CHAPTER 1

Expansions in Orthogonal Bases

1.1 Vector spaces

We will use without much further comment the idea of a *vector space*. Basically, a vector space is a set of *vectors* (these vectors will in fact often be functions) with the property that a linear combination of vectors is again a vector. Linear combinations involve *scalars*, which may be real numbers (for a *real vector space*) or complex numbers (for a *complex vector space*).

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a vector space, and α and β are scalars, then

$$\alpha\mathbf{u} + \beta\mathbf{v}$$

is also a vector in the space.

These linear combinations are required to satisfy a set of rules which any reasonable person would consider obvious. In particular there must be a *zero vector* $\mathbf{0}$ in the space such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u}$. See Section 9.6 of Greenberg, and in particular Definition 9.6.1, for a careful discussion of vector spaces.

For the moment we will consider only real vector spaces, returning to complex vector spaces at the end of these notes.

Example 1.1: One familiar vector space is $\mathbb{R}^n$, the set of all row vectors with n (real) components. If $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are two vectors in $\mathbb{R}^n$ then their linear combinations are constructed by making linear combinations of their components:

$$\alpha\mathbf{u} + \beta\mathbf{v} = (\alpha u_1 + \beta v_1, \ldots, \alpha u_n + \beta v_n). \tag{1.1}$$

The zero vector is $\mathbf{0} = (0, 0, \ldots, 0)$.

Example 1.2: Another example of a vector space, important for the theory of Fourier series and similar applications, is $C_p[a, b]$, the set of all piecewise continuous, real-valued functions $f(x)$ defined for $a \le x \le b$. (We defined piecewise continuity when we discussed the Laplace transform, and the concept is also is defined on page 249 of Greenberg.) As in any space of functions, the rule for linear combinations is

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \quad a \le x \le b. \tag{1.2}$$

The zero vector is the function which is identically 0: $\mathbf{0}(x) = 0$ for every x. An element f of $C_p[a, b]$ is of course a function but, since it belongs to a vector space, we may speak of it as a "vector" when we want to emphasize this context, as we did when we spoke of the zero vector above. Read Section 17.6 of Greenberg for more on $C_p[a,b]$ as a vector space.

Notation: When we speak of a general vector space in these notes we will denote typical vectors as in Example 1.1, using boldface letters: $\mathbf{u}$, $\mathbf{v}$, etc., and later $\mathbf{e}_1, \mathbf{e}_2, \ldots$. The reader should bear in mind, however, that what we say applies equally well when the vectors under consideration are functions, considered as members of a vector space like $C_p[a,b]$. When we are speaking specifically about functions we will denote them, as in Example 1.2, by the letters f, g, etc.

If there exists a finite number n of vectors, say $\mathbf{u}_1, \ldots, \mathbf{u}_n$, such that every vector $\mathbf{v}$ can be written as a linear combination of these,

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots c_n\mathbf{u}_n, \quad c_1, \ldots, c_n \text{ scalars}, \tag{1.3}$$

then the vector space is *finite dimensional.* In this case one may always choose n so small that the coefficients in (1.3) are unique, whatever the vector $\mathbf{v}$; we say then that the vector space *has dimension* n or *is* n*-dimensional*, and call the set $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of vectors a *basis* for the vector space. When no such n exists we say that the vector space is *infinite dimensional.* The space $\mathbb{R}^n$ of Example 1.1 has dimension n; a basis is $\mathbf{e}_1, \ldots, \mathbf{e}_n$, where $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ has entry 1 in the i^{th} place, with all other entries 0. The space $C_p[a,b]$ of Example 1.2 is infinite dimensional.

1.2 Inner products

An *inner product* in a vector space is a formula which assigns to any pair of vectors, say $\mathbf{u}$ and $\mathbf{v}$, a number $\langle \mathbf{u}, \mathbf{v} \rangle$, their inner product. When the vector space is $\mathbb{R}^n$ there is a familiar inner product (usually called the dot product):

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i. \tag{1.4}$$

When the vector space is $C_p[a,b]$ the most common inner product, used in the study of Fourier series, is

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx. \tag{1.5}$$

Inner products are discussed further in Section 9.6.2 of Greenberg.

