2.6 The Fourier transform

In this section we begin with a function $f(x)$ which is defined for all x but is *not* periodic; rather, our basic assumption throughout is that it makes sense to integrate $f(x)$ over all values of x:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \tag{2.81}$$

Our goal is to obtain a representation of $f(x)$ which is analogous to the expansion of a periodic function in a Fourier series. The place of the Fourier coefficients a_n and b_n (or equivalently c_n, for the complex form of the Fourier series) is taken by a new quantity $\hat{f}(\omega)$, called the *Fourier transform* of $f(x)$ and also written $\mathcal{F}\{f(x)\}$:

$$\hat{f}(\omega) = \mathcal{F}\{f(x)\}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx. \tag{2.82}$$

Here ω is a real variable which is analogous to the index n on the Fourier coefficients; we think of ω is a *frequency* variable, since (as a function of x) the exponential $e^{-i\omega x}$ oscillates with frequency ω. What is surprising is that the *inverse Fourier transform*, which takes us back from $\hat{f}(\omega)$ to $f(x)$, is given by a very similar integral:

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\, dx. \tag{2.83}$$

A more precise version of (2.83) is given in Theorem 2.1 below. Although we will not prove this theorem rigorously, we now give a heuristic discussion of why (2.83) it holds and of the relation of the Fourier transform to the Fourier series that we discussed earlier.

We begin by considering the Fourier series of a certain periodic function $f_\ell(x)$, where ℓ is a positive number, obtained from $f(x)$ in a two step process: we first let $h(x)$ be the restriction of $f(x)$ to the interval $[-\ell, \ell]$, and then let $f_\ell(x)$ be the periodic extension of $h(x)$ to all of $\mathbb{R}$:

$$\text{First:}\quad h(x) = f(x), \quad -\ell \le x \le \ell; \qquad\qquad \text{Then:}\quad f_\ell(x) = h_{\text{per}}(x).$$

See Figure 2.4. (Eventually we will take a limit $\ell \to \infty$.) Since $f_\ell(x)$ is periodic, with period 2ℓ, it has a (complex) Fourier series

$$f_\ell(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} \qquad (2.84),$$

with

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}\, dx. \tag{2.85}$$

The exponential $e^{-in\pi x/\ell}$ occurring in (2.85) corresponds to frequency $\omega_n = n\pi/\ell$, and comparing (2.85) and (2.82) we see that

$$c_n \approx \frac{1}{2\ell}\int_{-\infty}^{\infty} f(x)e^{-in\pi x/\ell}\, dx = \frac{1}{2\ell}\hat{f}\left(\frac{n\pi}{\ell}\right) = \frac{1}{2\ell}\hat{f}(\omega_n). \tag{2.86}$$

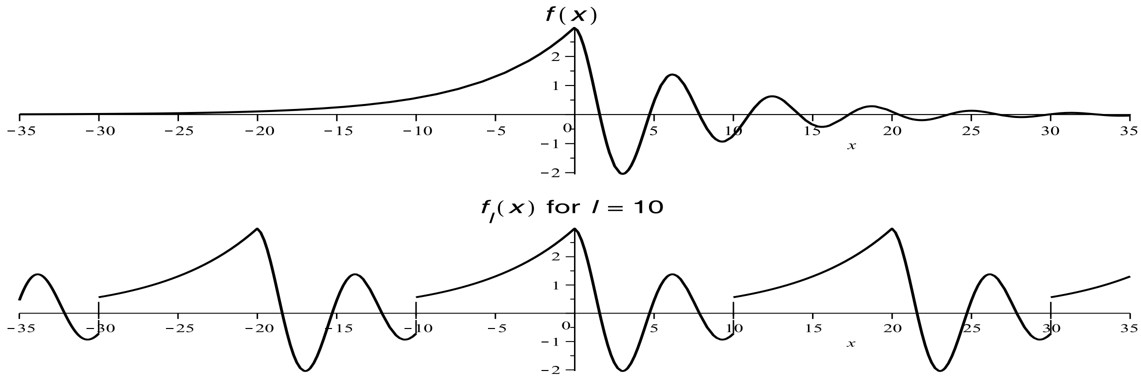


Figure 2.4: A function $f(x)$ satisfying (2.81) and the corresponding $f_\ell(x)$ for $\ell = 10$.

