CHAPTER 2

Fourier Series and Separation of Variables

2.1 Periodic functions and Fourier series

We first recall the elementary definitions of even, odd, and periodic functions (see Section 17.2 of Greenberg). A function $f(x)$ is *even* if it is defined for all x (or possibly in some interval symmetric about $x = 0$, that is, of the form $(-a, a)$ or $[-a, a]$) and satisfies $f(x) = f(-x)$; it is *odd* if it is similarly defined and satisfies $f(-x) = -f(x)$. We will frequently use the observation that if $f(x)$ is defined for $-a \le x \le a$ then

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f \text{ is odd;} \\ 2\displaystyle\int_0^a f(x)\,dx, & \text{if } f \text{ is even.} \end{cases} \tag{2.1}$$

This formula is easily derived by writing $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ and making the change of variable $y = -x$ in the first integral.

A function $f(x)$ defined on for all x is *periodic with period* T if $f(x+T) = f(x)$ for all x. A constant function is periodic with any period. Aside from this, the most important periodic functions are the trigonometric functions $\sin x$ and $\cos x$; these are each periodic with period 2π. Because of this, each of the functions $\cos(n\pi x/\ell)$ and $\sin(n\pi x/\ell)$ listed in (1.14) is periodic with period 2ℓ. Linear combinations of functions all having the same period T have period T, so that a Fourier series

$$S(x) = a_0 + \sum_{n=1}^{\infty}\Bigl[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\Bigr]. \tag{2.2}$$

(see (1.25)) is periodic with period 2ℓ.

It is also convenient to use the idea of the periodic extension of a given function: if f is defined on the interval $[a, b]$ then the *periodic extension* $f_{\rm per}$ of f, which has period $T = b - a$, is defined simply by "repeating" f in all the intervals $[a + nT, b + nT]$ for $n = 0, \pm1, \ldots$, so that for all x,

$$f_{\rm per}(x) = f(x - nT) \quad \text{whenever } a + nT < x \le b + nT, \quad n = 0, \pm1, \pm2, \ldots. \tag{2.3}$$

In Figure 2.1 we show a picture for $a = 1$, $b = 3$, and $f(x) = x - 3/2$:

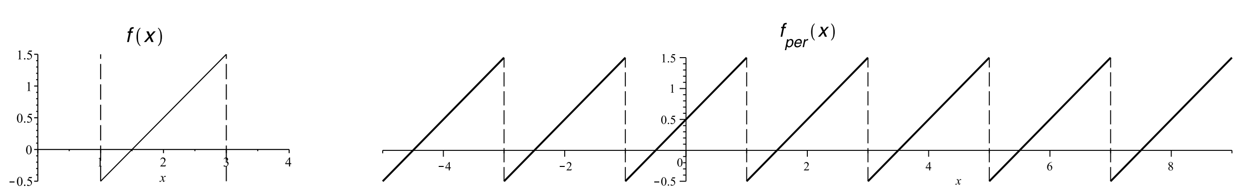

Figure 2.1: Periodic extension $f_{\rm per}(x)$ of a function $f(x)$ defined for $1 \le x \le 3$.

Note that $f_{\rm per}$ may be discontinuous at a, b, etc., even if f is continuous. A related fact is that in defining $f_{\rm per}$ we have taken $f_{\rm per}(a) = f(b)$ and not $f_{\rm per}(a) = f(a)$; some choice must be made but this has no effect in practice.

In Chapter 1 we discussed the Fourier series (2.2) as an expansion of a function f, let us say piecewise continuous, defined on the interval $[-\ell, \ell]$. Specifically, if we define the coefficients a_n and b_n by the formulas (1.26) that is, by

$$\begin{gathered} a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx, \qquad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1, \\ b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1, \end{gathered} \tag{2.4}$$

then we will write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\Big[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\Big] \qquad \text{for } -\ell \le x \le \ell, \tag{2.5}$$

to denote that the right hand side is the Fourier series of f. In fact, we know from Chapter 1 that the symbol $\sim$ in (2.5) can be replaced by an equality, if this is interpreted in the sense of one of the completeness statements of Section 1.5, that is, as in (1.31) or (1.32) (but recall that to be sure that (1.32) holds we need $f'(x)$ also to lie in $C_p[-\ell, \ell]$).

