e t eory of resonances as its origins in attempts to explain t e existence of metastable states in pysical systems. ese are states wic are localized or coerent for some long time period, called t e *li fetime*, and t en disintegrate. Examples abound and include unstable atoms and particles.

e mat ematical analysis of resonance penomena naturally leads to t e study of perturbations of self-adjoint operators wic ave embedded eigenvalues in t eir continuous spectra. An example of t is is in t e quantum t eory of t eelium atom, in wic t ere are long-lived Auger states [RSim]. e mat ematical study of t is problem proceeds by viewing as t e unperturbed self-adjoint operator, t e Hamiltonian governing two decoupled electron-proton systems. is system as many embedded eigenvalues. e perturbed Hamiltonian is t at wic includes t e effect of electron-electron repulsion. In Examples 3 and 4 of section 6, we discuss a class of problems wit t is structure. Anot er pysical problem in wic resonances play an important role is in t e setting of an atom coupled to t e poton-radiation field ([BFroSi], [JPi1,2], [Ki1,2]); see also Example 7 of section 6. Alt oug initially inspired by t e study of quantum pe- nomena, questions involving embedded eigenvalues ave been seen to arise, quite naturally in spectral geometry and number t eory [PS]. e sys - tematic mat ematical study of t e eff ects of perturbations on embedded eigenvalues was initiated by Friedric s [F].

e met od of analyzing t e resonance problem we developere is related to our work on t e large time be avior of nonlinear Sc r¨ odinger and nonlinear wave equations [SoWei3-5]. In t ese problems, certain states of t e system decay slowly as a result of resonant interactions generated by nonlinearity in t e e quations of motion. e met ods re quired are necessarily time-dependent as t e e quations are nonlinear and nonintegrable. ey are based on a direct approac to t e study of energy trans fer from discrete to continuum modes.

We consider t e following general problem. Suppose H is a self-adjoint operator in a Hilbert space $\mathcal{H} = L\ (\mathbb{R}^n)$, suc t at H as a simple eigenvalue, λ , wic is embedded in its continuous spectrum, wit associated eigenfunction, ψ :

$$H\ \psi\ = \lambda\ \psi\ , \qquad ||\psi\ ||\ = 1\,.$$

We now consider t e time-dependent Sc r¨ odinger equation, for t e perturbed self-adjoint Hamiltonian, $H = H\ + W$,

$$i\partial_t \phi = H \phi \tag{1.1}$$

were W is a perturbation wic is small in a sense to be specified. e c oice of decomposition of H into an unperturbed part, H , and a pertur-

[1]Some of the results of this paper were presented in the proceedings article [SoWe2] and in the preprint [SoWe3].

[2]Related to this is the observation that many nonlinear phenomena can be regarded as (generic) instabilities of embedded eigenvalues for suitable *linear* operators. This point of view is taken by I.M. Sigal in [Si1,2], who studies the non-existence of bifurcating time-periodic and spatially localized solutions of certain nonlinear wave and Schrödinger equations. The problem of absence of small amplitude *breathers* for Hamiltonian perturbations of the Sine-Gordon equation (see, for example, [SeKr] and [BiMWe]) can also be viewed in this context [Si2]. Other nonlinear wave phenomena, in which resonances have been shown to play a role, are studied in [PWe], [CrHi].

bation, W, depends on t e problem at and; see, for example, [D] .

PROBLEM. *Suppose we specify initial data, ϕ for* (1.1) *which are spectrally localized (relative to H) in a small interval Δ about λ . Describe the time-dynamics for $t \in (-\infty, \infty)$.*

We s all prove t at under quite general assumptions on H and W t at for small perturbations W,

(i) Has absolutely continuous spectrum in an interval about λ ,
(ii) t e solution wit suc data decays algebraically as $t \to \pm\infty$. For t e special case of initial conditions given by ψ , t e solution is c ar - acterized by transient exponential decay. e exponential rate, Γ (reciprocal of t e lifetime), can be calculated.

On t e more tec nical side, we ave imposed fairly relaxedypot eses on t e regularity of t e perturbation, W; in particular we do not require any condition on its commutators. is may be useful in problems like t e radiation problem and problems were Diric let decoupling is used.

e decay of solutions due to resonant coupling to t e continuum is revealed by decomposing t e solution of (1.1), wit data spectrally localized (relative to H) near λ , in terms of t e natural basis of t e unperturbed problem,

$$\phi(t) = a(t)\psi + \tilde{\phi}(t)\,, \quad \big(\psi\,, \tilde{\phi}(\cdot, t)\big) = 0\,. \tag{1.2}$$

After isolating t e key resonant contributions, t e system of equations governing $a(t)$ and $\tilde{\phi}$ is seen toave t e form

$$\begin{aligned} ia' &= (\Lambda - i\Gamma)a + \mathcal{C}\ (a, \tilde{\phi}) \\ i\partial_t\tilde{\phi} &= H\ \tilde{\phi} + \mathcal{C}\ (a, \tilde{\phi})\,, \end{aligned} \tag{1.3}$$

were t e $\mathcal{C}_j$, $j = 1, 2$, denote terms wic couple t e dynamics of a and $\tilde{\phi}$, and $\mathcal{C}$ lies in t e continuous spectral part of H . If t ese coupling terms are neglected, t en it is clear t at $a(t)$ is driven to zero provided $\Gamma > 0$. e quantity, Γ, is displayed in (2.7) and is always nonnegative. Its explicit formula, (2.7), is often referred to as t e *Fermi golden rule*. Generically, Γ is strictly positive. e exponential be avior suggested by t eseeuris - tics is, in general, only a transient; in general, e^{-iH_0t} as dispersive wave solutions, and coupling to t ese waves leads to (weaker) algebraic decay as $t \to \pm\infty$. At t is stage, we wis to point out t at alt oug presented in t e setting of a Scr¨ odinger type operator, acting in L $(\mathbb{R}^n)$, our results and t e approac we develop below can be carried out in t e setting of a general Hilbert space, $\mathcal{H}$, wit appropriate modifications made in t ey- pot eses. ese modifications are discussed in t e remark following our main t eorem, eorem 2.1. eir implementation is discussed in several of t e examples presented in section 6.

Historically, motivated by experimental observations, t e primary focus of mat ematical analyses of t e resonance problemas been on establis ing exponential decay at intermediate times. However, viewed as an infinite dimensional Hamiltonian system, t e asymptotic $(t \to \pm\infty)$ be avior of solutions is a fundamental question. Our met ods address t is question and are adaptable to nonautonomous linear, and nonlinear problems [SoWei4,5].

e time decay of suc solutions implies t at t e spectrum of t e per - turbed Hamiltonian, in a neigbor ood of λ , is absolutely continuous. is implies t e instabilit y of t e embedded eigenvalue. More precisely, under perturbation t e embedded eigenvalue moves off t e real axis and becomes a pole ("resonance pole" or "resonance energy") of t e resolvent analyt - ically continued across t e continuous spectrum onto a second Riemann s eet [Hu]. We will also s ow t at in a neigbor ood of suc embedded eigenvalues, t ere are no *new* embedded eigenvalues wic appear, and give an estimate on t e size of t is neigbor ood. Most importantly, we find t e time be avior of solutions of t e associated Sc r¨odinger type evolution equation for s ort, intermediate and long time scales. e *li fetime* of t e resonant state naturally emerges from our analysis. ese results are stated precisely ineorem 2.1.

Many different approac es to t e resonance problem in quantum me- c anics ave been developed over t e last 70 years and t e various c arac- terizations of resonance energies are expected to be equivalent; see [HSj]. e first (formal) approac to t e resonance problem, due to Weisskopf and Wigner [WeiWi], was introduced in t eir study of t e penomena of spon- taneous emission and t e instabilit y of excited states; see also [L]. eir approac plays a central role in today' s pysics literature; see for example [AlE], [LaLi]. It is time-dependent and our approac is close in spirit to t is met od.

Anot er approac , used bot by pysicists and mat ematicians is based on t e analytic properties of t e S-matrix in t e energy variable; see [LaxP]. Ot er approac es concentrated on t e be avior of a reduced Green's function, eit er by direct met ods, or by studying its analytic properties [Ho],[O].

e most commonly used approac is t at of analytic dilation or, more generally, analytic deformation [CyFKS], [HiSi]. is met od is very gen- eral, but requires a c oice of def ormation group adapted to t e problem at and, as well as tecnical analyticit y conditions wic do not appear to be necessary. In tis approac, te Hamiltonian of interest, H, is embedded in a one-parameter family of unitarily equivalent operators, $H(\theta)$, $\theta \in \mathbb{R}$. Under analytic continuation in θ te continuous spectrum of H is seen to move and te eigenvalue, wic was embedded in te continuum for te unperturbed operator, is now "uncovered" and isolated. us Rayleig-Scr¨odinger perturbation teory for an isolated eigenvalue can be applied, and used to conclude tat te embedded eigenvalue generically perturbs to a resonance. e nonvanising of te Fermi golden rule, (2.7), arises as a nondegeneracy condition ensuring tat we can see te motion of te embedded eigenvalue at second order in perturbation teory. In our work, it arises as a condition, ensuring te "damping" of states wic are spectrally localized (wit respect to H) about λ . Analytic deformation tecniques do not directly address te time beavior, wic require a separate argument [GeSi], [Hu], [Sk].

Additionally, "tresolds " may not be "uncovered" and terefore te metod of analytic deformation is unable to address te perturbation teory of suc points. Our time-dependent metod can yield information about tresolds, toug it may be problematic to ceck te local decay assumptions in intervals containing suc points; see owever Example 5 in section 7, concerning te instabilit y of a tresold eigenvalue of $-\Delta + V(x)$. Finally, in many cases, previous approaces ave required te potential to be dilation analytic, were we only require C beavior; see te concluding remarks of Appendix D for a discussion of tis point.

e paper is structured as follows. In section 2 te matematical framework is explained and te main teorem (eorem 2.1) is stated. In section 3 te solution is decomposed relative to te unperturbed operator, te key resonance is isolated and a dynamical system of te form (1.3) is derived. Sections 4 and 5 contained te detailed estimates of te large time beavior of solutions. In section 6 we outline examples and applications. Sections 7-11 are appendices. Section 7 (Appendix A) concerns te proof of te "singular" local decay estimate of Proposition 2.1, and section 10 (Appendix D) outlines a general approac to obtaining local decay estimates of te type assumed in te ypotesis (H4). In section 9 (Appendix C) we present te details of our expansion of te complex frequency, ω_* (see (2.12) and Proposition 3.3). In section 11 (Appendix E) we give results on boundedness of functions of self-adjoint operators in *weighted* function spaces wic may be of general interest.

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In t is section we first introduce certain necessary terminology and notation. We t en state t eypot eses (H) and (W) on t e unperturbed Hamiltonian, H , and on t e perturbation, W. e section t en concludes wit statements of t e main results.

For an operator, L, $||L||$ denotes its norm as an operator from L to itself. We interpret functions of a self adjoint operator as being defined by t e spectral t eorem. I n t e special case were t e operator is H , we omit t e argument, *i.e.* $g(H\ ) = g$.

For an open interval Δ, we denote an appropriate smoot ed c aracter - istic function of Δ by $g_\Delta(\lambda)$. In particular, we s all take $g_\Delta(\lambda)$ to be a nonnegative C^∞ function, wic is e qual to one on Δ and zero outside a neigbor ood of Δ. e support of its derivative is f urt ermore c osen to be small compared to t e size of Δ, *e.g.* less t an $-|\Delta|$. We furt er re quire t at $|g_\Delta^{n}(\lambda)| \leq c_n\ |\Delta|^{-n}$, $n \geq 1$.

P denotes t e projection on ψ , *i.e.* $P\ f = (\psi\ , f)\psi$.

$P_{\ b}$ denotes t e spectral projection on $\mathcal{H}_{pp} \cap \{\psi\ \}^\perp$, t e pure point spectral part of H ort ogonal to ψ . at is, $P\ _b$ projects onto t e subspace of $\mathcal{H}$ spanned by all eigenstates ot er t an ψ .

In our treatment, a central role is played by t e subset of t e spectrum of t e operator H , $\mathbf{T}^\#$, on wic a sufficiently rapid local decay estimate olds. For a decay estimate toold for $e^{\ -iH_0 t}$, one must certainly project out t e bound states of H , but t ere may be ot er obstructions to rapid decay. In scattering t eory t ese are called *threshold energies* [CyFKS]. Examples of t res olds are : (i) points of stationary pase of a constant coefficient principle symbol for two-body Hamiltonians; and (ii) for N-body Hamiltonians, zero and t e eigenvalues of subsystems. We will not give a precise definition of t res olds. For us it is sufficient to say t at away from t res olds t e favourable local decay estimates for H old.

Let Δ_* be union of intervals, disjoint from Δ, containing all t res olds of H , and a neigbor ood of infinity. We t en let

$$P \ = P_{\ b} + g_{\Delta_*}$$

were $g \ _{\Delta_*} = g_{\Delta_*}(H \ )$ is a smoot ed caracteristic f unction of t e set $\Delta \ _*$. We also define

$$\begin{aligned} \langle x\rangle \ &= 1 + |x| \ , \\ \overline{Q} &= I - Q \qquad \text{and} \\ P_c^{\#} &= I - P \ - P \ . \end{aligned} \tag{2.1}$$

us, $P \ _c^{\#}$ is a smoot ed out spectral projection of t e set $\mathbf{T}^{\#}$ defined as

$$\mathbf{T}^{\#} = \sigma(H \ ) - \{ \text{ eigenvalues, real neigbor oods of t res olds and infinit y} \}. \tag{2.2}$$

We expect e^{-iH_0t} to satisfy good local decay estimates on t e range of $P \ _c^{\#}$; see (H4) below.

Next we state our *hypotheses on* H .

(H1) H is a self adjoint operator wit dense domain $\mathcal{D}$, in $L \ (\mathbb{R}^n)$.

(H2) λ is a simple embedded eigenvalue of H wit (normalized) eigenfunction ψ .

(H3) ere is an open interval Δ containing λ and no ot er eigenvalue of H .

ere exists $\sigma \ > 0$ suc t at

(H4) *Local decay estimate*: Let $r \geq 2 + \varepsilon$ and $\varepsilon > 0$. If $\langle x\rangle^{\sigma} \ f \in L$ t en

$$\left\| \langle x\rangle^{-\sigma} e^{-iH_0t} P_c^{\#} f \right\| \ \leq C \langle t\rangle^{-r} \left\| \langle x\rangle^{\sigma} f \right\| \ , \tag{2.3}$$

(H5) By appropriate c oice of a real number c, t e L operator norm of $\langle x\rangle^{\sigma}(H \ + c)^{-} \ \langle x\rangle^{-\sigma}$ can be made sufficiently small.

Remarks. (i) We ave assumed t at λ is a simple eigenvalue to simplify t e presentation. Our met ods can be easily adapted to t e case of multiple eigenvalues.

(ii) Note t at Δ does not ave to be small and t at $\Delta \ _*$ can be c osen as necessary, depending on H .

(iii) In certain cases, t e above local decay conditions can be proved even wen λ is a t res old; see Ex ample 5 of section 6.

(iv) Regarding t e verification of t e local decayypot esis, one approac is to use tec ni ques based on t e Mourre estimate [JeMouPe], [SiSo].

If Δ contains no t res old values t en, quite generally, t e bound (2.3) olds wit r arbitrary and positive. See Appendix D.

We s all re quire t e f ollowing consequence of ypot esis (H4).