An inner product must satisfy three conditions:

(IP1) Linearity: for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and numbers α, β,

$$\langle \alpha\mathbf{u} + \beta v, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle \quad \text{and} \quad \langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle. \tag{1.6}$$

(IP2) Symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for any vectors $\mathbf{u}, \mathbf{v}$.

(IP3) Positivity $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for any vector $\mathbf{u}$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$.

It is easy to check that the inner products defined in (1.4) and (1.5) have these properties.

Later we will use inner products in $C_p[a,b]$ which are similar to (1.5) but have a more general form. Let $w(x)$ be a piecewise continuous function defined on $[a,b]$ which is strictly positive for all x, and for $f,g \in C_p[a,b]$ define

$$\langle f,g\rangle_w = \int_a^b f(x)g(x)\,w(x)\,dx. \tag{1.7}$$

Here $w(x)$ is called a *weight* function because it give more weight—more importance—to certain portions of the interval $[a,b]$: parts of the interval where $w(x)$ is large contribute more to $\langle f,g\rangle_w$ than parts where $w(x)$ is small. For example, we might take $w(x) = 1+x^2$ on the interval $[-1,1]$, thus defining an inner product $\langle f,g\rangle_w$ on $C_p[-1,1]$ by $\langle f,g\rangle_w = \int_{-1}^1 f(x)g(x)(1+x^2)\,dx$.

Example 1.3: To be explicit about how this works we check that in this example, that is, on $C_p[-1,1]$ with $w(x) = 1+x^2$, $\langle f,g\rangle_w$ satisfies properties (IP1)–(IP3) above. For (IP1) we just use the standard properties of integrals:

$$\begin{aligned}\langle \alpha f+\beta g,h\rangle_w &= \int_{-1}^1 \big(\alpha f(x)+\beta g(x)\big)h(x)\,(1+x^2)\,dx\\ &= \alpha\int_{-1}^1 f(x)h(x)\,(1+x^2)\,dx + \beta\int_{-1}^1 g(x)h(x)\,(1+x^2)\,dx\\ &= \alpha\,\langle f,h\rangle_w + \beta\,\langle g,h\rangle_w\,. \end{aligned}\tag{1.8}$$

Checking (IP2) is even easier:

$$\langle f,g\rangle_w = \int_{-1}^1 f(x)g(x)\,(1+x^2)\,dx = \int_{-1}^1 g(x)f(x)\,(1+x^2)\,dx = \langle g,f\rangle_w\,. \tag{1.9}$$

Finally, for (IP3), notice that

$$\langle f,f\rangle_w = \int_{-1}^1 f(x)^2\,(1+x^2)\,dx \ge 0, \tag{1.10}$$

because the integrand $f^2(x)(1+x^2)$ is nonnegative; moreover, such an integral with a nonnegative integrand can be zero only if the integrand is zero everywhere, which here means that $f(x) = 0$ for all x.

One can define similarly inner products in $\mathbb{R}^n$ that generalize (1.4): if $w_i > 0$ for all i, $1 \le i \le n$, then one defines

$$\langle \mathbf{u},\mathbf{v}\rangle_w = \sum_{i=1}^n u_i v_i w_i. \tag{1.11}$$

For example, in $\mathbb{R}^3$ the familiar dot product of two vectors $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{v} = (v_1,v_2,v_3)$ is $u\cdot v = u_1v_1 + u_2v_2 + u_3v_3$. If we want to assign more importance to the second coordinate than to the first, and yet more to the third coordinate, we might define $w_1 = 1$, $w_2 = 2$, and $w_3 = 3$, so that $\langle \mathbf{u}, \mathbf{v} \rangle_w = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$.