The approximation in the first step in (2.86) comes from replacing the integral from $-\ell$ to ℓ by an integral from $-\infty$ to ∞; we would expect this approximation to become more and more exact as $\ell \to \infty$.

Now we study the integral $(1/2\pi)\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega$ which we have claimed in (2.83) gives the inverse Fourier transform. For $-\ell < x < \ell$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega \approx \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(\omega_n)e^{i\omega_n x}\,\Delta\omega \;\approx \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = f(x). \tag{2.87}$$

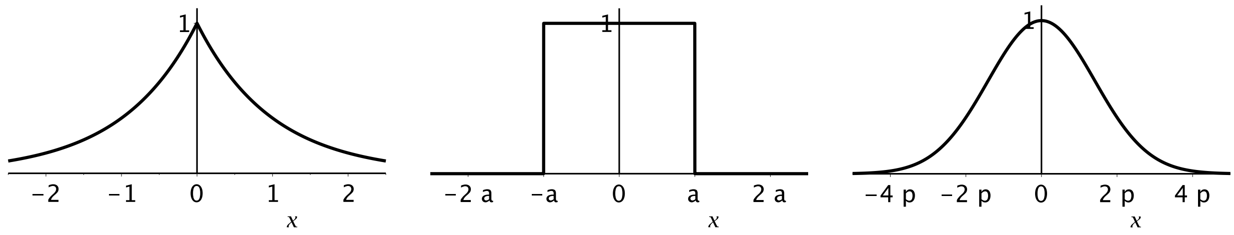
Here we have first approximated the integral as a Riemann sum, obtained by dividing the real line $(-\infty, \infty)$ into small intervals using the division points ω_n, $n = -\infty, \cdots, \infty$, so that the width of each interval is $\Delta\omega = \omega_{n+1} - \omega_n = \pi/\ell$, and have then used the approximation $\hat{f}(\omega_n) \approx 2\ell c_n$ from (2.86). The final equality, which holds for $-\ell < x < \ell$, is just (2.84). Now consider sending ℓ to infiniity. The approximations in (2.87) should become better and better in this limit, and the range $-\ell < x < \ell$ in which (2.87) holds will become the entire real line. Thus we expect that, for all x,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,dx. \tag{2.88}$$

This is the equation (2.83) which we wanted to justify. A more precise version of this result is given in the next theorem.

Theorem 2.1: *Suppose that $f(x)$ is defined for $-\infty < x < \infty$, that $f(x)$ and $f'(x)$ are piecewise continuous, and that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ (see (2.81)). Then*

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\}(x) = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x, \\ \dfrac{f(x-) + f(x+)}{2}, & \text{for all } x. \end{cases}$$



(a) (b) (c)

Figure 2.5: The functions of (a) Example 2.1, (b) Example 2.2, and (c) Example 2.4.

There is a small table of Fourier transforms in Appendix D of Greenberg. But they are easy to calculate.

Example 2.1: Suppose that $f_1(x) = e^{-|x|}$; see Figure 2.5(a). The condition (2.81) is certainly satisfied, since $f_1(x)$ decreases exponentially as $|x| \to \infty$. Then

$$\begin{aligned}\mathcal{F}\{f_1(x)\}(\omega) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}\,dx &= \int_{-\infty}^{0} e^{(1-i\omega)x}\,dx + \int_{0}^{\infty} e^{-(1+i\omega)x}\,dx \\ &= \left.\frac{e^{(1-i\omega)x}}{1-i\omega}\right|_{-\infty}^{0} - \left.\frac{e^{-(1+i\omega)x}}{1+i\omega}\right|_{0}^{\infty} \\ &= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}.\end{aligned}$$

Example 2.2: Suppose that $a > 0$ and let $f_2(x;a) = H(x+a) - H(x-a) = \begin{cases} 1, & \text{if } |x| < a, \\ 0, & \text{if } |x| > a. \end{cases}$ Then

$$\begin{aligned}\mathcal{F}\{f_2(x;a)\} = \hat{f}_2(\omega;a) &= \int_{-\infty}^{\infty} f_2(x;a)e^{-i\omega x}\,dx \\ &= \int_{-a}^{a} e^{-i\omega x}\,dx = \frac{1}{-i\omega}\big(e^{-i\omega x} - e^{i\omega x}\big)_{-a}^{a} = \frac{2}{\omega}\sin\omega a\end{aligned}$$

and

$$f_2(x;a) = \mathcal{F}^{-1}\{\hat{f}_2(\omega;a)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega a}{\omega}e^{i\omega x}\,d\omega.$$