PROPOSITION 2.1. *Let $r \geq 2+\varepsilon$ and $\varepsilon > 0$. Assume $\mu \in \mathbf{T}^{\#}$. Then, for $t \geq 0$*

$$\left\|\langle x\rangle^{-\sigma} e^{-iH_0 t}(H \;\; - \mu - i0)^{-} \;\; P_c^{\#} f\right\| \;\leq C\langle t\rangle^{-r} \quad \left\|\langle x\rangle^{\sigma} f\right\| \;\;, \tag{2.4}$$

For $t < 0$, estimate (2.4) *holds with $-i0$ replaced by $+i0$.*

e proof is given in Appendix A.

We now specify t e conditions we re quire of t e perturbation, W.

Conditions on W.

(W1) W is symmetric and $H = H \;\; + W$ is self-adjoint on $\mathcal{D}$ and t ere exists $c \in \mathbb{R}$ (wic can be used in (H5)), suc t at c lies in t e resolvent sets of H and H.

(W2) For some σ, wic can be c osen to be t e same as in (H4) and (H5),

$$|||W||| \equiv \left\|\langle x\rangle \;\; ^{\sigma} W g_{\Delta}(H \;\;)\right\| + \left\|\langle x\rangle^{\sigma} W g_{\Delta}(H \;\;)\langle x\rangle^{\sigma}\right\| + \left\|\langle x\rangle^{\sigma} W(H \;\; + c)^{-} \;\; \langle x\rangle^{-\sigma}\right\| \;< \infty\,, \tag{2.5}$$

and

$$\left\|\langle x\rangle^{\sigma} W(H \;\; + c)^{-} \;\; \langle x\rangle^{\sigma}\right\| < \infty\,, \tag{2.6}$$

(W3) *Resonance condition – nonvanishing of the Fermi golden rule*:

$$\Gamma \equiv \pi\big(W\psi \;\; , \delta(H \;\; - \tilde{\omega})(I - P \;\;)W\psi \;\; \big) \neq 0 \tag{2.7}$$

for $\tilde{\omega}$ near λ , and

$$\Gamma \geq \delta \;\; |||W||| \tag{2.8}$$

for some $\delta \;\; > 0$.

(W4) $|||W||| < \theta \;\; |\Delta|$ for some $\theta > 0$, sufficiently small, depending on t e properties of H , in particular t e local decay constants, but not on $|\Delta|$.

REMARK. Let $\mathcal{F}_c^{H_0}$ denote t e (generalized) Fourier transform wit respect to t e continuous spectral part of H . e resonance condition (2.7), can t en be expressed as

$$\Gamma \equiv \pi\left|\mathcal{F}_c^{H_0}[W\psi \;\;](\lambda \;\;)\right| \;\; > 0\,. \tag{2.9}$$

We can now state t e main result :

Theorem 2.1. *Let H satisfy the conditions* (H) *and the perturbation W satisfy the conditions* (W). *Then*

(a) $H = H \ + W$ *has no eigenvalues in* Δ.
(b) *The spectrum of* H *in* Δ *is purely absolutely continuous; in particular local decay estimates hold for* $e^{-iHt} g_\Delta(H)$. *Namely, for* ϕ *with* $\langle x \rangle^\sigma \phi \ \in L$, *as* $t \to \pm\infty$,

$$\left\| \langle x \rangle^{-\sigma} e^{-iHt} g_\Delta(H) \phi \ \right\| \ = \mathcal{O}\big(\langle t \rangle^{-r} \quad \big) . \tag{2.10}$$

(c) *For* ϕ *in the range of* $g_\Delta(H)$ *we have (for* $t \geq 0$*)*

$$e^{-iHt} \phi \ = (I + \mathcal{A}_W)\big(e^{-i\omega_* t} a(0) \psi \ + e^{-iH_0 t} \phi_d(0) \big) + \mathcal{R}(t) . \tag{2.11}$$

Here, $\|\mathcal{A}_W\|_{\mathcal{B} \ L^2} \ \leq C|||W|||$, $a(0)$ *is a complex number and* $\phi_d(0)$ *is a complex function in the range of* $P_c^\#$, *which are determined by the initial data; see* (3.1)-(3.2).

The complex frequency, ω_*, *is given by*

$$\omega_* = \omega - \Lambda - i\Gamma + \mathcal{O}\big(|||W||| \ \big) \, , \text{ were} \tag{2.12}$$

$$\omega \equiv \lambda \ + (\psi \ , W \psi \ ) , \tag{2.13}$$

$$\Lambda \equiv \big(W\psi \ , \ \text{P.V.} \ (H \ - \omega)^- \ W\psi \ \big) , \text{ and} \tag{2.14}$$

$$\Gamma \equiv \pi \big(W \psi \ , \delta(H \ - \omega)(I - P \ ) W \psi \ \big) . \tag{2.15}$$

We also have the estimates

$$\big\| \langle x \rangle^{-\sigma} \mathcal{R}(t) \big\| \ \leq C|||W||| , \quad t \geq 0 \tag{2.16}$$

$$\big\| \langle x \rangle^{-\sigma} \mathcal{R}(t) \big\| \ \leq C|||W|||^{\varepsilon} \langle t \rangle^{-r} \quad ,$$
$$t \geq |||W|||^{- \ \delta} \quad , \ \delta > 0 \, , \ \varepsilon = \varepsilon(\delta) > 0 . \tag{2.17}$$

REMARK. oug prased in t e setting of t e space $L \ \ (\mathbb{R}^n)$, our approac is quite general and our results old wit $L \ \ (\mathbb{R}^n)$ replaced by a Hilbert space, $\mathcal{H}$. In t is general setting, t e weigt function, $\langle x \rangle$, is to be replaced by a "weigting operator ", A, in t eypot eses (H), (W) and in t e definition of t e norm of W, $|||W|||$. Additionally, $P_c^\#$ can be taken to be a smoot ed out spectral projection onto t e subspace of $\mathcal{H}$ were t e local decay estimate (H4) olds.

Given an eigenstate ψ associated wit an embedded eigenvalue, λ , of t e unperturbed Hamiltonian, H , a quantity of pysical interest is t e *lifetime* of t e state ψ for t e perturbed dynamics. o find t e lif etime, consider t e quantum expectation value t at t e system is in t e resonant state, ψ ,

$$(\psi \ , e^{-itH} \psi \ ) . \tag{2.18}$$

Note t at

$$e^{-iHt} \psi \ = e^{-iHt} g_\Delta(H) \psi \ + e^{-iHt} \big(g_\Delta(H \ ) - g_\Delta(H) \big) \psi \ . \tag{2.19}$$

eorem 2 .1 and t e tec ni ques used in t e proof s of Propositions 3.1 and 3.2 yield t e following result concerning t e lif etime of t e state ψ .

COROLLARY 2.1. *Let*

$$H_* = H - \mathrm{Re}\omega_* I\,. \tag{2.20}$$

Then, for any $T > 0$ *there is a constant* $C_T > 0$ *such that for* $0 \leq t \leq T|||W|||^-$

$$\left|(\psi\ , e^{-iH_*t}\psi\ ) - e^{-\Gamma t}\right| \leq C_T|||W|||\,, \quad \text{as} \quad |||W||| \to 0\,. \tag{2.21}$$

3 e mps as la f es a ems

We begin wit t e following decomposition of t e solution of (1.1):

$$e^{-iHt}\phi\ = \phi(t) = a(t)\psi\ + \tilde{\phi}(t) \tag{3.1}$$

$$\left(\psi\ , \tilde{\phi}(t)\right) = 0 \qquad -\infty < t < +\infty\,. \tag{3.2}$$

Substitution into (1.1) yields

$$i\partial_t\tilde{\phi} = H\ \tilde{\phi} + W\tilde{\phi} - (i\partial_t a - \lambda\ a)\psi\ + aW\psi\ . \tag{3.3}$$

Recall now t at $I = \ P\ + P\ + P_c^\#$. aking t e inner product of (3.3) wit ψ gives t e amplitude e quation,

$$i\partial_t a = \left(\lambda\ + (\psi\ , W\psi\ )\right)a + (\psi\ WP\ \tilde{\phi}) + (\psi\ , W\phi_d)\,, \tag{3.4}$$

were,

$$\phi_d \equiv P_c^\#\tilde{\phi}\,. \tag{3.5}$$

e following e quation for ϕ_d is obtained by applying $P_c^\#$ to equation (3.3):

$$i\partial_t\phi_d = H\ \phi_d + P_c^\# W(P\ \tilde{\phi} + \phi_d) + aP_c^\# W\psi\ . \tag{3.6}$$

Our goal is to derive a closed system for $\phi_d(t)$ and $a(t)$. o ac ieve t is, we now propose to obtain an expression for $P\ \tilde{\phi}$, to be used in equations (3.4) and (3.6). Since $g_\Delta(H)\phi(\cdot,t) = \phi(\cdot,t)$, we find

$$\left(I - g_\Delta(H)\right)\phi = \left(I - g_\Delta(H)\right)[a\psi\ + P\ \tilde{\phi} + P_c^\#\tilde{\phi}] = 0 \tag{3.7}$$

or

$$\left(I - g_\Delta(H)g_I(H\ )\right)P\ \tilde{\phi} = -\overline{g}_\Delta(H)[a\psi\ + \phi_d]\,, \tag{3.8}$$

were $g_I(\lambda)$ is a smoot function, wic is identically e qual to one on t e support of $P\ (\lambda)$, and wic as support disjoint from Δ. erefore,

$$P\ \tilde{\phi} = -B\overline{g}_\Delta(H)\left(a\psi\ + \phi_d\right)\,, \tag{3.9}$$

were

$$B = \big(I - g_\Delta(H) g_I(H\ )\big)^-\ .$$

is computation is justified by t e following result wic is proved in Appendix B.

PROPOSITION 3.1. *The operator $B = (I - g_\Delta(H)g_I(H\ ))^-$ is a bounded operator on $\mathcal{H}$*

From (3.9) we get

$$\begin{aligned} \phi(t) &= a(t)\psi\ + \phi_d + P\ \tilde{\phi} \\ &= \tilde{g}_\Delta(H)\big(a(t)\psi\ + \phi_d(t)\big)\,, \end{aligned} \tag{3.10}$$

wit

$$\tilde{g}_\Delta(H) \equiv I - B\overline{g}_\Delta(H) = B g_\Delta(H)\big(I - g_I(H\ )\big)\,. \tag{3.11}$$

Alt oug $\tilde{g}(H)$ is not really defined as a function of H, we indulge in t is mild abuse of notation to empasize its dependence on H. In fact, we s all prove t at, in some sense, $\tilde{g}_\Delta(H) \sim g_\Delta(H) \sim g_\Delta(H\ )$.

Substitution of t e above expression (3.9) for $P\ \tilde{\phi}$ into (3.6) gives

$$i\partial_t\phi_d = H\ \phi_d + aP_c^{\#}W\tilde{g}_\Delta(H)\psi\ + P_c^{\#}W\tilde{g}_\Delta(H)\phi_d \tag{3.12}$$

and

$$\begin{aligned} i\partial_t a &= \big[\lambda\ + (\psi\ , W\tilde{g}_\Delta(H)\psi\ )\big]a + \big(\psi\ , W\tilde{g}_\Delta(H)\phi_d\big) \\ &= \omega a + (\omega\ - \omega)a + \big(\psi\ , W\tilde{g}_\Delta(H)\phi_d\big)\,, \end{aligned} \tag{3.13}$$

were

$$\omega = \lambda\ + (\psi\ , W\psi\ )\,, \tag{3.14}$$

$$\omega\ = \lambda\ + \big(\psi\ , W\tilde{g}_\Delta(H)\psi\ \big)\,. \tag{3.15}$$

e decay of $a(t)$ and $\phi\ _d$ is driven by a resonance. From equation (3.13), t e second term on t e rigt - and side of (3 .12) oscillates approximately like $e^{-i\lambda_0 t}$. Since λ lies in t e continuous spectrum of H , t is term res - onates wit t e continuous spectrum of H . o make explicit t e effect of t is resonance, we first write (3.12) as an equivalent integral equation.

$$\begin{aligned} \phi_d(t) = e^{-iH_0 t}\phi_d(0) - i\int^t e^{-iH_0\ t-s}\ a(s)P_c^{\#}W\tilde{g}_\Delta(H)\psi\ ds \\ - i\int^t e^{-iH_0\ t-s}\ P_c^{\#}W\tilde{g}_\Delta(H)\phi_d ds \\ \equiv \phi\ (t) + \phi_{res}(t) + \phi\ (t)\,. \end{aligned} \tag{3.16}$$

Our next goal is to obtain t e leading order be avior of $\phi\ _{res}(t)$. For $\epsilon > 0$ introduce t e f ollowing regularization:

$$\phi^\epsilon_{res}(t) = -i\int^t e^{-iH_0\ t-s}\ a(s)e^{\epsilon s}P_c^{\#}W\tilde{g}_\Delta(H)\psi\ ds\,. \tag{3.17}$$

en, $\phi\ ^{\epsilon}_{res}(t) \to \phi_{res}(t)$. o extract t e dominant oscillatory part of $a(t)$, we let

$$A(t) = e^{i\omega t} a(t)\,. \tag{3.18}$$

We now expand $\phi^{\epsilon}_{res}(t)$ using integration by parts,

$$\begin{aligned}
\phi^{\epsilon}_{res}(t) &= -i \int^{t} e^{-iH_0 t} e^{i\ H_0 - i\epsilon\ s} a(s) P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ ds \\
&= -i \int^{t} e^{-iH_0 t} e^{i\ H_0 - i\epsilon - \omega\ s} \big[e^{i\omega s} a(s)\big] P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ ds \\
&\equiv -ie^{-iH_0 t} \int^{t} e^{i\ H_0 - i\epsilon - \omega\ s} A(s) P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ ds \\
&= -e^{-iH_0 t}\ \big[(H\ \ - \omega - i\epsilon)^{-}\ e^{i\ H_0 - i\epsilon - \omega\ s} A(s) P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ \big|_{s=}^{s=t} \\
&\quad + e^{-iH_0 t} \int^{t} (H\ \ -\omega - i\epsilon)^{-}\ e^{i\ H_0 - i\epsilon - \omega\ s} \partial_t A(s) P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ ds\,.
\end{aligned} \tag{3.19}$$

Wit a view toward taking $\epsilon \downarrow 0$ we first note t at byypot esis (H) , since $|||W|||$ is assumed sufficiently small, weave t at $\omega \in \Delta$. e limit is t erefore singular, and we 'll find a resonant, purely imaginary, contribution coming from t e boundary term at $s = t$. Furt ermore, to study t e last term in (3.19) we will use t e e quation

$$\partial_t A = -ie^{i\omega t}\big(\psi\ , W\tilde{g}_{\Delta}(H)\phi_d\big) + i(\omega - \omega\ )A\,. \tag{3.20}$$

Now, taking $\epsilon \to 0$, we get in $L\ (\langle x\rangle^{-\ \sigma}\ dx)$,

Proposition 3.2. *The following expansion for $\phi_{res}(t)$ holds:*

$$\begin{aligned}
\phi_{res}(t) &= -a(t)(H\ \ - \omega - i0)^{-}\ P_c^{\#} W \tilde{g}_{\Delta}(H)\psi \\
&\quad + a(0)e^{-iH_0 t}(H\ \ - \omega - i0)^{-}\ P_c^{\#} W \tilde{g}_{\Delta}(H)\psi \\
&\quad - i\int^{t} e^{-iH_0\ t-s}\ (H\ \ - \omega - i0)^{-}\ P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ \cdot \big(\psi\ , W\tilde{g}_{\Delta}(H)\phi_d(s)\big) ds \\
&\quad + i(\omega - \omega\ ) \int^{t} e^{-iH_0\ t-s}\ (H\ \ - \omega - i0)^{-}\ P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ \cdot a(s) ds \\
&\equiv -a(t)(H\ \ - \omega - i0)^{-}\ P_c^{\#} W \tilde{g}_{\Delta}(H)\psi\ + \phi\ (t) + \phi\ (t) + \phi\ (t)\,.
\end{aligned} \tag{3.21}$$

Remark. o see t at t e terms in (3.21) are well defined we refer to t e proof of Proposition 2.1 in Appendix A. Localizing near and away from t e energy ω

$$\begin{aligned}
(H - \omega - i0)^{-}\ e^{-iH_0 t} P_c^{\#} &= (H - \omega - i0)^{-}\ e^{-iH_0 t} P_c^{\#} g_{\Delta} \\
&\quad + (H - \omega - i0)^{-}\ e^{-iH_0 t} \overline{g}_{\Delta}
\end{aligned}$$

$$\equiv T^t_{\Delta,} + S^t_{\Delta,} \ .$$

In Appendix A it is proved t at, for $\epsilon \geq 0$,

$$T^t_{\Delta,\epsilon}, S^t_{\Delta,\epsilon} : L \ \left(\langle x\rangle \ ^\sigma dx\right) \mapsto L \ \left(\langle x\rangle^- \ ^\sigma dx\right), \qquad t \geq 0 \,.$$

Substitution of (3.16) and (3.21) into (3.13) yields t e following equation for $a(t)$:

$$i\partial_t a(t) = \omega_* a(t) + \left(\psi \ , W\tilde{g}_\Delta(H)\{\phi \ (t) + \phi \ (t) + \phi \ (t) + \phi \ (t) + \phi \ (t)\}\right) . \tag{3.22}$$

Here,

$$\begin{aligned} \omega_* = \lambda \ &+ \left(\psi \ , W\tilde{g}_\Delta(H)\psi \ \right) \\ &- \left(\psi \ , W\tilde{g}_\Delta(H)(H \ - \omega - i0)^- \ P_c^\# W\tilde{g}_\Delta(H)\psi \ \right) . \end{aligned} \tag{3.23}$$

In order see t e resonant decay we must first consider t e be avior of t e *complex* frequency ω_* for small $|||W|||$. e next proposition contains an expression for ω_* wic depends explicitly on t e "data " of t e resonance problem, H and W, plus a controllable error.