Whatever the inner product in our vector space, we can use it to measure the size of vectors. For any vector $\mathbf{u}$ we define the *norm* $\|\mathbf{u}\|$ of $\mathbf{u}$ to be $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$. Note that this makes sense because, by (IP3), $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$; note also that, again by (IP3), $\|\mathbf{u}\| = 0$ only if $\mathbf{u}$ is the zero vector.

Finally, for any two vectors $\mathbf{u}$ and $\mathbf{v}$ their inner product always satisfies the *Cauchy-Schwarz inequality*:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|. \tag{1.12}$$

This should not be surprising; we know that in $\mathbb{R}^3$ the dot product (1.4) has a geometric interpretation: $\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$, where θ is the angle between the two vectors. The Cauchy-Schwarz inequality (1.12) then follows from the fact that $|\cos\theta| \leq 1$. A general proof may be given by noting that (1.12) is immediate if $\|\mathbf{v}\| = 0$ (that is, if $\mathbf{v} = \mathbf{0}$); if $\|\mathbf{v}\| > 0$ we may use the fact that the vector

$$\mathbf{w} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}$$

has nonnegative length:

$$\begin{aligned}
\|\mathbf{w}\|^2 &= \left\langle \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}, \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle - \left\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}, \mathbf{u} \right\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} \right\rangle \\
&= \|\mathbf{u}\|^2 - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\|\mathbf{v}\|^2} \geq 0.
\end{aligned}$$

1.3 Orthogonal sets of vectors

Suppose we are given a vector space with an inner product, which again we denote by $\langle \mathbf{u}, \mathbf{v} \rangle$. We say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. A set $\{\mathbf{e}_1, \ldots \mathbf{e}_k, \ldots\}$ of vectors in our space (there may be a finite or an infinite number of them) which are nonzero and mutually orthogonal,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \|\mathbf{e}_i\|^2 > 0, & \text{if } i = j, \end{cases} \tag{1.13}$$

is called an *orthogonal set*.

Example 1.4: The functions

$$1,\ \cos\frac{\pi x}{\ell},\ \sin\frac{\pi x}{\ell},\ \cos\frac{2\pi x}{\ell},\ \sin\frac{2\pi x}{\ell},\ \ldots,\ \cos\frac{n\pi x}{\ell},\ \sin\frac{n\pi x}{\ell},\ \ldots \tag{1.14}$$

form an orthogonal set in $C_p[-\ell, \ell]$ when we use the standard inner product (1.5). To see this, one just computes, as follows (we always assume here that $m, n \geq 1$):

$$\langle 1, 1 \rangle = \int_{-\ell}^{\ell} dx = 2\ell, \tag{1.15a}$$

$$\left\langle 1, \cos\frac{n\pi x}{\ell} \right\rangle = \int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\, dx = 0,$$

$$\left\langle 1, \sin\frac{n\pi x}{\ell} \right\rangle = \int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\, dx = 0, \tag{1.15b}$$

$$\left\langle \cos\frac{m\pi x}{\ell}, \cos\frac{n\pi x}{\ell} \right\rangle = \int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell} \cos\frac{m\pi x}{\ell}\, dx = \begin{cases} \ell, & \text{if } n = m \\ 0 & \text{, if } m \neq m. \end{cases} 0, \tag{1.15c}$$

$$\left\langle \sin\frac{m\pi x}{\ell}, \sin\frac{n\pi x}{\ell} \right\rangle = \int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell}\, dx = \begin{cases} \ell, & \text{if } n = m \\ 0 & \text{, if } m \neq m. \end{cases} 0, \tag{1.15d}$$

$$\left\langle \cos\frac{m\pi x}{\ell}, \sin\frac{n\pi x}{\ell} \right\rangle = \int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell}\, dx = 0. \tag{1.15e}$$

These integrals are perhaps most easily evaluated by substituting $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, and then using the formula, valid for m, n any integers,

$$\int_{-\ell}^{\ell} e^{-im\pi x/\ell} e^{in\pi x/\ell} = \begin{cases} 2\ell, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases} \tag{1.16}$$

