1,\alpha,\psi} > 0$ so that for $N \leq \frac{1}{(8C_{d,p_1}C)^r \varepsilon^{r-1}}$ the quantities defined by (3.1) and (3.2) satisfy

$$\forall n < N, \quad C_{d,p_1} \left\| \langle t\rangle^{d\left(\frac{1}{p_1}-\frac{1}{2}\right)} S_n(t) \right\|_{L^\infty} \leq \frac{1}{4}$$

$$\text{and} \qquad \forall n \leq N, \quad E_n := \left\| \langle t\rangle^{d\left(\frac{1}{p_1}-\frac{1}{2}\right)} \tilde{S}_{n,p_1}(t) \right\|_{L^\infty} \leq C'.$$

We also note that with $s > \left[\frac{d}{2}\right] + 2 > \frac{d}{2}$ and $\sigma > \frac{d}{2}$, there exists a constant $C_{p_1,\alpha,\psi}$ so that

$$\forall t \in \mathbb{R}, \quad \sup_{k\in\mathcal{K}} \left\| e^{it_0 H_0}\psi_k \right\|_{L^{p_1'}} \leq C_{p_1,\alpha,\psi} \langle t_0\rangle^{-d\left(\frac{1}{p_1}-\frac{1}{2}\right)}.$$

Then the identity (A.3) of Lemma A.2 applied with $A_0 = H_0$ and $A_{\mathcal{K}} = H_{\mathcal{K}}$ and the estimate yields

$$\begin{aligned}
\left\| e^{-itH_{\mathcal{K}}} \right\|_{\mathcal{L}(L^{p_1},L^{p_1'})} &\leq \left\| e^{-itH_0} \right\|_{\mathcal{L}(L^{p_1},L^{p_1'})} \\
&\quad + \sum_{n=1}^{N} E_n C_{d,p_1}^n \left(\prod_{m=0}^{n-1} \left\| \langle t\rangle^{d\left(\frac{1}{p_1}-\frac{1}{2}\right)} S_n(t) \right\|_{L^\infty} \right) C_{d,p_1} C_{p_1,\alpha,\psi}(N+1) \\
&\leq C_{p_1} t^{-d\left(\frac{1}{p_1}-\frac{1}{2}\right)} + \left(\sum_{n=1}^{N} \left(\frac{1}{4}\right)^n \right) C_{d,p_1}(C')^2 (N+1) \langle t\rangle^{-d\left(\frac{1}{p_1}-\frac{1}{2}\right)} \\
&\leq C_{p_1,r,\alpha,\psi}(N+1) t^{-d\left(\frac{1}{p_1}-\frac{1}{2}\right)}.
\end{aligned}$$

We conclude by interpolating with $\left\| e^{-itH_{\mathcal{K}}} \right\|_{\mathcal{L}(L^2)} = 1$ and $\frac{1}{p} = \theta\frac{1}{p_1} + (1-\theta)\frac{1}{2}$. ☐

End of the proof of Theorem 1.2. The proof is basically the same as the previous one. It suffices to notice that with $\langle x - x_0\rangle^{s'} u \in L^2(\mathbb{R})$, $s' > \frac{d}{2}$, and $\sigma > \frac{d}{2}$ the factor

$$\langle t_0\rangle^{d\left(\frac{1}{p_1}-\frac{1}{2}\right)} \sum_{k_0\in\mathcal{K}} \left| \langle \psi_{k_0}, e^{-itH_0}u\rangle \right| = \langle t_0\rangle^{d\left(\frac{1}{p_1}-\frac{1}{2}\right)} \sum_{k_0\in\mathcal{K}} \left| \langle \psi_{k_0}\langle D\rangle^{\sigma/2}, e^{-itH_0}\langle D\rangle^{-\sigma/2}u\rangle \right|$$

can be estimated by $C_{p_1,\alpha,\psi} + \frac{1}{4C_{d,p_1}}$ instead of $C_{p_1,\alpha,\psi}(N+1)$ by referring to Lemma 2.10. ☐

3.2 Strichartz estimates.

The strategy for the Strichartz estimate is the same as the one for the dispersion estimate. A curiosity is that it crucially relies on the endpoint Strichartz estimate of Keel and Tao ([9]) for which $q = q' = 2$.