PROPOSITION 3.3.

$$\omega_* = \lambda \ + (\psi \ , W\psi \ ) - \Lambda - i\Gamma + E(W)\,, \tag{3.24}$$

where

$$\begin{aligned} \Gamma &= \pi\left(W\psi \ , \delta(H \ - \omega)(I - P \ )W\psi \ \right), \\ \Lambda &= \left(W\psi \ , \text{P.V.}\ (H \ - \omega)^- \ W\psi \ \right), \\ E(W) &\leq C \ |||W||| \ \ , \end{aligned} \tag{3.25}$$

where ω is given by (2.13).

e term, Γ, in (3.25) is t e Fermi golden rule appearing in resonance ypot esis (W3) ($\Gamma \ \neq 0$).

e proof of Proposition 3.3 is a lengt y computation wic we present in Appendix C.

We conclude t is section wit a summary of t e coupled e quations for $\phi_d(t)$ and $a(t)$.

PROPOSITION 3.4.

$$i\partial_t a = \omega_* a + \left(\psi \ , W\tilde{g}_\Delta(H)\{\phi \ + \phi \ + \phi \ + \phi \ + \phi \ \}\right), \tag{3.26}$$

$$\begin{aligned} \phi_d(t) = e^{-iH_0 t}\phi_d(0) &- i\int^t e^{-iH_0 \ t-s} \ a(s) P_c^\# W\tilde{g}_\Delta(H)\psi \ ds \\ &- i\int^t e^{-iH_0 \ t-s} \ P_c^\# W\tilde{g}_\Delta(H)\phi_d(s)ds\,, \end{aligned} \tag{3.27}$$

where

$$\phi\ (t) = e^{-iH_0 t} P_c^{\#} \phi_d(0) \tag{3.28}$$

$$\phi\ (t) = -i \int^t e^{-iH_0\ t-s}\ P_c^{\#} W \tilde{g}_\Delta(H) \phi_d(s) ds \tag{3.29}$$

$$\phi\ (t) = -a(0) e^{-iH_0 t} (H\ \ - \omega - i0)^-\ P_c^{\#} W \tilde{g}_\Delta(H) \psi \tag{3.30}$$

$$\phi\ (t) = -i \int^t e^{-iH_0\ t-s}\ (H\ \ - \omega - i0)^-\ P_c^{\#} W \tilde{g}_\Delta(H) \psi \cdot \big(\psi\ , W \tilde{g}_\Delta(H) \phi_d(s)\big) ds \tag{3.31}$$

$$\phi\ (t) = i(\omega - \omega\ ) \int^t e^{-iH_0\ t-s}\ (H\ -\omega - i0)^-\ P_c^{\#} W \tilde{g}_\Delta(H) \psi\ \cdot a(s) ds \tag{3.32}$$

o prove t e main t eorem we estimate $a(t)$ and $\phi_d(t)$ from (3.26)-(3.32). Note t at since Im $\omega_* \sim -\mathrm{Im}\,\Gamma$ is negative, it is evident t at t is resonant contributionas t e effect of driving $a(t)$ to zero.

REMARK. Alt oug weave t e general result of eorem 2.1, in a given example it may prove beneficial to analyze t e system (3.26)-(3.32) directly in order to exploit special structure.

REMARK. Using t e above expansion and definitions, weave

$$\phi(t) = \ e^{-i\omega_* t} a(0) \psi\ + e^{-iH_0 t} (I - P\ ) g_\Delta \phi + \left[\tilde{g}_\Delta(H) - g_\Delta(H\ )\right] \left[e^{-i\omega_* t} a(0) \psi\ + e^{-iH_0 t} P_c^{\#} \phi\ \right] + \mathcal{R}(t)\,, \tag{3.33}$$

were

$$\mathcal{R}(t) = \tilde{g}(H) \left[\sum_{j=} R_j(t) \psi\ + \phi_{res}(t) + \phi\ (t) \right]. \tag{3.34}$$

See (4.10) and (4.8) for t e definition of R_j. e expansion in part (c) of eorem 2.1 is obtained by estimates of t e terms in (3.33) and (3.34). ese estimates are carried out in sections 4 and 5.

In t e next t wo sections we estimate t e solution over various time scales.

4 L al eayf l s

In t is section we begin our analysis of t e large time be avior of solutions. o prove local decay, we introduce t e norms

$$[a](T) \equiv \sup_{\leq s \leq T} \langle s \rangle^\alpha |a(s)| \quad \text{and} \quad [\phi_d]_{LD}(T) \equiv \sup_{\leq s \leq T} \langle s \rangle^\alpha \big\| \langle x \rangle^{-\sigma} \phi_d(s) \big\|\ ,$$

for wic we seek to obtain upper bounds t at are uni form in $T \in \mathbb{R}$. Because of terms like $\phi_j(t)$, $j = 2, 3, 4$ (see Proposition 3.4) and t e singular local decay estimate of Proposition 2.1, it is natural study t ese norms wit $\alpha = r - 1$. In t is section, it turns out t at we require t e restriction on α, $1 < \alpha < 3/2$. us, t rougout t is section we s all assume t e constraints

$$\alpha \equiv r - 1\,, \quad 1 < \alpha < 3/2\,.$$

In section 5 we relax t e upper bound on α.

Remark. In t e estimates immediat ely below and in subsequent sections we s all re quire bounds on t e following quantities like $\|\langle x\rangle^a W \tilde{g}_\Delta(H)\langle x\rangle^b\|$ wit $a, b \in \{0, \sigma\}$. at all t ese can be controlled in terms of t e norm $|||W|||$ is ensured by t e following proposition, wic is proved in Appendix B.

Proposition 4.1. *For* $a, b \in \{0, \sigma\}$,

$$\big\|\langle x\rangle^a W \tilde{g}_\Delta(H)\langle x\rangle^b\big\| \le C_{a,b}|||W|||\,. \tag{4.1}$$

We begin by estimating t e local decay norm of ϕ_d.

Local decay estimates for $\phi_d(t)$. From equation (3.27)

$$\begin{aligned}\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\| \;&\le \big\|\langle x\rangle^{-\sigma}e^{-iH_0 t}\phi_d(0)\big\| \\ &\quad+ \int^t \big|a(s)\big|\big\|\langle x\rangle^{-\sigma}e^{-iH_0\ t-s}\ P_c^{\#}W\tilde{g}_\Delta(H)\psi\ \big\|\ ds \\ &\quad+ \int^t \big\|\langle x\rangle^{-\sigma}e^{-iH_0\ t-s}\ P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(s)\big\|\ ds \\ &\le C\langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\| \ + C\big\|\langle x\rangle^{\sigma}W\tilde{g}_\Delta(H)\psi\ \big\| \int^t \langle t-s\rangle^{-r}\big|a(s)\big|ds \\ &\quad+ \big\|\langle x\rangle^{\sigma}W\tilde{g}_\Delta(H)\langle x\rangle^{\sigma}\big\|\int^t \langle t-s\rangle^{-r}\big\|\langle x\rangle^{-\sigma}\phi_d(s)\big\|\ ds\,.\end{aligned} \tag{4.2}$$

is implies, for $0 \quad \le t \le T$,

$$\begin{aligned}&\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\| \;\le C\langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\| \\ &+ C\langle t\rangle^{-\alpha}\big(\|\langle x\rangle^{\sigma}W\tilde{g}_\Delta(H)\psi\ \|\ [a](T) + \big\|\langle x\rangle^{\sigma}W\tilde{g}_\Delta(H)\langle x\rangle^{\sigma}\big\|[\phi_d]_{LD}(T)\big) \\ &\quad\le C\ \langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\| \ + C\ |||W|||\langle t\rangle^{-\alpha}\big([a](T) + [\phi_d]_{LD}(T)\big)\,.\end{aligned} \tag{4.3}$$

It follows t at

$$[\phi_d]_{LD}(T) \le C\ \big\|\langle x\rangle^{\sigma}\phi_d(0)\big\| \ + C\ |||W|||\big([a](T) + [\phi_d]_{LD}(T)\big)\,, \tag{4.4}$$

and t eref ore

$$\big(1 - C\ |||W|||\big)[\phi_d]_{LD}(T) \le C\ \big\|\langle x\rangle^{\sigma}\phi_d(0)\big\| \ + C\ |||W|||[a](T)\,. \tag{4.5}$$

An additional simple consequence of (4.2) and t e ort ogonalit y of t e decomposition (3.10), is

$$\left\|\langle x\rangle^{-\sigma}\phi_d(t)\right\| \ \le C\langle t\rangle^{-r}\left\|\langle x\rangle^{\sigma}\phi_d(0)\right\| \ + C|||W|||||\phi\ || \ . \tag{4.6}$$

Estimation of $a(t)$. We estimate $a(t)$ using equation (3.26). is equationas t e form

$$i\partial_t a = \omega_* a + \sum_{j=} F_j\,, \tag{4.7}$$

were

$$F_j(t) \equiv \left(\psi\ \ , W\tilde{g}_\Delta(H)\phi_j\right). \tag{4.8}$$

erefore,

$$a(t) = e^{-i\omega_* t}a(0) + \sum_{j=} R_j(t)\,, \tag{4.9}$$

were

$$R_j(t) = -i\int^t e^{-i\omega_*\ t-s}\ F_j(s)ds\,. \tag{4.10}$$

We next estimate eac R_j. In t e course of carrying out t e analysis we s all fre quently apply t e following :

LEMMA 4.1. *Let Γ, α and β denote real numbers such that $\Gamma > 0$ and $\beta > 1$. Define*

$$I_{\alpha,\beta}(t) = \langle t\rangle^\alpha \int^t e^{-\Gamma\ t-s}\ \langle s\rangle^{-\beta}ds\,. \tag{4.11}$$

Then,

(i)
$$I_{\alpha,\beta}(t) \le C(\langle t\rangle^\alpha e^{-\frac{1}{2}\Gamma t} + \langle t\rangle^{\alpha-\beta}\Gamma^{-}\ )\,. \tag{4.12}$$

(ii) *If $\alpha \le \beta$, we have*

$$\sup_{t\ge} I_{\alpha,\beta}(t) \le C(\Gamma^{-\alpha} + \Gamma^{-}\ )\,. \tag{4.13}$$

o prove t is lemma, note t at

$$\begin{aligned} I_{\alpha,\beta}(t) &= \langle t\rangle^\alpha\left(\int^{t/} + \int_{t/}^{t}\ \right)\{\cdots\}ds \\ &\le \langle t\rangle^\alpha e^{-\frac{1}{2}\Gamma t}\int^{t/}\ \langle s\rangle^{-\beta}ds + C\langle t\rangle^{\alpha-\beta}\int_{t/}^{t}\ e^{-\Gamma\ t-s}\ ds\,. \end{aligned}$$

Part (i) follows by explicitly carrying out t e integrals, using t at $\beta\ > 1$, and part (ii) follows by noting t at t e supremum over $t \ge 0$ of t e expres - sion obtained in (i).

Estimation of $R_0(t)$.

$$R\ (t) = -i\int^{t} e^{-i\omega_*\ t-s}\ \big(\psi\ , W\tilde{g}_\Delta(H)e^{-iH_0 s}P_c^{\#}\phi_d(0)\big)ds\,. \tag{4.14}$$

Estimation of t e integrand gives

$$\begin{aligned}\big|(\psi\ , W\tilde{g}_\Delta(H)e^{-iH_0 s}\phi_d(0))\big| &= \big|(\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ , \langle x\rangle^{-\sigma}e^{-iH_0 s}\phi_d(0))\big| \\ &\le \big\|\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ \big\|\ \big\|\langle x\rangle^{-\sigma}e^{-iH_0 s}\phi_d(0)\big\| \\ &\le C\big\|\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ \big\|\ \big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ \langle s\rangle^{-r} \\ &\le C|||W|||\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ \langle s\rangle^{-r}\,. \end{aligned} \tag{4.15}$$

Use of (4.15) in (4.14) yields

$$\big|R\ (t)\big| \le C|||W|||\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ \int^{t}\ e^{-\Gamma\ t-s}\ \langle s\rangle^{-r}ds\,. \tag{4.16}$$

Multiplication of (4.16) by $\langle t\rangle^{\alpha}$, use of Lemma 4.1 and t e lower bound for Γ, (2.8), yields t e bound

$$\langle t\rangle^{\alpha}\big|R\ (t)\big| \le C|||W|||\ ^{-\ \alpha}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ \ ,\quad t\ge 0\,. \tag{4.17}$$

It also follows from (4.16), since $r > 1$, t at

$$\big|R\ (t)\big| \le C|||W|||\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ \ . \tag{4.18}$$

Estimation of $R_1(t)$. We must bound t e expression

$$R\ (t) = -\int^{t} e^{-i\omega_*\ t-s}\ \left(\psi\ , W\tilde{g}_\Delta(H)\int^{s} e^{-iH_0\ \ s-\tau}\ P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)d\tau\right)ds. \tag{4.19}$$

is can be rewritten as

$$\begin{aligned} R\ (t) = \int^{t}\ e^{-i\omega_*\ t-s}\ ds\bigg(&\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ , \\ &\int^{s}\langle x\rangle^{-\sigma}e^{-iH_0\ \ s-\tau}\ P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)d\tau\bigg)\,. \end{aligned} \tag{4.20}$$

wic satisfies t e bound

$$\begin{aligned}\big|R\ (t)\big| \le C\big\|\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ \big\|\ &\int^{t}\ e^{-\Gamma\ t-s}\ ds \\ &\cdot\int^{s}\big\|\langle x\rangle^{-\sigma}e^{-iH_0\ \ s-\tau}\ P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)\big\|d\tau\,. \end{aligned} \tag{4.21}$$