Note that it is a consequence of (1.15) that

$$\|1\|^2 = 2\ell, \qquad \left\|\cos\frac{n\pi x}{\ell}\right\|^2 = \left\|\sin\frac{n\pi x}{\ell}\right\|^2 = \ell. \tag{1.17}$$

1.4 Best approximation

Now we again suppose that $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is an orthogonal set in some vector space, and ask the following fundamental question:

> **Question:** Given a vector $\mathbf{v}$, what linear combination
>
> $$\mathbf{u} = \sum_i c_i \mathbf{e}_i \tag{1.18}$$
>
> of the vectors in the orthogonal set gives the best approximation to $\mathbf{v}$? That is, how should the coefficients c_i be chosen to give this best approximation?

Before we can approach the question, we need to know in what sense the approximation is to be "best". The idea is to make the difference vector $\mathbf{v} - \mathbf{u}$ as small as possible, and, since we have introduced $\|\mathbf{w}\|$ as a measure of the size of the vector $\mathbf{w}$, this means to make $\|\mathbf{v}-\mathbf{u}\|$ as small as possible. So we may reformulate the question:

> **Question:** Given a vector $\mathbf{v}$, how should the coefficients c_i be chosen so that if $\mathbf{u}=\sum_i c_i\mathbf{e}_i$ then $\|\mathbf{v}-\mathbf{u}\|$ is as small as possible?

We will give several different ways to find the answer to this question.

Approach 1: Suppose first that it is possible to choose the coefficients c_i so that the vector $\mathbf{u}$ of (1.18) is in fact equal to $\mathbf{v}$, that is, so that the size of the error $\|\mathbf{v}-\mathbf{u}\|$ is zero. This means that we can write

$$\mathbf{v}=\sum_i c_i\mathbf{e}_i \tag{1.19}$$

for some coefficients c_i. Then it is easy to find what the coefficients must be: we take the inner product of both sides of this equation with the vector $\mathbf{e}_j$; this gives, using (IP1) and (1.13),

$$\langle \mathbf{v},\mathbf{e}_j\rangle=\left\langle \sum_i c_i\mathbf{e}_i,\mathbf{e}_j\right\rangle=\sum_i c_i\langle \mathbf{e}_i,\mathbf{e}_j\rangle=c_j\langle \mathbf{e}_j,\mathbf{e}_j\rangle=c_j\|e_j\|^2, \tag{1.20}$$

which implies that

$$c_j=\frac{\langle \mathbf{v},\mathbf{e}_j\rangle}{\|\mathbf{e}_j\|^2}. \tag{1.21}$$

As we will see in the next two approaches, formula (1.21) gives the “best” coefficients for approximating $\mathbf{v}$ even when one cannot write $\mathbf{v}$ as a linear combination of the $\mathbf{e}_i$.

Approach 2: Consider a simple example in which the vector space is $\mathbb{R}^3$—ordinary vectors in three dimensional space—and there are two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the orthonormal set. The set of vectors $\mathbf{u}$ which can be written as linear combinations of these two vectors—$\mathbf{u}=c_1\mathbf{e}_i+c_2\mathbf{e}_2$—forms a plane through the origin. The vector in that plane which best approximates $\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto the plane, so that if $\mathbf{u}$ is the best approximation then $\mathbf{v}-\mathbf{u}$ should be orthogonal to the plane, i.e., orthogonal to both the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. This geometric intuition in fact applies in general: the best approximating vector $\mathbf{u}$ in the form (1.18) should be such that $\mathbf{v}-\mathbf{u}$ is orthogonal to all the vectors $\mathbf{e}_j$:

$$\langle \mathbf{v}-\mathbf{u},\mathbf{e}_j\rangle=\left\langle \mathbf{v}-\sum_i c_i\mathbf{e}_i,\mathbf{e}_j\right\rangle=\langle \mathbf{v},\mathbf{e}_j\rangle-c_j\langle \mathbf{e}_j,\mathbf{e}_j\rangle=0. \tag{1.22}$$

This is just equation (1.20) again and leads again to (1.21).