For $n \in \{0, \dots, N-1\}$ we introduce the quantity

$$F_n = \sup_{\substack{\#\mathcal{K}' = n+1 \\ k_0 \in \mathcal{K}'}} \sum_{k \in \mathcal{K} \setminus \mathcal{K}'} \left\| \langle \psi_{k_0} | e^{-itH_{\mathcal{K}'}} | \psi_k \rangle \right\|_{L^1_t(\mathbb{R}_+)} . \tag{3.3}$$

Lemma 3.3. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma \geq \max\left(\frac{d-3}{2}, \frac{d-2}{4}\right)$ *and* $\alpha = \max_{k \in \mathcal{K}} \alpha_{j(k)}$ *fixed. Then there exists a constant* $C = C_{\alpha,\psi}$ *so that*

$$\left(N \leq \frac{1}{(4C)^{\frac{2d}{d+2}} \varepsilon^{d\frac{d-2}{d+2}}} \right) \Rightarrow \left(\forall n \in \{0, \dots, N-1\}, F_n \leq 2C\varepsilon^{\frac{d-2}{2}} N^{\frac{d+2}{2d}} \leq \frac{1}{2} \right).$$

Proof. Our induction now relies on the second identity (A.4) of Lemma A.2.

$n = 0$: For $H_{\mathcal{K}'} = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$, Proposition 2.11 gives the estimate

$$\sum_{k \neq k_0} \left\| \langle \psi_{k_0} | e^{-itH_{\mathcal{K}'}} | \psi_k \rangle \right\|_{L^1_t(\mathbb{R}_+)} \leq C_{\alpha,\psi} \leq \sum_{k \neq k_0} \frac{\varepsilon^{\frac{d}{2}-1}}{|k - k_0|^{\frac{d}{2}-1}} \leq C_{\alpha,\psi} \varepsilon^{\frac{d-2}{2}} N^{\frac{d+2}{2d}} = F_0.$$

$n \geq 1$: We assume that the constants F_m are known for $m < n$ and we take $\mathcal{K}' \subset \mathcal{K}$ with $\#\mathcal{K}' = n$ and $k_0 \in \mathcal{K}'$. We apply the identity (A.4) with $A_0 = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$ and $A_{\mathcal{K}' \setminus \{k_0\}} = H_{\mathcal{K}'}$. The L^1-estimates of convolutions on $\mathbb{R}_+$ yields

$$\sum_{k \in \mathcal{K} \setminus \mathcal{K}'} \left\| \langle \psi_{k_0} | e^{-itH_{\mathcal{K}'}} | \psi_k \rangle \right\|_{L^1_t(\mathbb{R}_+)} \leq \sum_{m=0}^{n} F_0 F_1 \dots F_m .$$

After taking the maximum with respect to $k_0 \in \mathcal{K}'$, we get the same estimate as in Lemma 3.1

$$F_n \leq \sum_{m=0}^{n} F_0 \dots F_m$$

which implies $F_n \leq 2F_0$ if $F_0 \leq \frac{1}{4}$.

□

We now introduce for $N \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^d)$ the quantity

$$\tilde{F}_N(u) = \sup_{\substack{\#\mathcal{K} = N+1 \\ k_0 \in \mathcal{K}}} \left\| \langle \psi_{k_0} | e^{-itH_{\mathcal{K}}} | u \rangle \right\|_{L^2_t(\mathbb{R}_+)} . \tag{3.4}$$

Lemma 3.4. *We assume Hypotheses 0)1)2)3) with* $s > \left[\frac{d}{2}\right] + 2$, $\sigma \geq \max\left(\frac{d-3}{2}, \frac{d-2}{4}\right)$ *and* $\alpha = \max_{k\in\mathcal{K}} \alpha_{j(k)}$ *fixed. If* $C = C_{\alpha,\psi}$ *denotes the constant of Lemma 3.3, there exists a constant* $C' = C'_{\alpha,\psi} > 0$ *so that*

$$\left(N \leq \frac{1}{(8C)^{\frac{2d}{d+2}} \varepsilon^{d\frac{d-2}{d+2}}}\right) \Rightarrow \left(\forall u \in L^2(\mathbb{R}^d), \quad \tilde{F}_N(u) \leq C' \|u\|_{L^2}\right).$$