Use of t e assumed local decay estimate (H4) gives t at $R\ (t)$ is bounded by

$$\begin{aligned} C\big\|\langle x\rangle^{\sigma}\tilde{g}_\Delta(H)W\psi\ \big\|\ \big\|\langle x\rangle^{\sigma}W\tilde{g}_\Delta(H)\langle x\rangle^{\sigma}\big\|&\int^{t} e^{-\Gamma\ t-s}\ ds \\ &\cdot\int^{s}\langle s-\tau\rangle^{-r}\big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|\ \ d\tau\,, \end{aligned} \tag{4.22}$$

and t eref ore

$$\big|R\ (t)\big| \le C|||W|||\ \int^{t} e^{-\Gamma\ t-s}\ ds \int^{s} \langle s-\tau\rangle^{-r}\langle\tau\rangle^{-\alpha} d\tau[\phi_d]_{LD}(T)\,. \tag{4.23}$$

Using Lemma 4.1, we ave

$$\langle t\rangle^{\alpha}\big|R\ (t)\big| \le C\big(1+|||W|||\ ^{-\ \alpha}\big)[\phi_d]_{LD}(T)\,. \tag{4.24}$$

Furt ermore, use of (4.6) in (4.22) gives

$$\big|R\ (t)\big| \le C|||W|||\|\langle x\rangle^{\sigma}\phi\ \|\ \ ,\quad t\ge 0\,. \tag{4.25}$$

Estimation of $R_2(t)$.

$$R\ (t) = ia(0)\int^{t} e^{-i\omega_*\ t-s}$$

$$\cdot\big(\psi\ , W\tilde g_\Delta(H)e^{-iH_0 s}(H\ \ -\omega-i0)^{-}\ P_c^{\#}W\tilde g_\Delta(H)\psi\ \big)ds\,. \tag{4.26}$$

erefore, by Proposition 2.1,

$$\big|R\ (t)\big| \le C\big|a(0)\big||||W|||\ \int^{t} e^{-\Gamma\ t-s}\ \langle s\rangle^{-r}\quad ds\,. \tag{4.27}$$

A first simple consequence, since $r>2$, is t at

$$\big|R\ (t)\big| \le C\big|a(0)\big||||W|||\ \ . \tag{4.28}$$

Next, multiplication of (4.27) by $\langle t\rangle^{\alpha}$, taking supremum over t e interval $0\le t\le T$ and applying Lemma 4.1 yields t e bound

$$\langle t\rangle^{\alpha}\big|R\ (t)\big| \le C\ \big|a(0)\big|\big(1+|||W|||\ ^{-\ \alpha}\big)\,, \tag{4.29}$$

Estimation of $R_3(t)$. We begin by recalling

$$R\ (t) = -i\int^{t}\ e^{-i\omega_*\ t-s}\ F\ (s)ds\,. \tag{4.30}$$

erefore,

$$\big|R\ (t)\big| \le C\int^{t} e^{-\Gamma\ t-s}\ \big|F\ (s)\big|ds\,. \tag{4.31}$$

$F\ (t) = (\psi\ , W\tilde g_\Delta(H)\phi\ (t))$ is given explicitly by t e expression

$$-\ i\int^{s} d\tau\big(\psi\ , W\tilde g_\Delta(H)e^{-iH_0\ s-\tau}\ (H\ \ -\omega-i0)^{-}\ P_c^{\#}W\tilde g_\Delta(H)\psi\ \big)$$

$$\times\ \big(\psi\ , W\tilde g_\Delta(H)\phi_d(\tau)\big)$$

$$= -i\int^{s}\ d\tau\big(\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi\ , \langle x\rangle^{-\sigma}e^{-iH_0\ s-\tau}\ (H\ \ -\omega-i0)^{-}\ P_c^{\#}\langle x\rangle^{-\sigma}$$

$$\cdot\ \langle x\rangle^{\sigma}W\tilde g_\Delta(H)\psi\ \big)\times\big(\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi\ , \langle x\rangle^{-\sigma}\phi_d(\tau)\big) \tag{4.32}$$

Estimation of $F\ (t)$ yields t e bound

$$\big|F\ (t)\big| \le C_W \int^s \big\|\langle x\rangle^{-\sigma} e^{-iH_0\ t-\tau}\ (H\ \ -\omega - i0)^-\ P_c^{\#}\langle x\rangle^{-\sigma}\big\| \\ \cdot \big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|\ \ d\tau\,, \quad (4.33)$$

were

$$C_W = \big\|\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi\ \big\|\ \ \cdot \big\|\langle x\rangle^{\sigma}W\tilde g_\Delta(H)\psi\ \big\|\ \ \le C|||W|||\ \ . \quad (4.34)$$

By Proposition 2.1 and (4.33)-(4.34),

$$\big|F\ (s)\big| \le C|||W|||\ \int^s \langle s-\tau\rangle^{-r}\quad \big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|\ d\tau\,. \quad (4.35)$$

If we bound $||\langle x\rangle^{-\sigma}\phi_d(\tau)||$ simply by $||\phi\ ||$ we obtain, from (4.35) and (4.31),

$$\big|R\ \ (t)\big| \le C|||W|||\,||\phi\ ||\ \ . \quad (4.36)$$

On t e ot er and, bounding $||\langle x\rangle^{-\sigma}\phi_d(\tau)||$ by $[\phi_d]_{LD}(T)\ \langle\tau\rangle^{-\alpha}$ $(\alpha = r-1)$ in (4.35) we obtain

$$\big|F\ (s)\big| \le C|||W|||\ \langle s\rangle^{-\alpha}[\phi_d]_{LD}(T)\,. \quad (4.37)$$

Finally, using (4.37) in (4.31) and applying Lemma 4.1 we ave

$$\langle t\rangle^{\alpha}\big|R\ \ (t)\big| \le C\big(|||W||| + |||W|||\ \ ^{-\ \alpha}\big)[\phi_d]_{LD}(T)\,. \quad (4.38)$$

Estimation of $R_4(t)$.

$$\begin{aligned} R\ \ (t) &= -i\int^t\ e^{-i\omega_*\ t-s}\ \ \big(\psi\ \ , W\tilde g_\Delta(H)\phi\ \ (s)\big)ds \\ &= (\omega - \omega\ \ )\int^t e^{-i\omega_*\ t-s}\ \Big(\tilde g_\Delta(H)W\psi\ \ , \\ &\qquad \int^s a(\tau)e^{-iH_0\ s-\tau}\ P_c^{\#}W\tilde g_\Delta(H)\psi\ \ \Big)d\tau \end{aligned}$$

By Proposition 2.1,

$$\begin{aligned} \big|R\ \ (t)\big| &\le |\omega - \omega\ \ |\,|||W|||\ \int^t e^{-\Gamma\ t-s}\ ds\int^s\langle s-\tau\rangle^{-r}\quad \big|a(\tau)\big|d\tau \\ &\le |\omega - \omega\ \ |\,|||W|||\ \int^t e^{-\Gamma\ t-s}\ \langle s\rangle^{-\alpha}ds[a](t)\,. \end{aligned}$$

We now estimate t e $|\omega - \omega\ \ |$. By (3.14)-(3.15),

$$\omega\ \ -\omega = \big(\psi\ \ , W\tilde g_\Delta(H)\psi\ \ \big) - \big(\psi\ \ , W\psi\ \ \big) \equiv \beta\,. \quad (4.39)$$

An explicit expression, (9.6), is obtained for β in Appendix C,

$$\beta = -\big(W\psi\ \ , B\bar g_\Delta(H)(H-\lambda\ \ )^-\ W\psi\ \ \big)\,. \quad (4.40)$$

Fromeorem 11. 1 of Appendix E and an argument along t e lines of t e proof of (4.1) weave $|\beta| \leq C|||W|||$. erefore, using Lemma 4.1, we find

$$\langle t\rangle^{\alpha}\big|R\ (t)\big| \leq C|||W|||\ ^{-\ \alpha}[a](T)\,. \tag{4.41}$$

If $\alpha < 3/2$, t en

$$\langle t\rangle^{\alpha}\big|R\ (t)\big| \leq C|||W|||[a](T)\,. \tag{4.42}$$

Closing the estimates and completion of the proof. We can now combine t e upper bounds (4. 17), (4.24), (4.29), (4.38) and (4.41) for t e $R_j(t), 0 \leq j \leq 4$ to obtain, via (4.9), t e following upper bound for $a(t)$ provided $|||W||| < 1/2$:

$$[a](T) \leq c\ \big|a(0)\big|\,|||W|||^{-\ \alpha} + c\ |||W|||\ ^{-\ \alpha}\big|\langle x\rangle^{\sigma}\phi\ \| \\ + c\ \big(1 + |||W|||\ ^{-\ \alpha}\big)[\phi_d]_{LD}(T)\,.$$

Substitution of t is bound into (4.4) gives t e f ollowing bound for ϕ_d:

$$[\phi_d]_{LD}(T) \leq C\ \big(1 + |||W|||\ ^{-\ \alpha}\big)\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ + c\ |||W|||\ ^{-\ \alpha}\big|a(0)\big| \\ + C\ \big(|||W||| + |||W|||\ ^{-\ \alpha}\big)[\phi_d]_{LD}(T)\,. \tag{4.44}$$

Use of (4.44) as a bound for t e last term in (4.43) yields a bound for $[a](T)$,

$$[a](T) \leq c\ \big|a(0)\big|\,|||W|||^{-\ \alpha} + c\ |||W|||\ ^{-\ \alpha}\big\|\langle x\rangle^{\sigma}\ \phi_d(0)\big\|\ . \tag{4.45}$$

Finally, for $|||W|||$ sufficiently small and $\alpha < 3/2$ weave

$$[\phi_d]_{LD}(T) \leq C\big(1 + |||W|||\ ^{-\ \alpha}\big)\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|\ + C|||W|||\ ^{-\ \alpha}\big|a(0)\big|\,. \tag{4.46}$$

aking $T \to \infty$ we conclude t e decay of $\phi(t)$, wit initial data ϕ in t e range of $P_{\Delta}(H)$, wit rate $\langle t\rangle^{-\alpha}$, $0 < \alpha < 3/2$. It follows [RSim] t at t e interval Δ consists of absolutely continuous spectrum of H, as asserted in parts (a) and (b) of eorem 2.1.

5 L al eayf l s f Lage r

In t e preceding subsection we proved t e decay of solutions, $\phi(t\ , x)$, in t e local decay sense, wit a slow rate of decay $\langle t\rangle^{-\alpha}$ wit $1 < \alpha < 3/2$; $\alpha = r - 1$. A consequence of t is result is t at, in t e interval Δ, t e spectrum of H is absolutely continuous. Now if Δ contains no t res olds of H, we expect decay as $t \to \infty$ at a rate wic is f aster t an any polynomial. (For example, t is is wat oneas for constant coefficient dispersive equations for energy intervals containing no points of stationary pase.) In t is section we s ow t at t is result olds in t e sense of (2.17) in eorem 2.1. is re quires some adaptation of t e met ods of section 4. We s all indicateere only t e re quired modifications to t e argument of t e previous section.

(1) e origin of t e restriction $\alpha < 3/2$ can be traced to our application of part (ii) of Lemma 4.1. In particular, in obtaining (4.13) we use t at

$$\sup_{t\geq} \langle t\rangle^{\alpha} e^{-\Gamma t} = \mathcal{O}(\Gamma^{-\alpha})\,. \tag{5.1}$$

It follows t at certain coefficients are f ound to be large for $|||W|||$ small, an obstruction to closing t e system of estimates for $[a]$ and $[\phi_d]_{LD}$, unless $\alpha < 3/2$. is is remedied by taking t e supremum in (5.1) over t in t e interval $[\Gamma^{- \ -\delta}, T]$, were $\delta > 0$.

LEMMA 5.1. *Let* $M \equiv \Gamma^{- \ -\delta} \sim |||W|||^{- \ \delta}$ *; see* (W3). *There exists* $\theta_* > 0$ *such that if* $|||W||| < \theta_*$ *and* $t \geq M$, *then*

(a) $\langle t\rangle^{r-} \int^t e^{-\Gamma \ t-s} \langle s\rangle^{-r} ds \leq C\Gamma^{\delta} \sim |||W||| \ ^{\delta}$.

(b) $\langle t\rangle^{r-} \int^t e^{-\Gamma \ t-s} \langle s\rangle^{-\alpha} ds \leq C\Gamma^{-}$.

(2) Assume $r > 2$ $(\alpha > 1)$. e analysis of section 4 yields a coupled system of integral inequalities for t e functions $a(t)$ and

$$L(t) \equiv \left\|\langle x\rangle^{-\sigma}\phi_d(t)\right\| \ . \tag{5.2}$$

e precise form of t ese ine qualities can be seen as follows. Let

$$I_r\{L\}(t) = \int^t \langle t-s\rangle^{-r} L(s) ds\,. \tag{5.3}$$

en, by (4.2), (4.9) and t e estimates f or $R_j(t)$, $j = 0,1,2,3,4$, t e inequalities for $L(t)$ and $a(t)$ take t e form

$$\begin{aligned} L(t) &\leq C \ \langle t\rangle^{-r} + C \ |||W|||I_r\{|a|\}(t) + C \ |||W|||I_r\{L\}(t) \\ |a(t)| &\leq A \ e^{-\Gamma t} + A \ |||W|||I_r\{e^{-\Gamma s}\}(t) + A \ |||W||| \ I_{r-} \ \{e^{-\Gamma s}\}(t) \\ &\quad + A \ |||W||| \int^t e^{-\Gamma \ t-s} \ I_r\{L\}(s) ds \\ &\quad + A \ |||W||| \int^t e^{-\Gamma \ t-s} \ I_{r-} \ \{L\}(s) ds \\ &\quad + A \ |||W||| \int^t e^{-\Gamma \ t-s} \ I_{r-} \ \{|a|\}(s) ds\,, \end{aligned} \tag{5.4}$$

were t e C_j and A_j denote positive constants.

(3) e procedure is first to consider t e functions $L(t)$ and $a(t)$ on a large but finite time interval, $0 \leq t \leq \Gamma^{- \ -\delta} \equiv M$, were δ is positive and suitably c osen. An explicit bound for $L(t)$ and $a(t)$ can be found by

iteration of t e ine qualities (5.4). For t is, we use t e following estimate of $I_r\{e^{-\Gamma s}\}$, wic is proved using integration by parts,

$$I_r\{e^{-\Gamma s}\} \leq c\ e^{-\Gamma t} + \sum_{k=}^{r-} c_k\ \Gamma^{k-}\ \langle t\rangle^{-r\quad k-}\ + c_{r-}\ \Gamma^{r-\ -\rho}\langle t\rangle^{-}\ , \tag{5.5}$$

were $\rho >\ \ 0$ is arbitrary.

(4) o s ow decay for arbitrary, in particular, large $\alpha = r - 1$, and t e estimates of $\mathcal{R}(t)$ of eorem 2.1, we introduce t e norms

$$[a]^{\Gamma}(T) \equiv \sup_{M\leq t\leq T} \langle t\rangle^{\alpha}\big|a(t)\big| \tag{5.6}$$

and

$$[\phi_d]^{\Gamma}_{LD}(T) \equiv \sup_{M\leq t\leq T} \langle t\rangle^{\alpha}\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\|\,. \tag{5.7}$$

We now reexpress t e system (5.4) for $L(t)$ and $a(t)$ by breaking t e time integrals in (5.4) into a part over t e interval $[\ 0, M]$ and a part over t e interval $[M, t]$. Using t e estimate of part (3) above, t e integrals over $[0, M]$ are estimated to be of order $|||W|||^{\varepsilon}\langle t\rangle^{-r}$ for some $\varepsilon = \varepsilon(\delta)$.