Example 2.3: Let $f_3(x) = e^{-ax}H(x)$, with $a > 0$. Then

$$\mathcal{F}\{f_3(x)\} = \hat{f}_3(\omega) = \int_{-\infty}^{\infty} f_3(x)e^{-i\omega x}\,dx = \int_{0}^{\infty} e^{-(a+i\omega)x}\,dx = \frac{1}{a+i\omega}.$$

Example 2.4: Suppose that $f_4(x;p) = e^{-x^2/4p^2}$; f_4 is a *Gaussian*, sometimes called a *bell curve*. See Figure 2.5(c). All that really matters is that we consider a function having the form e^{-Ax^2} for some positive constant A; the specific choice $A = 1/4p^2$ is used here for historical reasons. To work out the Fourier transform of $f_4(x;p)$ we need the *Gaussian integral*

$$\int_{-\infty}^{\infty} e^{-x^2/4p^2}\,dx = 2p\sqrt{\pi}, \tag{2.89}$$

as well as a simple completion of a square:

$$\frac{x^2}{4p^2} + i\omega x = \frac{1}{4p^2}(x + 2i\omega p)^2 + \omega^2 p^2.$$

Then we have

$$\begin{aligned} \mathcal{F}\{e^{-x^2/4p^2}\}(\omega) &= \int_{-\infty}^{\infty} e^{-x^2/4p^2} e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} e^{-(x^2/4p^2 + i\omega x)}\,dx \\ &= e^{-\omega^2 p^2}\int_{-\infty}^{\infty} e^{-(x+2i\omega)^2/4p^2}\,dx = e^{-\omega^2 p^2}\int_{-\infty}^{\infty} e^{-\xi^2/4p^2}\,d\xi \\ &= 2p\sqrt{\pi}e^{-\omega^2 p^2}, \end{aligned} \tag{2.90}$$

where we have made the substitution $\xi = x + 2i\omega p$.

Remark 2.12: Everyone should know the simple trick to derive (2.89); we pause to recall it:

Evaluation of a Gaussian integral

Let I denote the integral on the left hand side of (2.89). Then I^2 is a product of two integrals; the trick is to write these using two different (dummy) integration variables, x and y, so that the product becomes a double integral over the entire plane, and then to evaluate this double integral using a change of variables to polar coordinates:

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/4p^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/4p^2}\,dy\right) = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/4p^2}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/4p^2}\,r\,dr\,d\theta = (2\pi)(2p^2)\int_0^{\infty} e^{-u}\,du = 4\pi p^2, \end{aligned}$$

where in the last line we have made the substitution $u = r^2/4p^2$, with then $du = r\,dr/2p^2$. Taking the square root of $I^2 = 4\pi p^2$ yields (2.89).

Example 2.5: Suppose that $f_5(x;a) = \big(\delta(x-a) + \delta(x+a)\big)/2$ (this rather special Fourier transform will be needed below). Then

$$\mathcal{F}\{f_5(x;a)\}(\omega) = \int_{-\infty}^{\infty} \frac{\delta(x-a) + \delta(x+a)}{2} e^{-i\omega x}\,dx = \frac{e^{-ia\omega} + e^{ia\omega}}{2} = \cos a\omega.$$

The next proposition gives the most important properties of the Fourier transform.

Proposition 2.2: *(a) The Fourier transform and inverse Fourier transform are linear: if α and β are constants, then*

$$\begin{aligned} \mathcal{F}\{\alpha f(x)+\beta g(x)\} &= \alpha\mathcal{F}\{f(x)\}+\beta\mathcal{F}\{g(x)\}, \\ \mathcal{F}^{-1}\{\alpha\hat{f}(\omega)+\beta\hat{g}(\omega)\} &= \alpha\mathcal{F}^{-1}\{\hat{f}(\omega)\}+\beta\mathcal{F}^{-1}\{\hat{g}(\omega)\}, \end{aligned} \tag{2.91}$$

Proof: This follows immediately from the definitions (2.82) and (2.83).