Proof. For $N \in \mathbb{N}$, $N \leq \frac{1}{(8C)^{\frac{2d}{d+2}} \varepsilon^{d\frac{d-2}{d+2}}}$, Lemma 3.3 gives

$$\forall n < N, \quad F_n \leq \frac{1}{4}.$$

$N = 0$**:** The Strichartz estimate (2.9) for $H_{\{k_0\}} = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$, with $q = 2$ and $r = \frac{2d}{d-2}$, combined with $\psi_{k_0} \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ gives $\tilde{F}_0(u) \leq C''_{\alpha,\psi} \|u\|_{L^2}$.

$N \geq 1$**:** We first fix $k_0 \in \mathcal{K}$ and we use again the identity (A.4) with $A_0 = H_0 + |\psi_{k_0}\rangle\langle\psi_{k_0}|$ and $A_{\mathcal{K}\setminus\{k_0\}} = H_{\mathcal{K}}$. After taking the maximum with respect to $k_0 \in \mathcal{K}$, it yields

$$\tilde{F}_N(u) \leq \sum_{n=0}^{N} F_0 \dots F_{n-1} \tilde{F}_n(u) \leq \sum_{n=0}^{N} \tilde{F}_n(u) \left(\frac{1}{4}\right)^n$$

which implies $\tilde{F}_N(u) \leq \frac{3}{2}\tilde{F}_0(u) \leq \frac{3}{2} C''_{\alpha,\psi} \|u\|_{L^2}$.

□

End of the proof of Theorem 1.3. It is sufficient to prove the result for $q = 2$ and $r = \frac{2d}{d-2}$. The general result then follows by interpolation. For $u \in L^2(\mathbb{R}^d)$, the identity (A.4) of Lemma (A.3) applied with $A_0 = H_0$ and $A_{\mathcal{K}} = H_{\mathcal{K}}$ gives

$$e^{-itH_{\mathcal{K}}} = e^{-itH_0} + \int_0^t e^{-i(t-t')H_0} \Phi(t')\, dt'$$

with

$$\Phi(t') = \sum_{n=0}^{N} i^{n+1} \sum_{\#\{k_0,\dots,k_n\}=n+1} \int |\psi_{k_0}\rangle\langle\psi_{k_0}| e^{-it_0 H_{\{k_0\}}} |\psi_{k_1}\rangle\langle\psi_{k_1}| \dots$$
$$\dots |\psi_{k_n}\rangle\langle\psi_{k_n}| e^{-it_n H_{\{k_0,\dots,k_n\}}} |u\rangle \; D_{t'}(t_0, \dots, t_n).$$

Since $\psi \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, the set of functions ψ_{k_0} is uniformly bounded in $L^{\frac{2d}{d+2}}(\mathbb{R}^d)$. Then Lemma 3.3 and Lemma 3.4 state that

$$\|\Phi\|_{L^2_t L_x^{\frac{2d}{d+2}}} \leq \sum_{n=0}^{N} (N+1) F_0 \dots F_{n-1} \tilde{F}_n(u) \leq (N+1) \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n C'_{\alpha,\psi} \|u\|_{L^2},$$

if $N \leq \frac{1}{(8C_{\alpha\psi})^{\frac{2d}{d+2}}\varepsilon^{d\frac{d-2}{d+2}}}$. We conclude with the standard consequence of the Strichartz estimate

$$\left\| \int_0^t e^{-i(t-t')H_0}\Phi(t')\, dt' \right\|_{L_t^2 L^{\frac{2d}{d-2}}} \leq C \left\| \Phi \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

applied here with $\tilde{q}' = 2$ and $\tilde{r}' = \frac{2d}{d+2}$. □

End of the proof of Theorem 1.4. It is the same as the previous one if one notices that Proposition 2.11 and $\sigma > \frac{d-2}{2}$ provides the uniform estimate

$$\forall \varphi \in L^2(\mathbb{R}^d), \quad \sum_{k_0 \in \mathcal{K}} \left\| \langle \varphi | \langle x \rangle^{-s'} e^{-itH_0} | \psi_{k_0} \rangle \right\|_{L_t^1(\mathbb{R}_+)} \leq C_{s',\alpha,\psi} \left(1 + \varepsilon^{\frac{d-2}{2}} N^{\frac{d+2}{2d}} \right) \left\| \varphi \right\|_{L_x^2}.$$