Approach 3: Now we show explicitly that the choice (1.21) for the coefficients c_j makes $\|\mathbf{v}-\mathbf{u}\|$ as small as possible. Let $\mathbf{w}=\sum_i c_j\mathbf{e}_j$, with the c_j given by (1.21), and let $\mathbf{u}=\sum_i b_i\mathbf{e}_i$ be some other linear combination of the vectors $\mathbf{e}_i$; we want to show that the error $\mathbf{v}-\mathbf{u}$ is at least as big as the error $\mathbf{v}-\mathbf{w}$. The calculation in (1.22) shows that $\mathbf{v}-\mathbf{w}$

is orthogonal to all the vectors $\mathbf{e}_j$, and it is therefore orthogonal to any linear combination of these—in particular to $\mathbf{w}-\mathbf{u}=\sum_i(c_i-b_i)\mathbf{e}_i$, so that $\langle\mathbf{v}-\mathbf{w},\mathbf{w}-\mathbf{u}\rangle=0$. This lead to

$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= \|(\mathbf{v}-\mathbf{w})+(\mathbf{w}-\mathbf{u})\|^2\\ &= \langle(\mathbf{v}-\mathbf{w})+(\mathbf{w}-\mathbf{u}),(\mathbf{v}-\mathbf{w})+(\mathbf{w}-\mathbf{u})\rangle\\ &= \langle\mathbf{v}-\mathbf{w},\mathbf{v}-\mathbf{w}\rangle+\langle\mathbf{v}-\mathbf{w},\mathbf{w}-\mathbf{u}\rangle+\langle\mathbf{w}-\mathbf{u},\mathbf{v}-\mathbf{w}\rangle+\langle\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u}\rangle\\ &= \|\mathbf{v}-\mathbf{w}\|^2+\|\mathbf{w}-\mathbf{u}\|^2\\ &\geq \|\mathbf{v}-\mathbf{w}\|^2. \end{aligned} \tag{1.23}$$

Notice also that unless $\mathbf{u}=\mathbf{w}$ there is strict inequality in the last line of (1.23).

Summary: The best approximation $\mathbf{u}=\sum_i c_i\mathbf{e}_i$ to a given vector $\mathbf{v}$ is obtained by choosing the coefficients c_j according to

$$c_j=\frac{\langle\mathbf{v},\mathbf{e}_j\rangle}{\|\mathbf{e}_j\|^2}. \tag{1.24}$$

Example 1.5: Let us apply this to the vector space $C_p[-\ell,\ell]$ and the orthonormal set of trigonometric functions discussed in Example 1.4. According to (1.24), the best approximation to a function $f\in C_p[-\ell,\ell]$ of the form

$$S(x)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right] \tag{1.25}$$

is obtained by choosing

$$a_0=\frac{\langle f,1\rangle}{\|1\|^2}=\frac{1}{2\ell}\int_{-\ell}^{\ell}f(x)\,dx \tag{1.26a}$$

$$a_n=\frac{\left\langle f,\cos\frac{n\pi x}{\ell}\right\rangle}{\left\|\cos\frac{n\pi x}{\ell}\right\|^2}=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}\,dx,\qquad n\geq 1, \tag{1.26b}$$

$$b_n=\frac{\left\langle f,\sin\frac{n\pi x}{\ell}\right\rangle}{\left\|\sin\frac{n\pi x}{\ell}\right\|^2}=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}\,dx,\qquad n\geq 1, \tag{1.26c}$$

where we have used (1.17). The series (1.25), with the coefficients defined by (1.26), is called the *Fourier series* of the function f.