(b) Suppose that $f(x)$ is continuous, not just piecewise continuous, and has a derivative which is piecewise continuous and satisfies $\int_{-\infty}^{\infty}|f'(x)|\,dx<\infty$ (see (2.81)). Then

$$\mathcal{F}\{f'(x)\}(\omega)=i\omega\hat{f}(\omega) \tag{2.92}$$

Similarly, if $f(x), f'(x), \dots f^{(n-1)}(x)$ are continuous, $f^{(n)}$ is piecewise continuous, and $\int_{-\infty}^{\infty}|f^{(k)}(x)|\,dx<\infty$ for $k=0,\dots n$, Then

$$\mathcal{F}\{f^{(n)}(x)\}(\omega)=(i\omega)^n\hat{f}(\omega). \tag{2.93}$$

Proof: Equation (2.92) is obtained by integration by parts:

$$\mathcal{F}\{f'(x)\}(\omega)=\int_{-\infty}^{\infty}f'(x)e^{-i\omega x}\,dx=f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty}+i\omega\int_{-\infty}^{\infty}f(x)e^{-i\omega x}\,dx=i\omega\hat{f}(\omega).$$

Here we are justified in setting $f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty}$ equal to zero by the fact that $f(x)\to 0$ as $x\to\pm\infty$; the full proof of this is technical and we omit it.

To state the next property we define the *convolution* $(f*g)(x)$ of two functions $f(x)$ and $g(x)$, which are defined for all x and satisfy (2.81), by

$$(f*g)(x)=\int_{-\infty}^{\infty}f(\xi)g(x-\xi)\,d\xi. \tag{2.94}$$

This is very similar to the convolution we used in working with the Laplace transform, but differs because the range of integration in (2.94) is the entire line; there should be no confusion because we will never be using Laplace and Fourier transforms at the same time. The next part of our proposition shows that this convolution is related to the Fourier transform just as our earlier version was to the Laplace transform.

(c) Suppose that $f(x)$ and $g(x)$ satisfy (2.81) and have Fourier transforms $\hat{f}(\omega)$ and $\hat{g}(\omega)$ respectively. Then

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\}=f*g. \tag{2.95}$$

Proof: To verify (2.95) we compute the Fourier transform of $f * g$:

$$\begin{aligned}
\mathcal{F}\{(f*g)(x)\}(\omega) &= \int_{-\infty}^{\infty} (f*g)(x) e^{-i\omega x}\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) e^{-i\omega x}\, d\xi\, dx \\
&= \int_{-\infty}^{\infty} f(\xi) \int_{-\infty}^{\infty} g(x-\xi) e^{-i\omega x}\, dx\, d\xi \\
&= \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} \int_{-\infty}^{\infty} g(y) e^{-i\omega y}\, dy\, d\xi \\
&= \left(\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi}\, d\xi\right) \left(\int_{-\infty}^{\infty} g(y) e^{-i\omega y}\, dy\, d\xi\right) = \hat{f}(\omega)\hat{g}(\omega).
\end{aligned}$$

Here we have, in line-by-line order, (i) inserted the definition (2.94) of $f * g$, (ii) exchanged the order of the x and ξ integrals, (iii) made a change of variable from x to $y = x - \xi$ in the inner integral, using that then $dx = dy$ and $e^{-i\omega x} = e^{-i\omega y} e^{-i\omega \xi}$, and (vi) reorganized the integral to make it clear that we now have a product of the Fourier transforms $\hat{f}$ and $\hat{g}$.

(d) Suppose that $f(x)$ satisfies (2.81) and has Fourier transforms $\hat{f}(\omega)$. Then

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}\hat{f}(\omega) \qquad \text{and} \qquad \mathcal{F}^{-1}\{\hat{f}(\omega - a)\} = e^{iax} f(x). \tag{2.96}$$

Proof: We verify the first formula in (2.96):

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} f(y) e^{-i\omega(y+a)}\, dx = e^{-i\omega a}\hat{f}(\omega)$$

where we have made the change of integration variable $y = x - a$. The second formula is obtained similarly.