For $N \leq \frac{1}{(8C)^{\frac{2d}{d+2}}\varepsilon^{d\frac{d-2}{d+2}}}$ the same application of identity (A.4) as above leads to

$$\forall \varphi \in L^2(\mathbb{R}^d), \quad \left\| \langle \varphi | \langle x \rangle^{-s'} e^{-itH_{\mathcal{K}}} | u \rangle \right\|_{L_t^2} \leq C'_{s',\alpha,\psi} \left\| \varphi \right\|_{L^2} \left\| u \right\|_{L^2}.$$

□

A A variant of the Dyson expansion.

We introduce the notation $D_t(t_n, t_{n-1}, \ldots, t_0)$ for the measure on $\mathbb{R}^{n+1}$

$$D_t(t_n, t_{n-1}, \ldots, t_0) = \left(\prod_{k=0}^{n} 1_{\mathbb{R}_+}(t_k) \right) \delta(t_n + \cdots + t_0 = t).$$

We note the simplicial associativity relations

$$D_t(t_{n+1}, \ldots, t_0) = D_{t'_n}(t_{n+1}, t_n) D_t(t'_n, t_{n-1}, \ldots, t_0) \tag{A.1}$$

$$\text{and} \qquad D_t(t_n, \ldots, t_0) = D_t(t_n, t') D_{t'}(t_{n-1}, \ldots, t_0), \tag{A.2}$$

which is another way of writing the associativity of the convolution product on $\mathbb{R}_+$. Then the Dyson expansion (the iteration of Duhamel formula) writes for $A = A_0 + V$, with A_0 self-adjoint and $V \in \mathcal{L}(L^2)$, and $t \geq 0$

$$e^{-itA} = \sum_{n=0}^{\infty} (-i)^n \int e^{-it_n A_0} V e^{-it_{n-1} A_0} V \ldots V e^{-it_0 A_0} D_t(t_n, t_{n-1}, \ldots, t_0).$$

Remind also with this notation the standard estimate:

Lemma A.1. *For $\sigma > 1$, the estimate*

$$\forall n \in \mathbb{N}, \forall t \geq 0, \int \langle t_n \rangle^{-\sigma} \ldots \langle t_0 \rangle^{-\sigma} \; D_t(t_n, t_{n-1}, \ldots, t_0) \leq C_\sigma^n \langle t \rangle^{-\sigma},$$

holds with the constant $C_\sigma = 2^{\sigma+1} \int_0^\infty \langle t' \rangle^{-\sigma} \, dt'$.

Proof. It is a direct consequence of

$$\int_0^t \langle t-t'\rangle^{-\sigma}\langle t'\rangle^{-\sigma}\ dt' = 2\int_0^{t/2} \langle t-t'\rangle^{-\sigma}\langle t'\rangle^{-\sigma}\ dt' \le 2\left(\int_0^\infty \langle t'\rangle^{-\sigma}\ dt'\right)\langle t/2\rangle^{-\sigma}$$

and of the simplicial associativity (A.1)(A.2). □

We shall consider the case where the perturbations V_k, $k\in\mathcal{K}$, $\#\mathcal{K}\in\mathbb{N}$, are bounded operators and A_0 is a given self-adjoint operator. We set

$$A_K = A_0 + \sum_{k\in\mathcal{K}} V_k$$

and for $k_1,\dots,k_n$ all distinct in $\mathcal{K}$ ($n\le N=\#\mathcal{K}$)

$$A_{\{k_1,\dots,k_n\}} = A_0 + \sum_{l=1}^n V_{k_l}.$$

Lemma A.2. *With the above notations and assumptions we have for all $t\ge 0$ the identities:*

$$e^{-itA_\mathcal{K}} = \sum_{n=0}^{\#\mathcal{K}}(-i)^n \sum_{\#\{k_1,\dots,k_n\}=n} \int e^{-it_nA_{\{k_1,\dots,k_n\}}}V_{k_n}e^{-it_{n-1}A_{\{k_1,\dots,k_{n-1}\}}}V_{k_{n-1}}\cdots$$
$$\dots e^{-it_1A_{\{k_1\}}}V_{k_1}e^{-it_0A_0}D_t(t_n,t_{n-1},\dots,t_0) \quad \text{(A.3)}$$