In t is way, t e resulting system for $L(t)$ and $a(t)$ is now reduced to one wic can be studied using Lemma 5. 1 and t e approac of section 4. Using t is approac estimates for t e norms (5.6) and (5.7), and consequently of $\mathcal{R}(t)$ can be obtained.

6 xamples a ppl a s

In t is section we sketc examples and applications of eorem 2.1. Most of t ese examples ave been previously studied, under more stringent y-pot eses on H and W, *e.g.* some type of analyticity: dilation analyticity for t e Helium atom, translation analyticity for t e Stark Hamiltonian; see [CyFKS] and references cited t erein. eorem 2.1 enables us to relax t is requirement and gives t e detailed time-be avior of solutions near t e res - onant energy at all time-scales. Example 5 concerns t e instabilit y of an eigenvalue embedded at a t res old, a result wic we believe is new and not tractable by tec ni ques of dilation analyticity.

We begin wit t e remark t at in t e examples below, one can often replace t e operator, $-\Delta$ by $H \ \equiv \omega(p)$, were $p \ = -i\nabla$. e necessary ypot esis on local decay, (H4), is reduced to its verification for $H \ + V$. By t e general discussion of local decay estimates of Appendix D (see also [Si3]), we ave

Theorem 6.1. *The operator $H = H + V$ satisfies the required local decay estimates of (H4) under the following hypotheses:*

Hypotheses on $\omega(p)$:

(i) *$\omega(p)$ is real valued and $\omega(p) \to \infty$ as $|p| \to \infty$.*
(ii) *$\omega(p)$ is C^m function, $m \geq 4$.*
(iii) *$\nabla_p \omega = 0$ on at most finitely many points, in any compact domain.*

Hypotheses on $V(x)$: $V(x)$ is real valued and such that

(V1) *$V(x)$, $x \cdot \nabla V$, $(x \cdot \nabla) V$, $(x \cdot \nabla) V$ are all $g(H)$ bounded for $g \in C^\infty$.*
(V2) *$|V(x)| = 0(\langle x \rangle^{-\epsilon})$, $\epsilon > 0$, $|x| \to \infty$.*
(V3) *$\chi_R V$, $\chi_R(x \cdot \nabla)^m V$, $m{=}1, 2, 3$ are $g(H)$-compact, for $\chi_R \equiv \chi_{[R,\infty}(|x|)$ with some $R > 0$.*

e proof of t is result follows from t e procedure outlined in Appendix D were we use t eypot eses on $\omega(p)$ and V and t e c oice for t e operator A is

$$A = -(x \cdot \nabla_p \omega + \nabla_p \omega \cdot x) .$$

Remark. Due to lack of assumptions on analyticity of $\omega(p)$ or $V(x)$ one cannot simply apply t e tecni que of analytic deformation used in ot er approac es.

Example 1: *Dispersive Hamiltonian.* Wit t e above assumptions on $\omega(p)$ and $V(x)$, eorem 2.1 applies directly to t e operator $H \equiv \omega(p) + V(x)$.

Example 2: *Direct Sum.* Let

$$H = \begin{pmatrix} -\Delta_x & 0 \\ 0 & -\Delta_x + q(x) \end{pmatrix}$$

acting on $\mathbb{C} \otimes L (\mathbb{R}^n)$, were $q(x)$ is a well be aved potential aving some positive discrete eigenvalues. An example of t is t ype is considered in [W].

Consider, for example, t e case were $q(x) = P(x)$, is a polynomial wic is bounded below. I n t is case, t e spectrum of $-\Delta_x + P(x)$ is discrete and consists of an infinite set of eigenvalues $\lambda < \lambda \cdots$ wit corresponding eigenfunctions ψ , ψ , ... e spectrum of H is t en

$$\{\text{eigenvalues of } -\Delta_x + P(x)\} \cup [0, \infty)$$

and t erefore H as nonnegative eigenvalues embedded in its continuous spectrum.

Let

$$W = \begin{pmatrix} 0 & W(x) \\ W(x) & 0 \end{pmatrix}$$

wit W satisfying conditions (W).

Theorem 6.2. *For H and W as above, if for some strictly positive simple eigenvalue $\lambda > 0$ the resonance condition (Fermi golden rule) (2.7) holds, then in an interval Δ around λ, the spectrum of H is absolutely continuous and the other conclusions of Theorem 2.1 hold. Furthermore, if $n > 4$, Theorem 2.1 holds even when $\lambda = 0$ is an eigenvalue.*

Proof. In t is case local decay must be proved for $-\Delta_x$, wit $r > 2$. is is well known. Wat is more, i f t e spatial dimension is larger t an four, $n > 4$, t en $\lambda = 0$ is also allowed, since in t is case we use

$$\begin{aligned}\left\|\langle x\rangle^{-\frac{n}{2}-\epsilon} e^{i\Delta_x t}\psi\right\| &\le \left\|\langle x\rangle^{-\frac{n}{2}-\epsilon}\right\| \; \|e^{i\Delta_x t}\psi\|_\infty \\ &\le Ct^{-n/} \; \|\psi\| \; .\end{aligned}$$

Hence, for $\psi \in \mathcal{D}(\langle x\rangle^{n/ \quad \epsilon})$, and $n > 4$ we ave t e necessary decay,

$$\left\|\langle x\rangle^{-\frac{n}{2}-\epsilon} e^{i\Delta_x t}\langle x\rangle^{-\frac{n}{2}-\epsilon}\right\| \le Ct^{-n/}$$

wit $r = n/2 > 2$.

EXAMPLE 3: *Tensor Products.* Let $H = \mathbf{1} \otimes h + h \otimes \mathbf{1}$ act on $L(\mathbb{R}^n_{x_1}) \otimes L(\mathbb{R}^n_{x_2})$, were

$$h = -\Delta_{x_1} \quad \text{and} \quad h = -\Delta_{x_2} + q(x\ ) \; . \tag{6.1}$$

en,

$$\sigma(H\ ) = \{\lambda : \lambda = \lambda + \lambda\ ,\ \lambda \in \sigma(-\Delta_{x_1}) \text{ and } \lambda \in \sigma(-\Delta_{x_2} + q(x\ ))\} \, . \tag{6.2}$$

Let $W(x\ , x\ )$ act on $L \otimes L$, satisfying $(\mathbf{W})$, wit $\langle x\rangle \equiv 1 + |x\ | + |x\ |$.
en we ave

Theorem 6.3. *The embedded eigenvalues of H are unstable and Theorem 2.1 holds.*

EXAMPLE 4: *Helium Type Hamiltonians* [RSim]. Consider H as in Example 3 wit

$$h = -\Delta_{x_1} - |x\ |^{-}\ , \quad h = -\Delta_{x_2} - |x\ |^{-}\ . \tag{6.3}$$

Also, let W be of t e f orm

$$W(x\ , x\ ) = W(x\ - x\ )\, .$$

In t is case t e weigt $\langle x\rangle = 1 + |x\ | + |x\ |$. We now discuss t eypot - esis (W).

H as infinitely many negative eigenvalues embedded in t e continuous spectrum [CyFKS]. If Δ is a subinterval of t e negative real line containing exactly one negative eigenvalue, E, t en g_Δ is a sum of terms of t e form

$$g^c_{\Delta-E}(h\ ) \otimes \mathbf{P} \quad \text{and} \quad \mathbf{P} \otimes g^c_{\Delta-E}(h\ )\, . \tag{6.4}$$

Here, $g^c_{\Delta-E}(h_j)$ is a spectral projection onto t e continuous spectral part associated wit an interval $\Delta - E$, t e translate of Δ by $-E$, and $\mathbf{P}$ denotes a (negative) bound state projection. us, $g \quad _{\Delta}$ localizes eit er t e x or t e x variable, and so wile $\langle x \rangle\ ^{\sigma} W$ is not bounded we doave t at

$$\langle x \rangle\ ^{\sigma} W g_{\Delta}(H\ )$$

is bounded provided, for example, W is s ort range.

In t e case, were W is long range, *i.e.*

$$W(x\ \ - x\ \ ) = \mathcal{O}\big(\langle x\ \ - x\ \ \rangle^{-}\ \big) \tag{6.5}$$

we first prove a *minimal velocity bound* and t en use it to get local decay.

Going back to (3.27) we estimate

$$\left\| F\left(\tfrac{|x|}{t} \leq \eta\right) \phi_d(t) \right\| \tag{6.6}$$

using t e known propagation and minimal velocit y estimates for H [SiSo]. e problematic term, wic is t e last term on t e rigt - and side of (3.27) is t en bounded by

$$c \int^{t} \langle t-s \rangle^{-\ -\varepsilon} \| \langle x \rangle^{\frac{1}{2}\ \ \delta} W \tilde{g}_{\Delta}(H) \| \ \left\| F\left(\tfrac{|x|}{s} \leq \eta\right) \phi_d(s) \right\| \ ds$$

$$+ c \int^{t} \langle t-s \rangle^{-\ -\varepsilon} \left\| \langle x \rangle^{\frac{1}{2}\ \ \delta} W \tilde{g}_{\Delta}(H) F\left(\tfrac{|x|}{s} \geq \eta\right) \right\| \ \| \phi_d(s) \| \ ds\,.$$

Since

$$\left\| \langle x \rangle^{\frac{1}{2}\ \ \delta} W \tilde{g}_{\Delta}(H) F\left(\tfrac{|x|}{s} \geq \eta\right) \right\| \leq c|\ \ ||W|||\langle s \rangle^{-\frac{1}{2}\ \ \delta}\,, \tag{6.7}$$

we can close t e ine qualities and obtain

$$\langle t \rangle^{\frac{1}{2}-\delta} \left\| F\left(\tfrac{|x|}{t} \leq \eta\right) \phi_d(t) \right\| \leq \ c\ \ + c \ \sup_{\leq s \leq t} \left| \langle s \rangle^{\frac{1}{2}-\delta} a(s) \right|\ \ . \tag{6.8}$$

e above estimate, toget er wit t e estimates for $a(t)$ lead to local decay wit a rate $\langle t \rangle^{-\ \ /\ \ \delta}$. is rate is not sufficient to preclude singular continuous spectrum. However, t e Mourre estimateolds in t is interval for $H\ + W$ wic implies local decay and absence of singular continuous spectrum; seeeorem 10.1 and eorem 10.2 in Appendix D.

EXAMPLE 5: *Threshold Eigenvalues.* Let $H\ = -\Delta + V(x)$ in $L\ (\mathbb{R}^n)$, $n > 4$. Assume $V(x)$ is smoot and rapidly decaying for simplicit y. en, under certain conditions on t e spectrum of H , and t e beavior of its resolvent at zero energy, one can prove local decay and L^∞ decay wit a rate $r > 2$; see [JeK], [JoSoSog].

In suc cases, it f ollows byeorem 2.1 t at a t res old eigenvalue at $\lambda = 0$, if it exists, is unstable wit respect to small and generic perturbations, W.

EXAMPLE 6: *Stark Effect - an atom in a uniform electric field.* e Stark Hamiltonian is given by

$$H = -\Delta + V(x) + \vec{E} \cdot x \tag{6.9}$$

acting on L $(\mathbb{R}^n)$. If $V(x)$ is real valued and not too singular, t en for $\vec{E} \neq 0$ t e continuous spectrum of H is $(-\infty, \infty)$. o see t is apply eorem 9.1 wit $A \equiv \vec{E} \cdot p$, $p = -i\nabla$. us, if H as an eigenvalue, it is necessarily embedded in t e continuous spectrum.

Our results can be used to s ow t at any embedded eigenvalue is gener - ically unstable (*i.e.* provided t e Fermi golden rule resonance condition (W3) olds) and perturbs to a resonance.

o see t is, one can proceed by a decoupling argument; see [CHi]. is reduces t e problem to a direct sum of Hamiltonians, as in Example 2, wit Hamiltonians of t e f orm

$$\begin{aligned} H &= -\Delta + \vec{E} \cdot x + \tilde{V}(x, -i\nabla)\,, \\ H &= -\Delta + V_b(x)\,, \\ W &= \tilde{W}(x, -i\nabla)\,. \end{aligned}$$

e strategy is t en to use t e tec ni ques of Appendix B to verifyy- pot eses (W) and t e tec ni ques of Appendix D to prove t e necessary local decay estimates in (H) for t e t e operator H $= \mathrm{diag}(H\ , H\ )$.

EXAMPLE 7: *The Radiation Problem.* e radiation problem is t e fundamental problem wic motivated work on quantum resonances. See t e work of Weisskopf and Wigner [WeiWi], following Dirac [Di] and Landau [L]. We present ere a very brief description of t e problem and t e relation to our met ods. For a more detailed discussion of t e formulation see [BFroSi].

e free Hamiltonian, H , is t e direct sum operator acting on $\mathcal{H}_a \oplus \mathcal{H}_{photon}$. Here, $\mathcal{H}_a$ is t e Hilbert space associated wit an atom or molecule. $\mathcal{H}_{photon}$ is t e Fock space of free potons. H is t en t e Hamiltonian of a decoupled particle and free poton system

$$H \ = H_a \otimes I + I \otimes H_{photon}\,. \tag{6.10}$$

e next step is to introduce t e interaction term W t at couples t e poton-radiation field to t e atom. In quantum electrodynamics, t is cou- pling is given by t e standard minimal coupling, but in general it is suf- ficient to consider a simple approximation *e.g.* t e dipole approximation [AlE]. e goal is to s ow t at all eigenvalues of t e original atom, except t e ground state, are destabilized by t e coupling and become resonances.

is is t e penomenon of spontaneous emission. One is also interested in t e computation of t e *li fetime* and t e transition probabilities.

A simplified Hamiltonian wic incorporates t e essential mat ematical features of t e radiation problem is

$$H = H_a \otimes I + I \otimes H_{photon} + \lambda W\,, \tag{6.11}$$

were $H\ _a = -\Delta + V(x)$ acting on $L\ (\mathbb{R}^n)$ describes t e atom, and t e coupling is given by

$$W = \int \left(\omega(\vec{k})\right)^{-\ /\ } \left(g(\vec{k})e^{i\vec{k}x}a_{\vec{k}} + \overline{g}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}a^{\dagger}_{\vec{k}}\right)d\vec{k}\,. \tag{6.12}$$

e Hamiltonian associated wit t e poton field is given by

$$H_{photon} = \int \omega(\vec{k})a^{\dagger}_{\vec{k}}a_{\vec{k}}d\vec{k}\,, \tag{6.13}$$

t e second quantization of multiplication in Fourier space by $\omega(\vec{k})$ in $L\ (\mathbb{R}^n)$. Hence, H_{photon} acts on t e Fock space of bosons,

$$\mathcal{F} = \oplus_{m=}^{\infty}\ \otimes_{sym}^{m} L\ (\mathbb{R}\ )\,, \tag{6.14}$$

were $\otimes_{sym}^{m}$ denotes t e m-fold symmetric tensor product of $L\ (\mathbb{R}\ )$. e operator $a^{\dagger}_{\vec{k}}$ is t e creation operator on $\mathcal{F}$ and $a_{\vec{k}}$, its adjoint.