Remark 1.1: Often one works with orthogonal sets which have the added property that the vectors $\mathbf{e}_i$ have norm 1, that is, are *normalized*. Such vectors are called *unit vectors* and usually denoted with a hat, and a set $\{\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2\ldots\}$ of such vectors is called an *orthonormal set*. If $\{\mathbf{e}_1,\mathbf{e}_2\ldots\}$ is an orthogonal set then we may obtain an orthonormal set by defining $\hat{\mathbf{e}}_i=\mathbf{e}_i/\|\mathbf{e}_i\|$. The formula (1.24) for the coefficients c_j becomes

$$c_j=\langle\mathbf{v},\hat{\mathbf{e}}_j\rangle \tag{1.27}$$

when an orthonormal set is used.

1.5 Completeness

Now once again we suppose that $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is an orthogonal set in some vector space. Our final question is this: is it true that for every vector $\mathbf{v}$ in the space the best approximation $\mathbf{u} = \sum_i c_i \mathbf{e}_i$, with $c_i = \langle \mathbf{v}, \mathbf{e}_i \rangle / \|\mathbf{e}_i\|^2$, is actually equal to $\mathbf{v}$? In other words: are there are enough vectors in our orthogonal set to expand every vector $\mathbf{v}$ in terms of that set? If so, we say that the orthogonal set is *complete* or is a *basis*. To be more specific we may speak of an *orthogonal basis* or, or, if it is the orthogonal set is actually orthonormal, an *orthonormal basis*.

When the vector space is finite dimensional, say of dimension n, then the orthogonal set forms a basis if and only if it contains n vectors; that is, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthogonal set in an n dimensional vector space then for any vector v in that space we have

$$\mathbf{v} = \sum_{j=1}^{n} \frac{\langle \mathbf{v}, \mathbf{e}_j \rangle}{\|\mathbf{e}_j\|^2} \mathbf{e}_j. \tag{1.28}$$

Such an orthogonal basis is of course also a basis as defined in Section 1.1. When the vector space is infinite dimensional, we will need an infinite orthonormal set to have any hope of forming a basis; we are then asking whether every $\mathbf{v}$ in the space can be written as

$$\mathbf{v} = \sum_{j=1}^{\infty} \frac{\langle \mathbf{v}, \mathbf{e}_j \rangle}{\|\mathbf{e}_j\|^2} \mathbf{e}_j.$$

The question of whether or not a particular orthogonal set is complete is a delicate one, which must be considered separately in each case. Here we consider the question for the Fourier series described in Example 1.5.

The answer in this case is *yes*: the set of trigonometric functions (1.14) is indeed complete, which means that every function $f \in C_p[-\ell, \ell]$ is equal (more or less) to the sum $S(x)$ of its Fourier series, that is, that

$$f(x) = S(x) = \lim_{N\to\infty} \left[a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right] \right]. \tag{1.29}$$

In fact this is true (more or less) in two senses, depending on how the limit in (1.29) is interpreted; we say "more or less" because in the second interpretation we need to assume something about the derivative $f'(x)$.

Completeness statement I: Let us denote by S_N the partial sum of the Fourier series of f:

$$S_N(x) = a_0 + \sum_{n=1}^{N} \left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right]. \tag{1.30}$$

Then the norm of the difference between the function f and the approximation S_N goes to zero as $N \to \infty$:

$$\lim_{N\to\infty} \|S_N - f\| = \lim_{N\to\infty} \left(\int_{-\ell}^{\ell} [f(x) - S_N(x)]^2 \, dx \right)^{1/2} = 0. \tag{1.31}$$

In fact, this last statement is true not only for $f \in C_p[-\ell, \ell]$ but also for functions f belonging to a larger vector space: $L^2[-\ell, \ell]$, the space of all functions f for which the integral $\displaystyle\int_{-\ell}^{\ell} f(x)^2\, dx$ defining $\|f\|^2$ is finite.