2.7 Applications of the Fourier transform

The Fourier transform can be used to solve partial differential equations in which one of the variables varies over the entire line. We give two examples.

A1. The heat equation for a doubly infinite rod

We want to solve the initial value problem for a function $u(x,t)$:

$$\alpha^2 u_{xx}(x,t) = u_t(x,t); \qquad -\infty < x < \infty, \quad t > 0; \tag{PDE}$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty, \tag{IC}$$

We may think of this as the heat equation for an infinitely long rod, or as an approximation to the heat equation in a very long rod. The initial value $f(x)$ is assumed to be *localized*, that is, to fall off to zero as $x \to \pm\infty$; we will specifically assume that $f(x)$ satisfies the condition (2.81), so that we may use the Fourier transform. There are no explicit boundary conditions in the problem, but they arise implicitly because our solution $u(x,t)$ will satisfy $u(x,t) \to \infty$ as $x \to \pm\infty$.

Let $\hat{u}(\omega,t)$ denote the Fourier transform of $u(x,t)$ in the variable x:

$$\hat{u}(\omega,t) = \mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx. \tag{2.97}$$

Note that by (2.93) the Fourier transform of $u_{xx}(x,t)$ is $-\omega^2\hat{u}(\omega,t)$, and also that the transform of $u_t(x,t)$ is $\hat{u}_t(\omega,t)$, since we can carry a t derivative in (2.97) through the integral:

$$\hat{u}_t(\omega,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)e^{-i\omega x}\,dx = \mathcal{F}\{u_t(x,t)\}. \tag{2.98}$$

Thus the Fourier transform of (PDE) is

$$-\alpha^2\omega^2\hat{u}(\omega,t) = \hat{u}_t(\omega,t).$$

This is an ordinary differential equation for $\hat{u}(\omega,t)$ with solution $\hat{u}(\omega,t) = C(\omega)e^{-\alpha^2\omega^2 t}$, where $C(\omega)$ is some arbitrary function of ω. $C(\omega)$ may be evaluated from the initial condition:

$$C(\omega) = \hat{u}(\omega,0) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = \hat{f}(\omega).$$

thus

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-\alpha^2\omega^2 t}. \tag{2.99}$$

To complete the solution we must find $u(x,t) = \mathcal{F}^{-1}\{\hat{u}(\omega,t)\}$. Certainly we can write this down using the integral formula (2.83) for $\mathcal{F}^{-1}$:

$$u(x,t) = \mathcal{F}^{-1}\{\hat{u}(\omega,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-\alpha^2\omega^2 t}e^{i\omega x}\,d\omega.$$

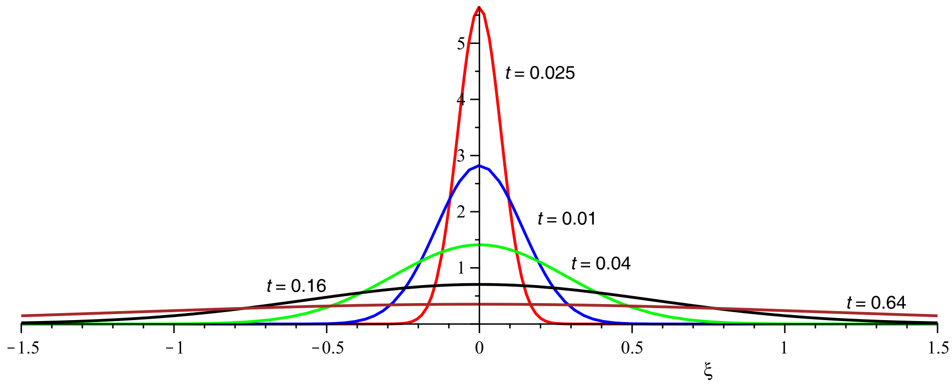


Figure 2.6: The function $K(\xi, t)$ plotted, with $\alpha^2 = 1$, for $t = 0.0025$ (red), $t = 0.01$ (blue), $t = 0.04$ (green), $t = 0.16$ (black), and $t = 0.4$ (brown).