For realistic potons, we must ave $g \equiv 1$ and $\omega(\vec{k}) \sim |\vec{k}|$ for $\vec{k}$ near zero. However, to make mat ematical sense of t e above Hamiltonian we need to introduce t e *ultraviolet cutoff*;

$$g = 0 \quad \text{for} \quad |\vec{k}| \gg 1\,. \tag{6.15}$$

Wen t e coupling constant λ is zero, and so $H = H$, it is fairly easy to verify our conditions (H) for H , even in t e massless poton case, $\omega(\vec{k}) = |\vec{k}|$ [BFroSi],[BFSS]. e conditions (W) owever fail wen $\omega(k) = |\vec{k}|$ since in t is case t e interaction $\lambda\ W$ is not localized. On t e ot er and, in t e massive case $\left(\omega(\vec{k}) = \sqrt{m\ + |\vec{k}|\ }\,,\ m \neq 0\right)$, t e interaction is localized for quite general $g(\vec{k})$; see [Ge]. In t is case, our conditions (W) can be verified and t eref ore t e results of eorem 2.1 can be applied.

7 ppe x : f f L al eayps 2.

Our aim is to prove local decay estimates for $e^{-iH_0 t}(H\ - \Lambda - i0)^{-}\ P_c^{\#}$ using t e given local decay estimates for $e^{-iH_0 t}P_c^{\#}$, were $\Lambda \in\ \mathbf{T}^{\#}$. e proof is split into two parts: analysis near Λ and analysis away from Λ.

Let Δ be a small interval about Λ and g_Δ denote a smoot ed out car - acteristic function of Δ and $\overline{g}_\Delta = 1 - g_\Delta$. We write

$$e^{-iH_0 t}(H \;\; - \Lambda - i0)^- \;\; P_c^{\#} = e^{-iH_0 t}(H \;\; - \Lambda - i0)^- \;\; P_c^{\#}(g_\Delta + \overline{g}_\Delta)$$
$$\equiv T_\Delta^t + S_\Delta^t\,.$$

We first estimate t e operator $T\,{}_\Delta^t$. Let $\varepsilon > 0$ and set

$$T_{\Delta,\varepsilon}^t = e^{-i \;\; H_0 - \Lambda - i\varepsilon \;\; t}(H \;\; - \Lambda - i\varepsilon)^- \;\; P_c^{\#} g_\Delta\,.$$

en, by (H4),

$$T_{\Delta,\varepsilon}^t = i\int_t^\infty e^{-i \;\; H_0 - \Lambda - i\varepsilon \;\; s} P_c^{\#} g_\Delta\, ds\,.$$

Let $\langle x\rangle^\sigma h \in L$. en,

$$\begin{aligned}
\big\|\langle x\rangle^{-\sigma} T_{\Delta,\varepsilon}^t h\big\| \;\; &\le \int_t^\infty \big\|\langle x\rangle^{-\sigma} e^{-i \;\; H_0 - \Lambda - i\varepsilon \;\; s} \;\; P_c^{\#} g_\Delta h\big\| \;\; ds \\
&\le \int_t^\infty e^{-\varepsilon s}\big\|\langle x\rangle^{-\sigma} e^{-iH_0 s} P_c^{\#} g_\Delta h\big\| \;\; ds \\
&\le \int_t^\infty e^{-\varepsilon s}\langle s\rangle^{-r}\big\|\langle x\rangle^{\sigma} h\big\| \;\; ds \\
&\le C\langle t\rangle \;\; ^{-r}\big\|\langle x\rangle^{\sigma} h\big\| \;\; .
\end{aligned}$$

erefore, taking $\varepsilon \downarrow 0$, we get

$$\big\|\langle x\rangle^{-\sigma} T_\Delta^t h\big\| \;\; \le C\langle t\rangle \;\; ^{-r}\big\|\langle x\rangle^{\sigma} h\big\| \;\; .$$

o estimate S_Δ^t, we exploit t at t e energy is localized away from Λ, and so t e resolvent $(H \;\; - \Lambda)^-$ is bounded,

$$\begin{aligned}
\langle x\rangle^{-\sigma} S_\Delta^t \langle x\rangle^{-\sigma} &= \langle x\rangle^{-\sigma} e^{-iH_0 t}(H \;\; - \Lambda - i0)^- \;\; P_c^{\#}\overline{g}_\Delta\langle x\rangle^{-\sigma} \\
&= \langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^{\#}\langle x\rangle^{-\sigma}\cdot\langle x\rangle^{\sigma}(H \;\; - \Lambda - i0)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma}\,.
\end{aligned} \tag{7.1}$$

For t e operator norm we t enave t e bound

$$\begin{aligned}
\big\|\langle x\rangle^{-\sigma} S_\Delta^t\langle x\rangle^{-\sigma}\big\| \le \big\|\langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^{\#}\langle x\rangle^{-\sigma}\big\| \\
\cdot\big\|\langle x\rangle^{\sigma}(H \;\; - \Lambda - i0)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma}\big\|\,.
\end{aligned} \tag{7.2}$$

We bound t e first factor in (7.2) using t e assumed local decay estimate (H4). e second factor is controlled as follows. Note t at

$$\begin{aligned}
\langle x\rangle^{\sigma}(H \;\; - \Lambda - i0)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma} = \langle x\rangle^{\sigma}(H \;\; + c)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma} \\
+ (\Lambda + c)\langle x\rangle^{\sigma}(H \;\; + c)^- \;\; \langle x\rangle^{-\sigma}\cdot\langle x\rangle^{\sigma}(H \;\; - \Lambda - i0)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma}\,.
\end{aligned} \tag{7.3}$$

aking operator norms and usingypot esis **(H5)** andeorem 11.2 of Appendix E we obtain t e following bound on t e second factor in (7.2)

$$\big(1 - |\Lambda + c|\|\langle x\rangle^{\sigma}(H \;\; + c)^- \;\; \langle x\rangle^{-\sigma}\|\big)\big\|\langle x\rangle^{\sigma}(H \;\; - \Lambda - i0)^- \;\; \overline{g}_\Delta\langle x\rangle^{-\sigma}\big\| \le$$

$$\left\|\langle x\rangle^{\sigma}(H +c)^{-} \overline{g}_{\Delta}\langle x\rangle^{-\sigma}\right\| \leq \left(1+\|\langle x\rangle^{\sigma}g_{\Delta}\langle x\rangle^{-\sigma}\|\right)\left\|\langle x\rangle^{\sigma}(H +c)^{-} \langle x\rangle^{-\sigma}\right\|.$$

is completes t e proof .

8 ppe x B: peam smaes vlv g $g_{\Delta}(H)$

In t is section we prove Propositions 3.1 and 4.1. ese propositions require some operator calculus.

Let $\hat{h}(\lambda)$ denote t e Fourier trans form of t e function g, wit t e nor - malization,

$$\hat{h}(\mu) = (2\pi)^{-} \int e^{i\mu\lambda}h(\mu)d\mu .$$

Proof of Proposition 3.1. Recall t at λ denotes an embedded eigenvalue of t e unperturbed operator, H , g_{Δ} is a smoot ed out c aracteristic function of t e interval Δ, and I is an open set wic contains t e support of P and is disjoint from Δ.

We need to s ow t at

$$B = \left(I - g_{\Delta}(H)g_I(H)\right)^{-} \tag{8.1}$$

is bounded and we do t is by s owing t at $\|g_{\Delta}(H)g_I(H)\|$ as small norm. We use tec ni ques of [SiSo].

Let $\tilde{\Delta}$ be an interval wic contains and is sligtly larger t an Δ. en

$$\begin{aligned} g_{\Delta}(H)g_I(H) &= g_{\Delta}(H)\left(I - g_{\tilde{\Delta}}(H)\right)g_I(H) \\ &= g_{\Delta}(H)g_{\Delta'}(H)g_I(H) \\ &= g_{\Delta}(H)\left(g_{\Delta'}(H) - g_{\Delta'}(H)\right)g_I(H) , \end{aligned} \tag{8.2}$$

were Δ and Δ' are disjoint.

We now obtain, an expression for t e above difference, wic is easily estimated. Using t e Fourier trans form weave t at

$$g_{\Delta'}(H) - g_{\Delta'}(H) = \int (e^{i\mu H_0} - e^{i\mu H})\hat{g}_{\Delta'}(\mu)d\mu . \tag{8.3}$$

Furt ermore,

$$\begin{aligned} e^{i\mu H_0} - e^{i\mu H} &= (I - e^{i\mu H}e^{-i\mu H_0})e^{i\mu H_0} \\ &= -\int^{\mu} \tfrac{d}{ds}e^{isH}e^{-isH_0}\,ds\, e^{i\mu H_0} \\ &= -\int^{\mu} e^{isH}i(H - H)e^{-isH_0}\,ds\, e^{i\mu H_0} \\ &= -i\int^{\mu} e^{isH}We^{-isH_0}\,ds\, e^{i\mu H_0} . \end{aligned} \tag{8.4}$$

Substitution of (8.4) into (8.3) yields

$$g_{\Delta'}(H\ ) - g_{\Delta'}(H) = -i \int \hat{g}_{\Delta'}(\mu) e^{i\mu H} d\mu \int^{\mu} e^{-isH} W e^{isH_0} ds\,. \tag{8.5}$$

We now apply t e operator $g_{\Delta}(H)$ to t e expression in (8.5) and estimate

$$\begin{aligned} \big\|g_{\Delta}(H)(g_{\Delta'}(H\ ) - g_{\Delta'}(H)\big\| &\leq \int \big|\hat{g}_{\Delta'}(\mu)\big| \int^{\mu} \big\|g_{\Delta}(H)W\big\| ds\, d\mu \\ &\leq \int \big|\hat{g}_{\Delta'}(\mu)\big| |\mu| d\mu \big\|g_{\Delta}(H)W\big\| \\ &\leq C|\Delta|^{-}\ \big\|g_{\Delta}(H)W\big\| \leq C|\Delta|^{-}\ |||W|||\,. \end{aligned} \tag{8.6}$$

erefore,

$$\big\|g_{\Delta}(H)g_I(H\ )\big\| \leq C|\Delta|^{-}\ |||W|||\,.$$

and $(I - g_{\Delta}(H)g_I(H\ ))^{-}$ is bounded provided $|\Delta|^{-}\ |||W||| < \theta$ is sufficiently small; see (W4).

Proof of Proposition 4.1. We estimate t e norm of t e operator

$$\mathcal{G} = \langle x\rangle^{\sigma} W \tilde{g}_{\Delta}(H) \langle x\rangle^{\sigma} \tag{8.7}$$

in terms of $|||W|||$, defined in (W2).

Recall t at by (3.1 1)

$$\tilde{g}_{\Delta}(H) = g_{\Delta}(H)\big(I - g_{\Delta}(H)g_I(H\ )\big)^{-}\ \overline{g}_I(H\ )\,. \tag{8.8}$$

Using (8.8) we express $\mathcal{G}$ as t e product of operators

$$\begin{aligned} &\langle x\rangle^{\sigma} W \tilde{g}_{\Delta}(H) \langle x\rangle^{\sigma} = \mathcal{G}\ \cdot \mathcal{G}\ \cdot \mathcal{G} \\ &\equiv \langle x\rangle^{\sigma} W g_{\Delta}(H) \langle x\rangle^{\sigma} \cdot \langle x\rangle^{-\sigma}\big[I - g_{\Delta}(H)g_I(H\ )\big]^{-}\ \langle x\rangle^{\sigma} \cdot \langle x\rangle^{-\sigma}\overline{g}_I(H\ )\langle x\rangle^{\sigma}\,. \end{aligned} \tag{8.9}$$

eref ore it suffices to obtain upper bounds for $\|\mathcal{G}_j\|$, $j = 1, 2, 3$. We s all use some general operator calculus estimates of Appendix E, especially eorem 11.1.

Bound on $\mathcal{G}_3$: is follows from eorem 11.1 of Appendix E, wit $A = H$ and $\varphi = \overline{g}_I$, a function wic is smoot and rapidly decaying at infinity.

Bound on $\mathcal{G}_2$: By our ypot eses and t e proof of Proposition 3.1, $\|g_{\Delta}(H)g_I(H\ )\|$ is small and

$$\big(I - g_{\Delta}(H)g_I(H\ )\big)^{-}\ = \sum_{n=}^{\infty} \big(g_{\Delta}(H)g_I(H\ )\big)^{n} \tag{8.10}$$

converges in t e norm. We need to s ow t is in t e weigted norms. For t is, we will s ow t at t e norm of $g\ \ _{\Delta}(H)g_I(H\ )$ is small in t e weigted norm, *i.e.*

$$\langle x\rangle^{\sigma} g_{\Delta}(H) g_I(H\ )\langle x\rangle^{-\sigma} = \mathcal{O}\big(|||W|||\big)\,. \tag{8.11}$$

Since t e supports of $g\ _{\Delta}$ and g_I are disjoint

$$\begin{aligned}\langle x\rangle^{\sigma} g_{\Delta}(H) g_I(H\ )\langle x\rangle^{-\sigma} &= \langle x\rangle^{\sigma}\big(g_{\Delta}(H) - g_{\Delta}(H\ )\big) g_I(H\ )\langle x\rangle^{-\sigma}\\ &= \langle x\rangle^{\sigma}\big(g_{\Delta}(H) - g_{\Delta}(H\ )\big)\langle x\rangle^{-\sigma}\cdot\langle x\rangle^{\sigma} g_I(H\ )\langle x\rangle^{-\sigma}\,.\end{aligned}$$

By parts (a) and (b), respectively, of eorem 11.1 bot

$$\begin{aligned}\big\|\langle x\rangle^{\sigma} g_I(H\ )\langle x\rangle^{-\sigma}\big\| &< \infty \text{ and}\\ \langle x\rangle^{\sigma}\big(g_{\Delta}(H) - g_{\Delta}(H\ )\big)\langle x\rangle^{-\sigma} &= \mathcal{O}\big(|||W|||\big)\,.\end{aligned} \tag{8.12}$$

Bound on $\mathcal{G}_1$: Expanding about t e unperturbed operator, $H\ $, we ave

$$\begin{aligned}\mathcal{G}\ &= \langle x\rangle^{\sigma} W g_{\Delta}(H)\langle x\rangle^{\sigma}\\ &= \langle x\rangle^{\sigma} W (H+c)^{-}\ \langle x\rangle^{\sigma}\cdot\langle x\rangle^{-\sigma}(H+c) g_{\Delta}(H\ )\langle x\rangle^{\sigma}\,.\end{aligned} \tag{8.13}$$

aking norms, we get

$$\|\mathcal{G}\ \| \le \big\|\langle x\rangle^{\sigma} W (H+c)^{-}\ \langle x\rangle^{\sigma}\big\|\cdot\big\|\langle x\rangle^{-\sigma}(H+c) g_{\Delta}(H\ )\langle x\rangle^{\sigma}\big\| \tag{8.14}$$

Consider t e first factor in (8.14). We s ow t at it is of order $|||W|||$ as $|||W|||\to 0$. Note t at

$$\begin{aligned}\langle x\rangle^{\sigma} W (H+c)^{-}\ \langle x\rangle^{\sigma} = \langle x\rangle^{\sigma} W (H\ +c)^{-}\ \langle x\rangle^{\sigma} &- \langle x\rangle^{\sigma} W (H\ +c)^{-}\ \langle x\rangle^{-\sigma}\\ &\cdot\langle x\rangle^{\sigma} W (H+c)^{-}\ \langle x\rangle^{\sigma}\,.\end{aligned}$$

aking norms we obtain

$$\begin{aligned}\big(1 - \|\langle x\rangle^{\sigma} W (H\ +c)^{-}\ \langle x\rangle^{-\sigma}\|\big)\big\|\langle x\rangle^{\sigma} W (H+c)^{-}\ \langle x\rangle^{\sigma}\big\|&\\ \le \big\|\langle x\rangle^{\sigma} W (H\ +c)^{-}\ \langle x\rangle^{\sigma}\big\|&\,.\end{aligned} \tag{8.15}$$

erefore, if $|||W||| < 1/2$ t e first factor of (8.14) is bounded by $2|||W|||$. e second factor of (8.14) is bounded byeorem 11.1.