Completeness statement II: Suppose that *both* $f(x)$ and $f'(x)$ belong to $C_p[-\ell, \ell]$. Then for each $x \in [-\ell, \ell]$ the series (1.25) defining $S(x)$ converges, that is, $S(x) = \lim_{N\to\infty} S_N(x)$ exists, and it is given by

$$S(x) = \begin{cases} f(x), & \text{if } f \text{ is continuous at the point } x \in (-\ell, \ell), \\ \dfrac{f(x+) + f(x-)}{2}, & \text{if } f \text{ is discontinuous at the point } x \in (-\ell, \ell). \\ \dfrac{f(\ell-) + f(-\ell+)}{2}, & \text{if } x = \ell. \end{cases} \tag{1.32}$$

Here $f(x+) = \lim_{y\to x,\, y>x} f(y)$ and $f(x-) = \lim_{y\to x,\, y<x} f(y)$. That is, the Fourier series converges to $f(x)$ wherever $f(x)$ is continuous, and at a point x where f takes a jump it converges to the average of the values of f from the right and left of the jump. The endpoints of the interval need special treatment: there $S(x)$ converges to the average of the limit (from the left) of f at ℓ and the limit (from the right) of f at $-\ell$.

1.6 Complex vector spaces

Sometimes it is convenient to consider vector spaces formed by complex-valued functions or by row vectors with complex entries; see Example 1.6 below. When we form linear combinations $\alpha f + \beta g$ or $\alpha\mathbf{u} + \beta\mathbf{v}$ in this setting the numbers α and β can be complex. Most of what we said above goes through, but with one key change. We still want the inner product to satisfy $\langle \mathbf{u}, \mathbf{u}\rangle \ge 0$ or $\langle f, f\rangle \ge 0$, so that for row vectors (the vector space is now $\mathbb{C}^n$) we define

$$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_i u_i \overline{v}_i, \tag{1.33}$$

where the bar denotes complex conjugation; now $\langle \mathbf{u}, \mathbf{u}\rangle = \sum_i u_i \overline{u}_i = \sum_i |u_i|^2 \ge 0$. Similarly we define, for f, g complex-valued functions piecewise continuous on $[a, b]$,

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx, \tag{1.34}$$

so that again $\langle f, f\rangle = \int_a^b |f(x)|^2\, dx \ge 0$.

The general rules for an inner product on a complex vector space are

(IP1′) Linearity: for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and complex numbers α, β,

$$\langle \alpha\mathbf{u} + \beta v, \mathbf{w}\rangle = \alpha\,\langle \mathbf{u}, \mathbf{v}\rangle + \beta\,\langle \mathbf{v}, \mathbf{w}\rangle \quad \text{and} \quad \langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}\rangle = \overline{\alpha}\,\langle \mathbf{u}, \mathbf{v}\rangle + \overline{\beta}\,\langle \mathbf{u}, \mathbf{w}\rangle\,. \tag{1.35}$$

(IP2′) Symmetry: $\langle \mathbf{u}, \mathbf{v}\rangle = \overline{\langle \mathbf{v}, \mathbf{u}\rangle}$ for any vectors $\mathbf{u}, \mathbf{v}$.

(IP3′) Positivity: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for any vector $\mathbf{u}$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$.

Note that the inner product is now not linear but *antilinear* in its second argument (second equation in (IP1′)); this is necessary for (IP1′) to be consistent with (IP2′). One can check that this is exactly how the inner products (1.33) and (1.34) behave.

We say that a set $\{e_1, \ldots\}$ of vectors in a complex vector space is an orthogonal set exactly as for the real case, that is, when (1.13) holds. Moreover, exactly the same arguments tell us the the coefficients c_j which make $\sum c_j \mathbf{e}_j$ the best approximation to a vector $\mathbf{v}$ are given by (1.24): $c_j = \langle \mathbf{v}, \mathbf{e}_j \rangle / \|\mathbf{e}_j\|^2$. In using this formula, however, one must now be a little cautious, because $\langle \mathbf{v}, \mathbf{e}_j \rangle \neq \langle \mathbf{e}_j, \mathbf{v} \rangle$.