Remark 2.1: One may use (2.1) to considerably simplify the formulas (2.4) when f is even or odd. For example, if f is even then $f(x)\cos(n\pi x/\ell)$ is even and $f(x)\sin(n\pi x/\ell)$ is odd, so that from (2.1),

$$a_0 = \frac{1}{\ell}\int_0^{\ell} f(x)\,dx, \qquad a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \qquad b_n = 0, \qquad n \ge 1. \tag{2.6}$$

Similarly, if f is odd one has

$$a_0 = 0, \qquad a_n = 0, \qquad b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1. \tag{2.7}$$

Let us emphasize that in (2.4)–(2.7) we are considering the Fourier series of a function defined on the interval $[-\ell, \ell]$. We now want to relate these series to the Fourier series of periodic functions; there are two complementary ways of doing so.

Approach 1. Suppose that we are given a periodic piecewise continuous function $g(x)$, defined for all x; for the moment we assume that $g(x)$ which has period 2ℓ, i.e., that $g(x + 2\ell) = g(x)$. In particular, $g(x)$ is defined for x in the interval $[-\ell, \ell]$ and thus by restricting x to lie in this interval we obtain a function $f \in C_p[-\ell, \ell]$ (specifically, $f(x) = g(x)$ for $-\ell \le x \le \ell$ and $f(x)$ is undefined for other values of x). Then by the paragraph above we know that if we define a_n and b_n by (1.26), then f is the sum of its Fourier series in the sense of (1.32) (again assuming that $f'(x)$ is also piecewise continuous). But now the periodic extension $f_{\rm per}(x)$ of $f(x)$ is just $g(x)$, and the right hand side of (2.4) is already a periodic function with period 2ℓ. Thus we will also say that (2.2) is the Fourier series of $g(x)$:

$$g(x) \sim a_0 + \sum_{n=1}^{\infty}\Bigl[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\Bigr] \equiv S(x) \qquad \text{for all real } x. \tag{2.8}$$

As above, $\sim$ can be replaced by $=$ in the sense of (1.32), which here becomes

$$S(x) = \begin{cases} g(x), & \text{if } g \text{ is continuous at } x,\\ \dfrac{g(x+)+g(x-)}{2}, & \text{if } g \text{ is discontinuous at the point } x. \end{cases} \tag{2.9}$$

Note that in (2.9) we no longer need any special consideration for the endpoint of an interval, as we did in (1.32).

Approach 2. A second way to look at the connection of Fourier series on an interval with the Fourier series of periodic functions is to start with a function f defined only on the interval $[-\ell,\ell]$, say $f \in C_p[-\ell,\ell]$. Then $f_{\rm per}$, a periodic function of period 2ℓ, can play the role of g above; in particular, the Fourier series of f converges to $f_{\rm per}$ everywhere, in our usual sense:

$$S(x) = \begin{cases} f_{\rm per}(x), & \text{if } f_{\rm per} \text{ is continuous at } x,\\ \dfrac{f_{\rm per}(x+)+f_{\rm per}(x-)}{2}, & \text{if } f_{\rm per} \text{ is discontinuous at the point } x. \end{cases} \tag{2.10}$$

Remark 2.2: (a) Suppose that we are in the situation described above: $g(x)$ is periodic with period 2ℓ, $f(x)$ is defined on $[-\ell,\ell]$, and $g(x) = f_{\rm per}(x)$ or, equivalently, $f(x)$ is the restriction of $g(x)$ to the interval $[-\ell,\ell]$. Then since $f(x) = g(x)$ for $-\ell \le x \le \ell$, f can be replaced by g in the definition (2.4) of the Fourier coefficients:

$$\begin{gathered} a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} g(x)\,dx, \qquad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} g(x)\cos\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1,\\ b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} g(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1, \end{gathered} \tag{2.11}$$

Thus we never need to think about $f(x)$ at all: everything can be expressed in terms of $g(x)$. Furthermore, since each integrand in (2.11) is now periodic with period 2ℓ, the interval $[-\ell,\ell]$ over which the integration is carried out may be replaced by any other interval of the same length: for any X,

$$\begin{gathered} a_0 = \frac{1}{2\ell}\int_{X}^{X+2\ell} g(x)\,dx, \qquad a_n = \frac{1}{\ell}\int_{X}^{X+2\ell} g(x)\cos\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1,\\ b_n = \frac{1}{\ell}\int_{X}^{X+2\ell} g(x)\sin\frac{n\pi x}{\ell}\,dx, \qquad n \ge 1. \end{gathered} \tag{2.12}$$