An alternate approach is to use the convolution property (2.95) of the Fourier transform to compute $\mathcal{F}^{-1}\{\hat{u}(\omega, t)\}$. (2.99) expresses $\hat{u}(\omega, t)$ as product; since $\mathcal{F}^{-1}\{\hat{f}(\omega)\} = f(x)$ and, from Example 2.4 with $p = \sqrt{\alpha^2 t}$, $\mathcal{F}^{-1}\{e^{-\alpha^2\omega^2 t}\} = e^{-x^2/4\alpha^2 t}/2\sqrt{\alpha^2\pi t}$, (2.95) yields (see (2.95))

$$u(x,t) = \frac{1}{2\sqrt{\alpha^2\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4\alpha^2 t}}\, d\xi. \tag{2.100}$$

We may interpret the solution (2.100) as follows. From (2.89),

$$\frac{1}{2\sqrt{\alpha^2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\alpha^2 t}}\, d\xi = \frac{1}{2\sqrt{\alpha^2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4\alpha^2 t}}\, d\xi = 1.$$

Thus (2.100) tells us that the value $u(x, t)$ is an average of the initial values $f(\xi)$, averaged with weight function $K(x - \xi, t)$, where

$$K(\xi, t) = \frac{1}{2\sqrt{\alpha^2\pi t}} e^{-\frac{\xi^2}{4\alpha^2 t}}.$$

$K(\xi, t)$ is called the *heat kernel*. The graphs of $K(\xi, t)$ for several different values of t, with $\alpha^2 = 1$, are shown in Figure 2.6; the graphs of $K(x - \xi, t)$ would look the same but be shifted to have center at $\xi = x$. We see that for small t the averaging will take place over values of $f(\xi)$ with ξ very close to x, while for large t the averaging will be over a very wide range of values. (The first statement is not really true, since the heat kernel $K(\xi, t)$ is never zero, but it is effectively zero outside a range which grows with t.)

As a specific example, suppose that $f(x)$ is the square pulse $f_2(x; a)$ considered in Example 2.2; see Figure 2.5(b). In this case the solutions (2.100) are plotted, with $a = 2$, as functions of x for various values of t, in Figure 2.7. For small values of t the averaging discussed above does not affect values $u(x, t)$ of the solution when x is near the center of the pulse or some distance outside it; for those values of x essentially all the values $f(\xi)$ being averaged are the same, either 1 or 0 respectively. Near the edge of the pulse, however, the averaging extends over values of $f(\xi)$ which are both 1 and 0, and this produces a rounding of the edges of the pulse. After a longer time the averaging includes both 1 and 0 values for all x shown in the figure, and the resulting solution looks very much like the weight function itself.