Example 1.6: Consider again $C_p[-\ell, \ell]$, now allowing complex valued functions. Equation (1.16) tells us that the functions

$$\varphi_n(x) = e^{in\pi x/\ell}, \qquad n = 0, \pm 1, \pm 2 \ldots \tag{1.36}$$

form an orthogonal set. Since the orthogonal set (1.14) of trigonometric functions is complete, and since these trigonometric functions can be expressed in terms of the complex exponentials (1.36) via

$$\begin{aligned} 1 &= \varphi_0(x), \\ \cos \frac{n\pi x}{\ell} &= \frac{\varphi_n(x) + \varphi_{-n}(x)}{2}, \quad n \geq 1, \\ \sin \frac{n\pi x}{\ell} &= \frac{\varphi_n(x) + \varphi_{-n}(x)}{2}, \quad n \geq 1, \end{aligned} \tag{1.37}$$

the complex exponentials are also a complete set, so that any (complex) function $f \in C_p[-\ell, \ell]$ can be expanded in a *complex Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}, \tag{1.38}$$

with

$$c_n = \frac{\left\langle f(x), e^{in\pi x/\ell} \right\rangle}{\left\| e^{in\pi x/\ell} \right\|^2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\, dx. \tag{1.39}$$

Notice that in writing the second form in (1.38) we have taken the complex conjugation in (1.34) into account, using the fact that $\overline{e^{in\pi x/\ell}} = e^{-in\pi x/\ell}$. We can of course also use the complex Fourier series formulas (1.38) and (1.39) if f happens to be real, since a real function is just a special case of a complex function. In that case it follows from (1.39) that $c_{-n} = \overline{c_n}$.

1.7 Exercises

1. All parts of this question refer to the vector space $C_p[0,2]$ and two inner products on this space:

$$\langle u, v\rangle = \int_0^2 u(x)v(x)\,dx \qquad \text{and} \qquad \langle u, v\rangle_w = \int_0^2 u(x)v(x)w(x)\,dx,$$

where $w(x) = x^2 - 2x + 2$. We correspondingly write

$$\|u\| = \sqrt{\langle u, u\rangle} \qquad \text{and} \qquad \|u\|_w = \sqrt{\langle u, u\rangle_w}.$$

Let $f(x) = 1$, $g(x) = 1 - x$, and $h(x) = x^2$.

(a) Show that f and g are orthogonal in the inner product $\langle\cdot,\cdot\rangle$, and find $\|f\|$ and $\|g\|$. Then compute the constants c_1, d_1 that make $h_1 = c_1 f + d_1 g$ the approximation of h with the smallest error $\|h - h_1\|$, and compute this error.

(b) Show that f and g are orthogonal in the inner product $\langle\cdot,\cdot\rangle_w$, and find $\|f\|_w$ and $\|g\|_w$. Then compute the constants c_2, d_2 that make $h_2 = c_2 f + d_2 g$ the approximation of h with the smallest error $\|h - h_2\|_w$, and compute this error.

(c) Compute $\|h - h_2\|$ and show that, as expected, this error is larger than $\|h - h_1\|$.

2. In this problem we consider continuous functions defined on the (infinite) interval $[0, \infty)$. For these we introduce the inner product and norm

$$\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx, \qquad \|f\| = \sqrt{\langle f, f\rangle}.$$

(a) Find constants a, b, and c such that if $f_1(x) = 1$, $f_2(x_) = x + a$, and $f_3(x) = x^2 + bx + c$ then $\{f_1(x), f_2(x), f_3(x)\}$ is an orthogonal set. In the remainder of the problem we assume that f_1, f_2, and f_3 are defined with these constants.

(b) Let $g(x) = x^3$. Compute the constants c_1, c_2, and c_3 such that

$$g_1(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$$

is the best approximation to $g(x)$ by a linear combination of f_1, f_2, and f_3, in the sense that the error $\|g - g_1\|$ is as small as possible. Compute this error.