(b) In the discussion above we have always taken $f(x)$ to be defined on $[-\ell, \ell]$, but this is not necessary. We can start with $f(x)$ defined on any interval $[a, b]$, obtain its periodic extension $f_{\rm per}(x) = g(x)$, and then expand $g(x)$ in a Fourier series. Suppose that $b-a = T$, so that $g(x)$ has period T; then we may use (2.12), with $2\ell = T$, to find the Fourier coefficients of $g(x)$. In particular, if we take $X = a$ then $X + 2\ell = b$ and we obtain expressions involving the intergral of $g(x)$, or equivalently $f(x)$, over the original interval $[a, b]$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \Bigl[a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}\Bigr] \tag{2.13}$$

with

$$\begin{gathered} a_0 = \frac{1}{T}\int_a^b f(x)\,dx, \qquad a_n = \frac{2}{T}\int_a^b f(x)\cos\frac{2n\pi x}{T}\,dx, \qquad n \geq 1, \\ b_n = \frac{2}{T}\int_a^b f(x)\sin\frac{2n\pi x}{T}\,dx, \qquad n \geq 1. \end{gathered} \tag{2.14}$$

Finally, everything said above applies also to the complex form of the Fourier series: a function $g(x)$, periodic with period 2ℓ, has a complex Fourier series

$$g(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}, \tag{2.15}$$

with

$$c_n = \frac{1}{2\ell}\int_X^{X+2\ell} g(x)e^{-in\pi x/\ell}\,dx. \tag{2.16}$$

and with convergence in the sense of (2.9).

2.2 Separation of variables

To provide motivation for further study of Fourier series we will discuss here a very simple case of the method of *separation of variables* for solving partial differential equations (PDE). More examples of this method will be considered in Section 2.3 and Section 2.5.

Remark 2.3: All the equations that we will study will be *linear*; this means that, if $u(x, t)$ is the unknown function that we want to find, every term in the equation will be either

(i) u itself or some partial derivative of u, possibly multiplied by function of x and/or t, or

(ii) a term independent of u, that is, a constant or some function of x and/or t.

For example, the *heat equation* on an interval, which we will consider in this section, is

$$u_t(x, t) - \alpha^2 u_{xx}(x, t) = f(x, t), \qquad 0 < x < L, \quad t > 0. \tag{2.17}$$

When the equation contains no term independent of u, that is, no term of type (ii), it is called *homogeneous*; otherwise it is called *inhomogeneous*. (2.17) is inhomogeneous (unless f is zero); the corresponding homogeneous equation is

$$u_t(x,t) - \alpha^2 u_{xx}(x,t) = 0, \qquad 0 < x < L, \quad t > 0. \tag{2.18}$$

In these notes we will use separation of variables only for solving homogeneous PDE. We will solve inhomogeneous PDE using a particular solution; see Chapter 3. I believe that this approach is much clearer than that of Greenberg, who sometimes uses separation of variables for inhomogeneous problems.

The (homogeneous) heat equation (2.18), also called the *diffusion equation*, describes the temperature of a rod of length L. It is written in terms of a coordinate system along the rod for which the coordinate x varies from $x = 0$ at the left end of the rod to $x = L$ at the right end. The rod is assumed to be of such small cross section that we can regard the temperature as depending only on this coordinate x and the time t, neglecting any variation of the temperature in directions perpendicular to the axis of the rod; the variable $u(x,t)$ denotes this temperature. We also assume that the lateral surface of the rod is well insulated, so that heat can flow into or out of the rod only through the ends. Under these assumptions, $u(x,t)$ satisfies (2.18).

Let us rewrite (2.18) slightly as

$$\text{PDE:} \qquad u_t(x,t) = \alpha^2 u_{xx}(x,t), \qquad 0 < x < L, \quad t > 0. \tag{2.19}$$

We emphasize that here subscripts denote partial derivatives, that is, $u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$. α^2 is a constant, the *diffusion constant* or *thermal diffusivity*, which depends on the properties of the material of which the rod is formed. See Section 18.2.3 of Greenberg for a derivation of the heat equation and a discussion of this constant.