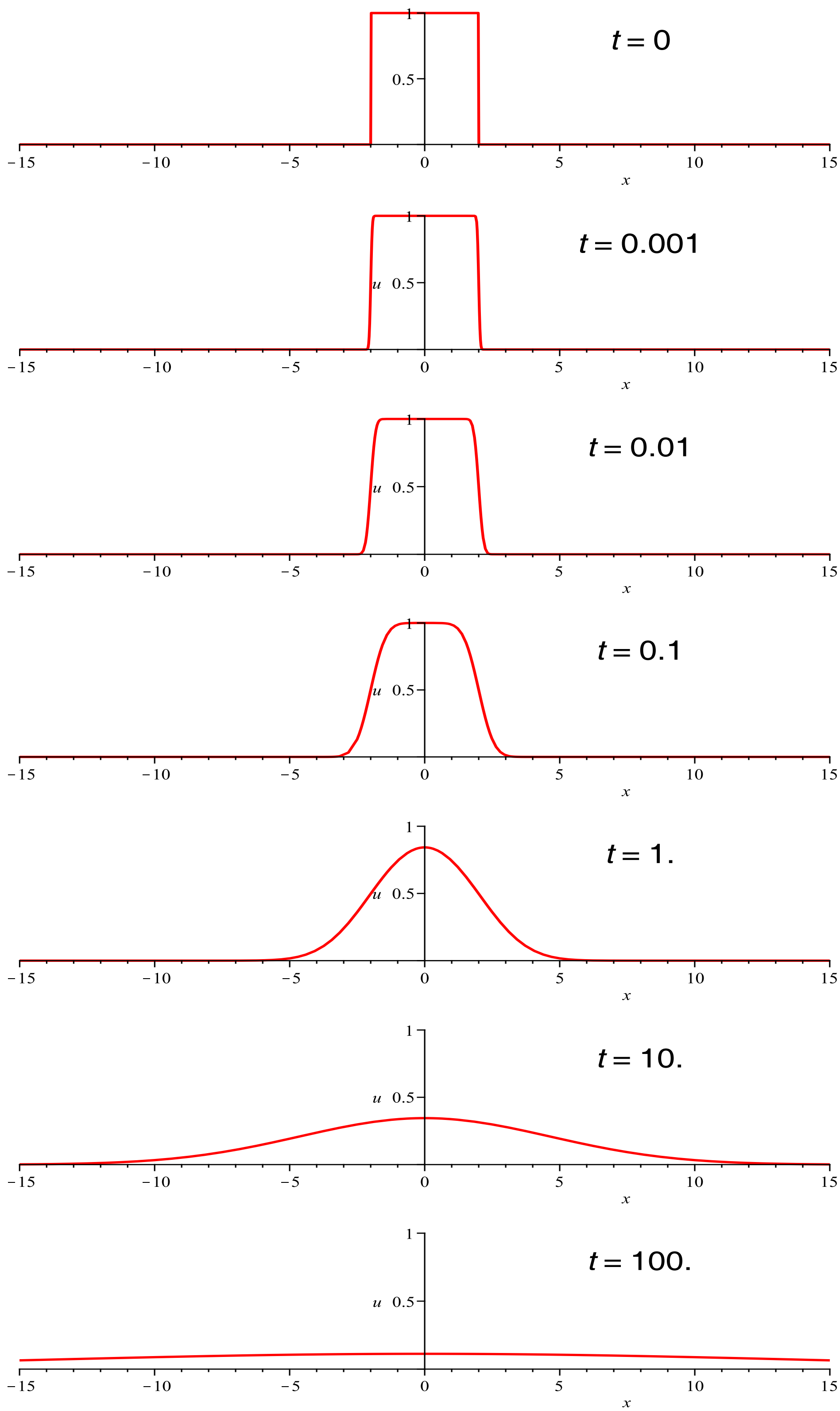


Figure 2.7: Evolution of a square pulse under the heat equation.

A2. The wave equation on the real line

We want to solve the initial value problem for a function $u(x,t)$:

$$c^2 u_{xx}(x,t) = u_{tt}(x,t); \qquad -\infty < x < \infty, \quad t > 0; \tag{PDE}$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad -\infty < x < \infty. \tag{IC}$$

Again we suppose that f and g satisfy (2.81): $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ and $\int_{-\infty}^{\infty} |g(x)|\,dx < \infty$. As in **A.1** above we let $\hat{u}(\omega,t)$ denote the Fourier transform, in x, of $u(x,t)$, and by taking the Fourier transform of (PDE) obtain

$$-c^2\omega^2\hat{u}(\omega,t) = \hat{u}_{tt}(\omega,t) \qquad \Leftrightarrow \qquad \hat{u}_{tt} + (c\omega)^2\hat{u} = 0.$$

Again we have obtained an ordinary differential equation for $\hat{u}(\omega,t)$; now the solution is

$$\hat{u}(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t,$$

with $A(\omega)$ and $B(\omega)$ determined by the initial conditions:

$$\hat{u}(\omega,0) = A(\omega) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = \hat{f}(\omega) \quad \Rightarrow \quad A(\omega) = \hat{f}(\omega),$$
$$\hat{u}_t(\omega,0) = c\omega B(\omega) = \mathcal{F}\{u_t(x,0)\} = \mathcal{F}\{g(x)\} = \hat{g}(\omega) \quad \Rightarrow \quad B(\omega) = \frac{1}{c\omega}\hat{g}(\omega).$$

Thus

$$\hat{u}(\omega,t) = \hat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\hat{g}(\omega)\sin c\omega t. \tag{2.101}$$