Finally, we note t at t e above bounds on $\mathcal{G}_j$ complete t e proof of Proposition 4.1.

9 ppe x : x pa s fhe mplex Feqey, ω_*

In t is section we prove Proposition 3.3, in wic an expansion of t e complex frequency, ω_*, is presented. In particular, our goal will be to obtain an expansion of ω_* wic is explicit to second order in t e perturbation, W, wit an error term of order $|||W|||\ $.

Recall t at

$$\omega_* = \lambda \ + \omega_A - \omega_B \, , \tag{9.1}$$

were

$$\omega_A = \big(\psi \ , W\tilde{g}_\Delta(H)\psi \ \big) = \omega \ - \lambda \quad \text{(see (3.15))}\, , \tag{9.2}$$

and

$$\omega_B = \big(W\psi \ , \tilde{g}_\Delta(H) R_{H_0}(\omega + i0) P_c^{\#} W \tilde{g}_\Delta(H)\psi \ \big)\, . \tag{9.3}$$

Expansion of ω_A:

$$\begin{aligned}
\omega_A &\equiv \big(\psi \ , W\tilde{g}_\Delta(H)\psi \ \big) \\
&= (\psi \ , W\psi \ ) + \big(W\psi \ , B[g_\Delta(H) - g_\Delta(H \ )]\psi \ \big) \\
&\equiv (\psi \ , W\psi \ ) + \beta\, .
\end{aligned} \tag{9.4}$$

In wat follows, we s all fre quently use t e notation $(H - \lambda)^-$ and $\frac{}{H-\lambda}$ interc angeably.

PROPOSITION 9.1.

$$\big[g_\Delta(H) - g_\Delta(H \ )\big]\psi \ = -\bar{g}_\Delta(H - \lambda \ )^- \ W\psi \tag{9.5}$$

Proof. Noting t at $H \ - H \ = W$, weave t e expansion formula

$$\begin{aligned}
g_\Delta(H) - g_\Delta(H \ ) &= \int \hat{g}_\Delta(\lambda)(e^{i\lambda H} - e^{i\lambda H_0}) d\lambda \\
&= \int \hat{g}_\Delta(\lambda) e^{i\lambda H}(1 - e^{-i\lambda H} e^{i\lambda H_0}) d\lambda \\
&= i \int \hat{g}_\Delta(\lambda) e^{i\lambda H} \int^\lambda e^{-isH} W e^{isH_0} ds\, d\lambda\, .
\end{aligned}$$

We next apply t is expansion to ψ , were $H \quad \psi \ = \lambda \ \psi$ and obtain

$$\begin{aligned}
\big(g_\Delta(H) - g_\Delta(H \ )\big)\psi \ &= i \int \hat{g}_\Delta(\lambda) e^{i\lambda H} \int^\lambda e^{-isH} W e^{is\lambda_0} \psi \ ds\, d\lambda \\
&= i \int \hat{g}_\Delta(\lambda) e^{i\lambda H} \int^\lambda e^{-isH \quad is\lambda_0} W\psi \ ds\, d\lambda \\
&= i \int \hat{g}_\Delta(\lambda) e^{i\lambda H} \frac{e^{-i\lambda H \quad i\lambda\lambda_0} - 1}{-iH + i\lambda} W\psi \ d\lambda \\
&= -\int \hat{g}_\Delta \frac{e^{i\lambda\lambda_0}}{H - \lambda} d\lambda\, W\psi \ + \int \hat{g}_\Delta(\lambda) \frac{e^{i\lambda H}}{H - \lambda} W\psi \ d\lambda \\
&= -g_\Delta(\lambda \ ) \frac{1}{H - \lambda} W\psi \ + g_\Delta(H) \frac{1}{H - \lambda} W\psi \\
&= -\big(1 - g_\Delta(H)\big) \frac{1}{H - \lambda} W\psi \ , \quad \big(g_\Delta(\lambda \ ) = 1\big) \\
&\equiv -\bar{g}_\Delta(H)(H - \lambda \ )^- \ W\psi \ .
\end{aligned}$$

is completes t e proof of t e proposition.

Substitution of (9.5) into t e above expression f or β yields

$$\beta = -\big(W\psi\ , B\bar{g}_\Delta(H)(H-\lambda\ )^{-}\ W\psi\ \big)\,. \tag{9.6}$$

Let $h(\lambda)$ be a function wic is equal to one on t e support of $\bar{g}_\Delta$ and is zero outside a small neigbor ood of t e support of $\bar{g}_\Delta$. eref ore, $(H\ -\lambda\ )^{-}\ h(H\ )$ is bounded. A computation yields

Proposition 9.2.

$$\begin{aligned}
\omega_A &= (\psi\ , W\psi\ ) + \beta \\
&= (\psi\ , W\psi\ ) - \big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\lambda\ )^{-}\ \ W\psi\ \big) \qquad (9.7)\\
&\quad - \big(W\psi\ , \bar{g}_\Delta(H)(H-\lambda\ )^{-}\ (h(H) - I)W\psi\ \big)\\
&\quad - \big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\lambda\ )^{-}\ [h(H) - h(H\ )]W\psi\ \big)\\
&\quad + \big(W\psi\ , [g_\Delta(H) - g_\Delta(H\ )]h(H)(H-\lambda\ )^{-}\ W\psi\ \big)\\
&\quad + \big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\lambda\ )^{-}\ W(H-\lambda\ )^{-}\ h(H)W\psi\ \big)\\
&\quad - \big(W\psi\ , Bg_\Delta(H)g_I(H\ )\bar{g}_\Delta(H)(H-\lambda\ )^{-}\ W\psi\ \big)\,. \qquad (9.8)
\end{aligned}$$

Note also t at t e second term in (9.7) can be expressed as

$$\begin{aligned}
\big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\lambda\ )^{-}\ W\psi\ \big) &= \big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\omega)^{-}\ W\psi\ \big)\\
&\quad - (W\psi\ , \psi\ )\cdot\big(W\psi\ , \bar{g}_\Delta(H\ )(H\ -\lambda\ )^{-}\ (H\ -\omega)^{-}\ W\psi\ \big) \qquad (9.9)
\end{aligned}$$

Expansion of ω_B. Let $R_H(\lambda) \equiv (H-\lambda)^{-}$. Recall t at ω_B is given by t e expression

$$\omega_B = \big(W\psi\ , \tilde{g}_\Delta(H)R_{H_0}(\omega + i0)P_c^{\#}W\tilde{g}_\Delta(H)\psi\ \big)\,,$$

and $\tilde{g}_\Delta(H) = Bg_\Delta(H)(I - P\ )$. We find after some computation

Proposition 9.3.

$$\begin{aligned}
\omega_B &= \big(W\psi\ , g_\Delta(H\ )R_{H_0}(\omega+i0)P_c^{\#}W\psi\ \big)\\
&\quad + \big(W\psi\ , [B-I]g_\Delta(H\ )R_{H_0}(\omega+i0)P_c^{\#}WBg_\Delta(H)\psi\ \big)\\
&\quad + \big(W\psi\ , [g_\Delta(H) - g_\Delta(H\ )]R_{H_0}(\omega+i0)P_c^{\#}W\tilde{g}_\Delta(H)\psi\ \big)\\
&\quad + \big(W\psi\ , g_\Delta(H\ )R_{H_0}(\omega+i0)P_c^{\#}WB[g_\Delta(H) - g_\Delta(H\ )]\psi\ \big)\\
&\quad + \big(W\psi\ , (B-I)[g_\Delta(H) - g_\Delta(H\ )]\overline{g}_I(H\ )R_{H_0}(\omega+i0)P_c^{\#}WBg_\Delta(H)\psi\ \big)\,.
\end{aligned} \tag{9.10}$$

Here, we ave used t at $B\psi\ = \psi$. More generally, $(B-I)g_\Delta(H\ ) = Bg_\Delta(H)g_I(H\ )g_\Delta(H\ ) = 0$, and t eref ore t e second term in (9.10) is zero.

It follows from (9.4), (9.9) and (9.10) t at

$$\omega_* = \lambda\ + \omega_A - \omega_B$$

$$\equiv \lambda \ + (\psi \ , W\psi \ ) - (\Lambda + i\Gamma) + \sum_{j=} E_j \, , \tag{9.11}$$

were

$$\Lambda + i\Gamma = \big(W\psi \ , \bar{g}_\Delta(H \ )(H \ - \omega)^- \ W\psi \ \big)$$
$$+ \big(W\psi \ , g_\Delta(H \ )(H \ - \omega - i0)^- \ P_c^\# W\psi \ \big) \, . \tag{9.12}$$

and

$$\begin{aligned}
E \ &= (W\psi \ , \psi \ ) \cdot \big(W\psi \ , \bar{g}_\Delta(H \ - \lambda \ )^- \ (H \ - \omega)^- \ W\psi \ \big) \\
E \ &= \big(W\psi \ , (H - \lambda \ )^- \ \big(h(H) - I\big)\bar{g}_\Delta(H)W\psi \ \big) \\
E \ &= -\big(W\psi \ , \bar{g}_\Delta(H \ )(H \ - \lambda \ )^- \ [h(H) - h(H \ )]W\psi \ \big) \\
E \ &= \big(W\psi \ , [g_\Delta(H) - g_\Delta(H \ )]h(H)(H - \lambda \ )^- \ W\psi \ \big) \\
E \ &= \big(W\psi \ , \bar{g}_\Delta(H \ )(H \ - \lambda \ )^- \ W(H - \lambda \ )^- \ h(H)W\psi \ \big) \\
E \ &= \big(W\psi \ , Bg_\Delta(H)g_I(H \ )[g_\Delta(H \ ) \\
&\qquad - g_\Delta(H)]\bar{g}_I(H \ )R_{H_0}(\omega + i0)P_c^\# WBg_\Delta(H)\psi \ \big) \\
E_7 &= \big(W\psi \ , g_\Delta(H \ )R_{H_0}(\omega + i0)P_c^\# WB[g_\Delta(H \ ) - g_\Delta(H)]\psi \ \big) \\
E \ &= \big(W\psi \ , [g_\Delta(H \ ) - g_\Delta(H)]R_{H_0}(\omega + i0)P_c^\# WBg_\Delta(H)\psi \ \big) \\
E \ &= \big(W\psi \ , B[g_\Delta(H \ ) - g_\Delta(H)]g_I(H \ )\bar{g}_\Delta(H)(H - \lambda \ )^- \ W\psi \ \big)
\end{aligned}$$

We now claim t at t e terms E_j, $j = 1, \ldots, 9$, are all of order $|||W|||$. Consider first $E \ = E^a \cdot E^b$. Estimation of t e first factor gives

$$|E^a| \leq C|||W||| \, , \tag{9.13}$$

by Proposition 4.1.

Estimation of t e second factor gives

$$\begin{aligned}
|E^b| &= \big|(Wg_\Delta(H \ )\psi \ , \bar{g}_\Delta(H \ )(H \ - \lambda \ )^- \ (H \ - \omega)^- \ Wg_\Delta(H \ )\psi \ )\big| \\
&\leq \big\|\langle x\rangle^\sigma Wg_\Delta(H \ )\psi \ \big\| \ \big\|\langle x\rangle^{-\sigma}\bar{g}_\Delta(H \ )(H \ - \lambda \ )^- \ (H \ - \omega)^- \ \langle x\rangle^{-\sigma}\big\| \\
&\leq C|||W||| \ \big\|\langle x\rangle^{-\sigma}\bar{g}_\Delta(H \ )(H \ - \lambda \ )^- \ (H \ - \omega)^- \ \langle x\rangle^{-\sigma}\big\| \leq C|||W||| \ ,
\end{aligned}$$

byeorem 11.1. erefore, $|E \ | \leq C|||W|||$.

e term E is zero; $(h - 1)\bar{g}_\Delta \equiv 0$ since $h \equiv 1$ on t e support of $\bar{g}_\Delta$.

e term E can be treated by t e same type of estimates as E . e remaining terms are E_j, $j = 2, 3, 4, 6, 7, 8, 9$. Eac of t ese expressions as two explicit occurrences of t e perturbation, W, as well as a difference of operators: $g_\Delta(H) - g_\Delta(H \ )$ or $h(H) - h(H \ )$. By (8.12), t ese differences are $\mathcal{O}(|||W|||)$, so we expect eac of t ese terms to be $\mathcal{O}(|||W||| \ )$. We carry t is argument out f or t e term E_7. e ot er terms are similarly estimated.

Consider E_7. Let $\tilde{\Delta}$ be an interval properly containing Δ so restricted to t e interval Δ, $g_{\tilde{\Delta}} \equiv 1$ and $g_\Delta = g_\Delta g_{\tilde{\Delta}}$. en,

$$
\begin{aligned}
|E_7| &= \big|(\langle x\rangle^{\sigma} g_\Delta(H) W\psi \ , \langle x\rangle^{-\sigma} g_{\tilde{\Delta}}(H) R \ (\omega + i0) P_c^{\#} \langle x\rangle^{-\sigma} \\
&\qquad \cdot \langle x\rangle^{\sigma} W B [g_\Delta(H \ ) - g_\Delta(H)]\psi \ )\big| \\
&\le \big|(\langle x\rangle^{\sigma} g_\Delta(H) W\psi \ , \langle x\rangle^{-\sigma} g_{\tilde{\Delta}}(H \ ) R \ (\omega + i0) P_c^{\#} \langle x\rangle^{-\sigma} \\
&\qquad \cdot \langle x\rangle^{\sigma} W B [g_\Delta(H \ ) - g_\Delta(H)]\psi \ )\big| \\
&+ \big|(\langle x\rangle^{\sigma} g_\Delta(H) W\psi \ , \langle x\rangle^{-\sigma} [g_{\tilde{\Delta}}(H) - g_{\tilde{\Delta}}(H \ )] \langle x\rangle^{\sigma} \langle x\rangle^{-\sigma} R \ (\omega + i0) P_c^{\#} \langle x\rangle^{-\sigma} \\
&\qquad \cdot \langle x\rangle^{\sigma} W B [g_\Delta(H \ ) - g_\Delta(H)]\psi \ )\big| . \qquad (9.14)
\end{aligned}
$$

Using Proposition 4.1, eorem 11.1 and (8.12) we ave t at

$$
|E_7| \le C|||W||| \ \big\|\langle x\rangle^{-\sigma}(H \ - \omega - i0)^{-} \ P_c^{\#}\langle x\rangle^{-\sigma}\big\| . \qquad (9.15)
$$

at t e term $\|\langle x\rangle^{-\sigma}(H \ - \omega - i0)^{-} \ \langle x\rangle^{-\sigma}\|$ is finite is a consequence of Proposition 2.1 wit $t = 0$. us we ave t e f ollowing proposition from wic Proposition 3.4 follows.

Proposition 9.4.

(1) $\Lambda + i\Gamma = \big(W\psi \ , \mathrm{P.V.}(H \ - \omega)^{-} \ W\psi \ \big)$
$\qquad + i\pi\big(W\psi \ , \delta(H \ - \omega)(I - P \ )W\psi \ \big)$,

(2) $|E_j| \le C|||W||| \ , \qquad 1 \le j \le 9$,

(3) $\omega_* = \lambda \ + (\psi \ , W\psi \ ) - \Lambda - i\Gamma + E(W)$, were $|E(W)| \le C|||W||| \ .$ (9.16)

It remains to verify part (1). is follows from an application of t e well-known distributional identity

$$
(x \mp i0)^{-} \ \equiv \lim_{\varepsilon \to \ +} (x \mp i\varepsilon)^{-} \ = \mathrm{P.V.}\ x^{-} \ \pm \ i\pi\delta(x) \qquad (9.17)
$$

to t e second term in equation (9.12) and t e identity $g_\Delta(H \ )P_c^{\#} = I - P$.