To determine $u(x,t)$, however, we need more than the heat equation alone; as indicated above, heat can enter or leave the rod through its ends, and we must specify *boundary conditions* which determine this heat flow. For the moment let us suppose that the temperature at each end is held constant and equal to zero:

$$\text{BC:} \qquad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \qquad t > 0. \tag{2.20}$$

These are called *homogeneous Dirichlet boundary conditions:* they are *homogeneous* because both equations in (2.20) have 0 on the right hand side, and the term *Dirichlet* here refers to the fact that the boundary condition at a particular boundary, $x = 0$ or $x = L$, involves only the *value* of the temperature at that boundary. Other naturally occurring boundary conditions, which we will discuss later, involve also the derivative $u_x(x,t)$ at the boundary $x = 0$ or $x = L$.

Finally, we must give an *initial condition* specifying the temperature when the process starts, say at $t = 0$:

$$\text{IC:} \qquad u(x,0) = f(x), \qquad 0 < x < L, \tag{2.21}$$

where $f(x)$ is some given function defined for $0 < x < L$.

The PDE (2.19), boundary conditions (2.20), and initial condition (2.21) form one *initial/boundary value problem* which we wish to solve to determine $u(x,t)$ for all (x,t) with $0 < x < L$ and $t > 0$. At the risk of redundancy, we summarize:

Problem 1: Find a function $u(x,t)$ satisfying

$$\begin{array}{rccr} \text{PDE:} & u_t(x,t) = \alpha^2 u_{xx}(x,t), & 0 < x < L, \quad t > 0, & \\ \text{BC:} & u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, & t > 0 & (2.22) \\ \text{IC:} & u(x,0) = f(x), & 0 < x < L. & \end{array}$$

The method we will use is *separation of variables*, which may be broken down into three steps:

Step 1: Find nonzero solutions of the partial differential equation (2.19) which have a product form

$$u(x,t) = X(x)T(t). \tag{2.23}$$

Step 2: Select from among the solutions found in Step 1 those solutions which satisfy the boundary condition (2.20). There will typically be an infinite sequence of these:

$$u_n(x,t) = X_n(x)T_n(t), \qquad n = 1, 2, \ldots. \tag{2.24}$$

Step 3: Observe that, because the PDE and BC are linear and homogeneous, any linear combination of solutions of these will again be a solution. Thus for any choice of coefficients $c_1, c_2, \ldots$ the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) \tag{2.25}$$

will again be a solution of the PDE and BC (assuming the series converges). Choose the constants c_n so that $u(x,t)$ satisfies the initial condition (2.21).

Remark 2.4: In Step 1 we specified a nonzero solution. This is because the zero function $u(x,t) = 0$ is always a solution of our problem (2.19) with (2.20), because both of these equations are linear and homogeneous. However, this solution cannot help us satisfy the initial condition (2.21) (unless $f(x) = 0$ for all x, in which case $u(x,t) = 0$ is a solution and we are done). For this reason we do not include the zero solution in carrying out Steps 2 and 3.

Let us now apply this program to our problem (2.19)–(2.21). For the first step we substitute the product form (2.23) into the PDE (2.19), and see that $X(x)$ and $T(t)$ must satisfy $X(t)T'(t) = \alpha^2 X''(x)T(t)$ or, dividing through by $\alpha^2 X(x)T(t)$,

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{2.26}$$

Now the left side of (2.26) depends only on t—it is independent of x—and the right side depends only on x, yet the equality holds for all x and t. This can happen only if both sides are constant—say both equal to $-\lambda$. We conclude that $u(x,t) = X(x)T(t)$ is a solution of (2.19) if and only if for some constant λ,

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

that is,

$$T'(t) = -\alpha^2 \lambda T(t) \tag{2.27a}$$

$$X''(x) = -\lambda X(x) \tag{2.27b}$$

Now technically λ could be complex, but this possibility will never interest us. There are thus three cases to consider: those in which λ is positive, negative, or zero; when $\lambda > 0$ we write $\lambda = \kappa^2$ and when $\lambda < 0$ we write $\lambda = -k^2$. Then from (2.27) we obtain the solutions in these three cases (A and B are arbitrary constants, not both zero by Remark 2.4):

		Equation	$X(x)$	$T(t)$	
(I)	$\lambda = \kappa^2 > 0$	$X'' + \kappa^2 X = 0$	$A\cos\kappa x + B\sin\kappa x$	$e^{-\alpha^2\kappa^2 t}$	
(II)	$\lambda = 0$	$X'' = 0$	$A + Bx$	1	(2.28)
(III)	$\lambda = -k^2 < 0$	$X'' - k^2 X = 0$	$A\cosh kx + B\sinh kx$	$e^{\alpha^2 k^2 t}$	

In each case, $u(x,t) = X(x)T(t)$.