Finally, we can find $u(x,t)$ by taking the inverse Fourier transform of (2.101); we will do this using the convolution property (2.95). If we recall the functions f_2 and f_5 from Example 2.2 and Example 2.5, and their Fourier transforms as computed in those examples, we see that (2.101) can be written as

$$\hat{u}(\omega,t) = \mathcal{F}\{f_4(x;ct)\}\mathcal{F}\{f(x)\} + \frac{1}{2c}\mathcal{F}\{f_2(x;ct)\}\mathcal{F}\{g(x)\}$$

so that by (2.95),

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}\{\hat{u}(\omega,t)\} &= (f_4 * f)(x,t) + \frac{1}{2a}(f_2 * g)(x,t) \\ &= \int_{-\infty}^{\infty} f_4(\xi;ct)f(x-\xi)\,d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} f_2(\xi;ct)g(x-\xi)\,d\xi. \end{aligned} \tag{2.102}$$

Let us consider the two terms in (2.102) separately. For the first we find,

$$\int_{-\infty}^{\infty} f_4(\xi;ct)f(x-\xi)\,d\xi = \frac{1}{2}\int_{-\infty}^{\infty}\big(\delta(\xi-ct)+\delta(\xi+ct)\big)f(x-\xi)\,d\xi = \frac{1}{2}\big(f(x-ct)+f(x+ct)\big).$$

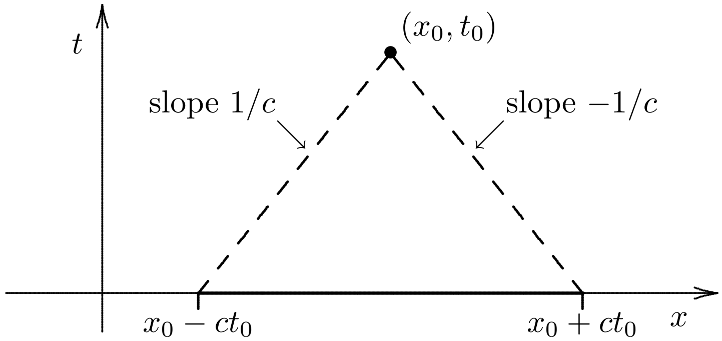


Figure 2.8: The x-t plane and the domain of dependence of $u(x,t)$.

For the second,

$$\frac{1}{2c}\int_{-\infty}^{\infty} f_2(\xi;ct)g(x-\xi)\,d\xi = \frac{1}{2c}\int_{-ct}^{ct} g(x-\xi)\,d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\eta)\,d\eta$$

where we have made the change of variable $\eta = x-\xi$ and, since then $d\eta = -d\xi$, exchanged the upper and lower limits. In summary,

$$u(x,t) = \frac{1}{2}\big(f(x-ct)+f(x+ct)\big) + \frac{1}{2a}\int_{x-ct}^{x+ct} g(\eta)\,d\eta. \tag{2.103}$$

The result (2.103) is called *d'Alembert's solution* of the wave equation on the infinite line. There is an important geometric interpretation of this formula. Consider Figure 2.8, which shows the x-t plane; the initial data are specified at $t=0$, that is, along the x axis in this figure. One particular point, (x_0,t_0), is shown with a heavy dot, and we are interested in how $u(x_0,t_0)$ depends on the initial data $f(x)$ and $g(x)$. The dashed lines project backwards in time from this point; their slopes are $dt/dx = \pm 1/c$, which correspond to velocities $dx/dt = \pm c$, and they intersect the x axis at $x = x_0 - ct_0$ and $x = x_0 + ct_0$. Now (2.103) shows that the *only* values of $f(x)$ which can influence $u(x_0,t_0)$ are the values at these two points: $f(x_0-ct_0)$ and $f(x_0+ct_0)$. Similarly, the *only* values of $g(x)$ which can influence $u(x_0,t_0)$ are the values $g(x)$ with $x_0 - ct_0 \le x \le x_0 + ct_0$, that is, the values at points on the portion of the x axis marked with the heavy line. *Data at points x not on this heavy line, that is, with $x < x_0 - ct_0$ or $x > x_0 + ct_0$, cannot influence $u(x_0,t_0)$.* This portion of the x axis is called the *domain of dependence* of $u(x_0,t_0)$. Another way to say this is that the initial information $f(x)$ and $g(x)$ cannot propagate with speed greater than c. The situation is quite different for the heat equation: the initial value $u(x,0) = f(x)$ will affect $u(x_0,t_0)$ for *every* (x_0,t_0), although the effect will be extremely small unless (x_0,t_0) is close to $(x,0)$ in some appropriate sense.