0 ppe x : Ge eal pp ah L al eay smae s

Hypot esis (H4) for our main t eorem is one requiring t at our unper - turbed operator, H , satisfy a suitable local decay estimate, (2.3). In t is section we give an outline to a very general approac to obtaining suc estimates based on a tec ni que originating in t e work of Mourre [Mou]; see also [PeSiSim]. In t e following general discussion we s all let H denote self-adjoint operator on a Hilbert space, $\mathcal{H}$, keeping in mind t at our application is to t e unperturbed operator H . Let $E \in \sigma(H)$, and assume t at an operator A can be found suc t at A is self-adjoint and $\mathcal{D}(A) \cap \mathcal{H}$ is dense in $\mathcal{H}$. Let Δ denote an open interval wit compact closure. We s all use t e notation

$$\mathrm{ad}_A^n(H) = \left[\cdots [H, A], A, \cdots A\right], \tag{10.1}$$

for t e n-fold commutator.

Assume t e t wo conditions

(M1) e operators

$$g_\Delta(H)\mathrm{ad}_A^n(H) g_\Delta(H), \quad 1 \le n \le N \tag{10.2}$$

can all be extended to a bounded operator on $\mathcal{H}$.

(M2) *Mourre estimate*:

$$g_\Delta(H) i[H, A] g_\Delta(H) \ge \theta g_\Delta(H) \ + K \tag{10.3}$$

for some $\theta > 0$ and compact operator, K.

Theorem 10.1 (Mourre; see [CyFKS, eorem 4.9]). *Assume conditions* (M1)-(M2), *with* $N = 2$. *Then, in the interval* Δ, H *can only have absolutely continuous spectrum with finitely many eigenvalues of finite multiplicity. Moreover, the operator*

$$\langle A\rangle^{-} \ g_\Delta(H)(H - z)^{-} \ \langle A\rangle^{-} \tag{10.4}$$

is uniformly bounded in z, *as an operator on* $\mathcal{H}$. *If* $K = 0$, *then there are no eigenvalues in the interval* Δ.

Theorem 10.2 (Sigal–Soffer; see [SiSo],[GeSi], [HuSi]). *Assume conditions* (M1)-(M2) *with* $N \ge 2$ *and* $K = 0$. *Then, for all* $\varepsilon > 0$

$$\left\| F\left(\tfrac{|A|}{t} < \theta\right) e^{-iHt} g_\Delta(H)\psi \right\| \ \le C\langle t\rangle^{-\frac{N}{2} \ \varepsilon} \left\| |A|^{\frac{N}{2}} \psi \right\| \ , \tag{10.5}$$

and therefore

$$\left\| \langle A\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\| \ \le C\langle t\rangle^{-\sigma} \left\| |A|^{N/ \ } \psi \right\| \ , \tag{10.6}$$

for $\sigma < N/2$. *Here,* F *is a smoothed out characteristic function, and* $F(|A|/t < \theta)$ *is defined by the spectral theorem.*

Let Δ denote an open interval containing t e closure of Δ.

Corollary 10.1. *Assume that* $\langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma}$ *is bounded. Then, in the above theorems we can replace the weight* $\langle A\rangle^{-\sigma}$ *by* $\langle x\rangle^{-\sigma}$.

e strategy for using t e above results to prove local decay estimates like t at in (H4) is as follows. en

$$\begin{aligned} \left\| \langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\| \ &= \left\| \langle x\rangle^{-\sigma} g_{\Delta_1}(H) e^{-iHt} g_\Delta(H)\psi \right\| \\ &= \left\| \langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma} \cdot \langle A\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma}\right\| \cdot \left\|\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \\
&\leq C\left\|\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \\
&\leq C \ \left\|F\left(\tfrac{|A|}{t} < \theta\right)\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \\
&\quad + C \ \left\|F\left(\tfrac{|A|}{t} \geq \theta\right)\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \ .
\end{aligned} \tag{10.7}
$$

eorem 10.2 is used to obtain t e decay of t e first term on t e rigt - and side of (10.7), wile we can replace $|A|$ by θt in t e second term.

Remark. Here we return to our comment in t e introduction on t e relation between our assumption (H4) (local decay for $e^{-iH_0 t}$) and t e y- pot esis of dilation analyticit y, used in previous works. Dilation analyticity or its generalization, analytic deformation, is t e re quirement t at t e map,

$$d(\theta) : \theta \mapsto (\ e^{i\theta A} H \ e^{-i\theta A} f, f) \,, \tag{10.8}$$

as analytic continuation to a strip, for f in a dense subset of $\mathcal{H}$. Since t e n^{th} derivative of $d(\theta)$ at $\theta = 0$ is $(\mathrm{ad}_A^n(H \ )f, f)$, by t e above local decay result, t e assumption (H4) is t e re quirement t at t e mapping, $d(\theta)$ be of class C .

ppe x : We ghem smaes f F s f pea s

In Appendices A, B and C we frequently require facts and estimates of functions of a self-adjoint operator. In t is section we give some basic definitions and provide t e statements and proofs of suc estimates. We s all refer to certain known results and our basic references are [RSim] and [AmMoG].

Let A denote a self-adjoint operator wit domain $\mathcal{D}$ wic is dense in a Hilbert space $\mathcal{H}$. en we ave t at for any bounded continuous complex-valued function, $\varphi \in L \ (\mathbb{R})$,

$$\varphi(A) = \text{weak} - \lim_{\varepsilon \downarrow} \pi^{-} \int \varphi(\lambda) \Im R_A(\lambda + i\varepsilon)^{-} \, d\lambda \,, \tag{11.1}$$

were $R_A(\lambda) = (A - \lambda)^{-}$ denotes t e resolvent of A. Here and t roug- out t is section all regions of integration are assumed to be over $\mathbb{R}$ unless explicitly stated ot erwise.

Theorem 11.1. *Let $\tilde{A}$ and $\tilde{B}$ denote bounded self-adjoint operators, and let Γ be a contour in the complex plane, not passing through the origin, surrounding $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$ and lying in the strip $|\Im\zeta| < 1$.*

(a) *Let* $\psi : \mathbb{R} \to \mathbb{C}$ *be a* $W^{\,,}$ *function. Suppose*

$$\eta_{\tilde{A}} \equiv \left\| \langle x \rangle^{\sigma} \tilde{A} \langle x \rangle^{-\sigma} \right\| < - \min \left[\text{distance}(\Gamma, 0), 1 \right] . \tag{11.2}$$

Then, there exists a positive number $C \ = C \ (\|\psi\|_{W^{2,1}}, \eta_{\tilde{A}})$ *such that*

$$\left\| \langle x \rangle^{\sigma} \psi(\tilde{A}) \ \langle x \rangle^{-\sigma} \right\| < C \ \ . \tag{11.3}$$

(b) *Let* ψ *be as in part* (a). *Assume that* $\tilde{A}$ *and* $\tilde{B}$ *both satisfy condition* (11.2). *Then, there is a constant* $C \ = C \ (\|\psi\|_{W^{2,1}}, \eta_{\tilde{A}}, \eta_{\tilde{B}})$ *such that*

$$\left\| \langle x \rangle^{\sigma} [\psi(\tilde{A}) - \psi(\tilde{B})] \langle x \rangle^{-\sigma} \right\| \leq C \ \left\| \langle x \rangle^{\sigma} (\tilde{A} - \tilde{B}) \langle x \rangle^{-\sigma} \right\| . \tag{11.4}$$

e following result s ows t at t e case of unbounded self-adjoint operators is reducible toeorem 11.1.

Theorem 11.2. *Suppose that Theorem 11.1 holds, and let* Γ *and* φ *be as in Theorem 11.1. Furthermore, assume that* $x \ \varphi''(x)$ *and* $x\varphi'(x)$ *are* L *functions. Let* A *and* B *be densely defined self-adjoint operators for which* $(A + c)^{-}$ *and* $(B + c)^{-}$ *are bounded for some real number* c *and satisfy the estimate* (11.2). *Then,*

$$\left\| \langle x \rangle^{\sigma} \varphi(A) \langle x \rangle^{-\sigma} \right\| < C \ (\psi) . \tag{11.5}$$

and

$$\left\| \langle x \rangle^{\sigma} [\varphi(A) - \varphi(B)] \langle x \rangle^{-\sigma} \right\| \leq C \ (\psi) \left\| \langle x \rangle^{\sigma} (A - B) \langle x \rangle^{-\sigma} \right\| , \tag{11.6}$$

where the constants $C \ (\psi)$ *and* $C \ (\psi)$ *are as in Theorem 11.1, with* $\psi(x) \equiv \varphi(x^{-} \ - c)$.

Proof. Let $\tilde{A} = (A + c)^{-}$ and note t at $\varphi(A) = \varphi(\tilde{A}^{-} \ - c) = \psi(\tilde{A})$. It suffices to s ow t at $\psi(x) = \varphi(x^{-} \ - c)$ satisfies t eypot eses of eorem 11.1. It is simple to c eck t at $\partial_{x}^{\,j} \psi(x) \in L$, for $j = 0, 1, 2$. is proves eorem 11.2.

We now embark on t e proof of eorem 11.1. A key tool is an expansion formula for $\varphi(A)$; see Proposition 6.1.4 on page 239 of [AmMoG].

Theorem 11.3. *Let* A *be a densely defined self-adjoint operator and* φ *be as in the statement of Theorem 11.1. Then,*

$$\begin{aligned} \varphi(A) = \tfrac{}{\pi} \int \varphi(\lambda) \Im R_A(\lambda + i) d\lambda + \tfrac{}{\pi} \int \varphi'(\lambda) \Im i R_A(\lambda + i) d\lambda \\ + \tfrac{}{\pi} \int \ \tau \, d\tau \int \varphi''(\lambda) \Im i \ R_A(\lambda + i\tau) d\lambda \\ \equiv \varphi \ + \varphi \ + \varphi \ \ . \end{aligned} \tag{11.7}$$

where all integrals exist in the norm of the space of bounded operators on $\mathcal{H}$.

o prove eorem 11.1 we first obtain a simple expression for t e t ird summand in (11.7) by interc anging order of integration. We begin wit a calculation of t e τ-integral

$$\int \tau\, d\tau \Im\big(i \quad R_A(\lambda + i\tau)\big) = -\int \tau \tfrac{}{i}\big(R_A(\lambda+i\tau) - R_A(\lambda - i\tau)\big)$$

$$= -\int d\tau\, \tau \quad \big[(A-\lambda) \quad + \tau \quad\big]^{-}$$

$$= f(A;\lambda) - 1\,,$$

were

$$f(z;\lambda) = (z-\lambda)\int^{\frac{1}{z-\lambda}} (1+\mu \quad)^{-} \, d\mu\,. \tag{11.8}$$

For eac λ in t e support of φ, t e function $f(z;\lambda)$ is analytic in t e strip $|\Im z| < 1$; t is corresponds to c oice an appropriate branc of t e function $z \mapsto (z-\lambda)\arctan(z-\lambda)^{-}$. By (11.7)

$$\varphi \quad (A) = \tfrac{}{\pi}\int \varphi''(\lambda) f(A;\lambda) d\lambda\,. \tag{11.9}$$

e strategy is as f ollows:

First, we observe t at $\langle x\rangle^{\sigma}\varphi_j(\tilde{A})\langle x\rangle^{-\sigma}$ is bounded for $j = 1, 2$. is is true because $\varphi, \varphi' \in L$ and (11.2) can be used to bound t e weigted norm of t e resolvent by a convergent geometric series. erefore, it remains to bound t e operator $\varphi \quad (\tilde{A})$, were φ is given explicitly (11.9).

Lemma 11.1. *Let $\tilde{A}$ and $\tilde{B}$ denote bounded self-adjoint operators and $f(\zeta)$ be a function which is defined and analytic in a neighborhood of $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$. Let Γ be a smooth contour in the domain of analyticity of f, surrounding $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$, not passing through the origin and such that the estimate,*

$$\big\|\langle x\rangle^{\sigma} M \langle x\rangle^{-\sigma}\big\| \leq \tfrac{}{}\min_{\zeta\in\Gamma} |\zeta|\,, \tag{11.10}$$

holds with $M = \tilde{A}$ and $M = \tilde{B}$. Then, there exist positive constants C and C such that

$$\big\|\langle x\rangle^{\sigma} f(\tilde{A})\langle x\rangle^{-\sigma}\big\| \leq C \tag{11.11}$$

$$\big\|\langle x\rangle^{\sigma}[f(\tilde{A}) - f(\tilde{B})]\langle x\rangle^{-\sigma}\big\| \leq C \;\big\|\langle x\rangle^{\sigma}(\tilde{A}-\tilde{B})\langle x\rangle^{-\sigma}\big\| \tag{11.12}$$

Proof. By t e Cauc y integral f ormula we ave

$$f(\tilde{A}) = (2\pi i)^{-} \int_{\Gamma} f(\zeta)(\tilde{A} - \zeta I)^{-} \, d\zeta\,. \tag{11.13}$$

Part (a) follows by use of (11.10) to expand t e resolvent in a geometric series and by termwise estimation in t e weigted norm.

Part (b) follows by t e same met od; by (11.13) applied to $\tilde{B}$ and computation of t e difference, we get

$$f(\tilde{A}) - f(\tilde{B}) = (2\pi i)^{-} \int_{\Gamma} f(\zeta)\big[(\tilde{A} - \zeta I)^{-} - (\tilde{B} - \zeta I)^{-}\big]d\zeta$$
$$= (2\pi i)^{-} \int_{\Gamma} f(\zeta)\big[(\tilde{A} - \zeta I)^{-} (\tilde{A} - \tilde{B})(\tilde{B} - \zeta I)^{-}\big]d\zeta\,.$$

Estimation in t e weigted space yields (11.12). is completes t e proof of t e lemma .

o complete t e proof s of eorems 1 1.1 and eorem 11.2, we need to estimate t e operator $\langle x\rangle^{\sigma}\varphi\ (\tilde{A})\langle x\rangle^{-\sigma}$, were $\tilde{A}$ is t e bounded sel f-adjoint operator defined by $\tilde{A} = (A + c)^{-}$. We accomplis t is by applying t e previous lemma to t e function $f(\zeta;\lambda)$ defined in (11.8), were λ is in t e support of φ. e function $f(\zeta;\lambda)$ is analytic in t e strip $|\Im\zeta| < 1$, and Γ is, byypot esis, a contour in its domain of analyticit y, surrounding $\sigma(\tilde{A})$ (respectively, $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$,) and so t at (1 1.10) olds. en, by Lemma 11.1 weave t at $f(\tilde{A};\lambda)$ and $f(\tilde{B};\lambda)$ satisfy (11.11) and (11.12). Finally, using t e representation f ormula for φ , (11.9), weave

$$\big\|\langle x\rangle^{\sigma}\varphi\ (\tilde{A})\langle x\rangle^{-\sigma}\big\| \le C\ \|\varphi''\|_{L^1}\,,$$
$$\big\|\langle x\rangle^{\sigma}[\varphi\ (\tilde{A}) - \varphi\ (\tilde{B})]\langle x\rangle^{-\sigma}\big\| \le C\ \|\varphi''\|_{L^1}\big\|\langle x\rangle^{\sigma}[\tilde{A} - \tilde{B}]\langle x\rangle^{-\sigma}\big\|\,. \tag{11.14}$$

is completes t e proof .

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