$$e^{-itA_\mathcal{K}} = \sum_{n=0}^{\#\mathcal{K}} i^n \sum_{\#\{k_1,\dots,k_n\}=n} \int e^{-it_0A_0}V_{k_1}e^{-it_1A_{\{k_1\}}}V_{k_2}e^{-it_1A_{\{k_1,k_2\}}}\dots$$
$$\dots V_{k_n}e^{-it_nA_{\{k_1,\dots,k_n\}}}D_t(t_n,t_{n-1},\dots,t_0). \quad \text{(A.4)}$$

Proof. For $t\ge 0$ we set $N=\#\mathcal{K}$ and

$$B(t) = \left(e^{-itA_\mathcal{K}}e^{itA_0}-\mathrm{Id}\right) - \sum_{k\in\mathcal{K}}\left(e^{-itA_{\{k\}}}e^{itA_0}-\mathrm{Id}\right).$$

We have $B(0)=0$ and the derivative equals

$$i\partial_t B(t) = e^{-itA_\mathcal{K}}\left(\sum_{k\in\mathcal{K}}V_k\right)e^{itA_0} - \left(\sum_{k\in\mathcal{K}}e^{-itA_{\{k\}}}V_k\right)e^{itA_0}$$
$$= \sum_{k\in\mathcal{K}}\left(e^{-itA_\mathcal{K}}-e^{-itA_{\{k\}}}\right)V_ke^{itA_0}.$$

Hence, we have by taking $k_1 = k$

$$e^{-itA_{\mathcal{K}}} - e^{-itA_0} = \sum_{k_1\in\mathcal{K}} \Bigg[\left(e^{-itA_{\{k_1\}}} - e^{-itA_0}\right) \\ - i\int \left(e^{-it_1A} - e^{-it_1A_{\{k_1\}}}\right) V_{k_1} e^{-it_0A_0}\ D_t(t_1,t_0)\Bigg].$$

We iterate after noticing that the factor $\left(e^{-it_1A_{\mathcal{K}}} - e^{-it_1A_{\{k_1\}}}\right)$ is the same as the left-hand side with $A_{\mathcal{K}} = A_{\{k_1\}} + \sum_{k\neq k_1} V_k$. We obtain for all $M \leq N$

$$e^{-itA_{\mathcal{K}}} - e^{-itA_0} = \\ \sum_{n=1}^{M}(-i)^{n-1} \sum_{\#\{k_1,\ldots,k_n\}=n} \Bigg[\int \left(e^{-it_nA_{\{k_1,\ldots,k_n\}}} - e^{-it_nA_{\{k_1,\ldots,k_{n-1}\}}}\right) V_{k_{n-1}} e^{-it_{n-1}A_{\{k_1,\ldots,k_{n-2}\}}} \ldots \\ \ldots e^{-it_1A_{\{k_1\}}} V_{k_1} e^{-it_0A_0}\ D_t(t_{n-1},\ldots,t_0)\Bigg] \\ + (-i)^M \sum_{\#\{k_1,\ldots,k_M\}=M} \int \left(e^{-it_MA_{\mathcal{K}}} - e^{-it_MA_{\{k_1,\ldots,k_M\}}}\right) V_{k_M} e^{-it_{M-1}A_{\{k_1,\ldots,k_{M-1}\}}} \ldots \\ \ldots e^{-it_1A_{\{k_1\}}} V_{k_1} e^{-it_0A_0}\ D_t(t_{M-1},\ldots,t_0).$$

We conclude with the identity $A_{\mathcal{K}} = A_{\{k_1,\ldots,k_N\}}$ $(M = N)$ and the Duhamel formula $(n \leq N)$

$$\left(e^{-itA_{\{k_1,\ldots,k_n\}}} - e^{-itA_{\{k_1,\ldots,k_{n-1}\}}}\right) = -i\int e^{-it_1A_{\{k_1,\ldots,k_n\}}} V_{k_n} e^{-it_0A_{\{k_1,\ldots,k_{n-1}\}}}\ D_t(t_0,t_1).$$

□

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