Now we pass to the second step: determining which solutions in (2.28) satisfy the boundary conditions (2.20). Consider the condition that $u(0,t) = 0$ for all t; since $u(t,0) = X(0)T(t)$ this requires that $X(0) = 0$; the condition is similar for $x = L$ and we conclude that (2.20) reduces to

$$X(0) = X(L) = 0. \tag{2.29}$$

We analyze (2.29) separately for the three cases of (2.28).

(I) Since $X(x) = A\cos\kappa x + B\sin\kappa x$, $X(0) = A$ and (2.29) tells us that $A = 0$. But then $X(L) = B\sin\kappa L$, and $X(L) = 0$ requires either that $B = 0$ or $\sin\kappa L = 0$. The first possibility would lead to $X(x) = 0$ for all X, an uninteresting case (see Remark 2.4), so we consider only the second possibility, which requires that $\kappa = n\pi/L$ or $\lambda = (n\pi/L)^2$, $n = 1, 2, \ldots$. Thus from case I we have the solutions

$$u_n(x,t) = \sin\frac{n\pi x}{L} e^{-(\alpha n\pi/L)^2 t}, \qquad \lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, \ldots. \tag{2.30}$$

(II) Since $X(x) = A + Bx$, $X(0) = A$ and (2.29) requires that $A = 0$. Then $X(x) = Bx$ and $X(L) = BL$; since L is not zero, $X(L) = 0$ requires that $B = 0$, so that $X(x) = 0$ for all x; we reject this solution (see Remark 2.4). Case II leads to no interesting solutions.

(III) Since $X(x) = A\cosh kx + B\sinh kx$, $X(0) = A$ and (2.29) requires that $A = 0$. Then $X(x) = B\sinh kx$ and $X(L) = B\sinh kL$; since $\sinh u = 0$ only if $u = 0$, $X(L) = 0$ again requires that $B = 0$ and we have only the (uninteresting) zero solution.

We conclude that the only solutions of our PDE and BC found by separation of variables are those of (2.30).

We now turn to the third step of our program, and thus ask: Can we find constants $c_1, c_2, \ldots$ such that

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} e^{-(\alpha n\pi/L)^2 t}, \tag{2.31}$$

which we know solves our PDE and BC, also satisfies the initial condition $u(x,0) = f(x)$? Since $u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L)$, this is equivalent to asking:

> **Q1:** Given a function $f(x)$ defined on $[0,L]$, do there exist constants $c_1, c_2, \ldots$ such that
>
> $$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}, \qquad \text{for} \quad 0 \le x \le L \quad ? \tag{2.32}$$

If the answer is "yes" then (2.31) furnishes a solution to our problem (2.19)–(2.21). We analyze this question in the next section.

Remark 2.5: We may also take a slightly different, but equivalent, point of view on our procedure of separation of variables; specifically we may combine the differential equation for $X(x)$ in (2.27b) with (2.29) to obtain a *boundary value problem*

$$\text{ODE:} \qquad X''(x) = -\lambda X(x) \quad \text{for some } \lambda, \qquad 0 < x < L, \tag{2.33a}$$

$$\text{BC:} \qquad X(0) = 0, \quad X(L) = 0. \tag{2.33b}$$

After we have solved this problem, producing the λ_n and correspondingly the solutions $X_n(x) = \sin(n\pi x/L)$, we solve (2.27a) to obtain $T_n(t)$, and then $u_n(x,t) = X_n(x)T_n(t)$.

2.3 Half range and quarter range Fourier series

In this section we describe several useful Fourier-type series associated with a function $f(x)$ defined on the interval $[0,L]$, say $f \in C_p[0,L]$. Each of these is associated with some extension of f to a periodic function defined on the entire real line; graphs of these various extensions are shown in Figure 2.2 when $L = \pi$ and the function f is given by $f(x) = x$, $0 \le x \le \pi$.

A. The Fourier series of f. This is the series discussed in Section 2.1; see in particular Remark 2.2(b). Since $f(x)$ is defined for $0 \le x \le L$ it is the Fourier series of the periodic extension $f_{\rm per}$ of f to a function of period L. It has the form (2.13) with $\ell = L/2$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \Big[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L} \Big], \tag{2.34}$$

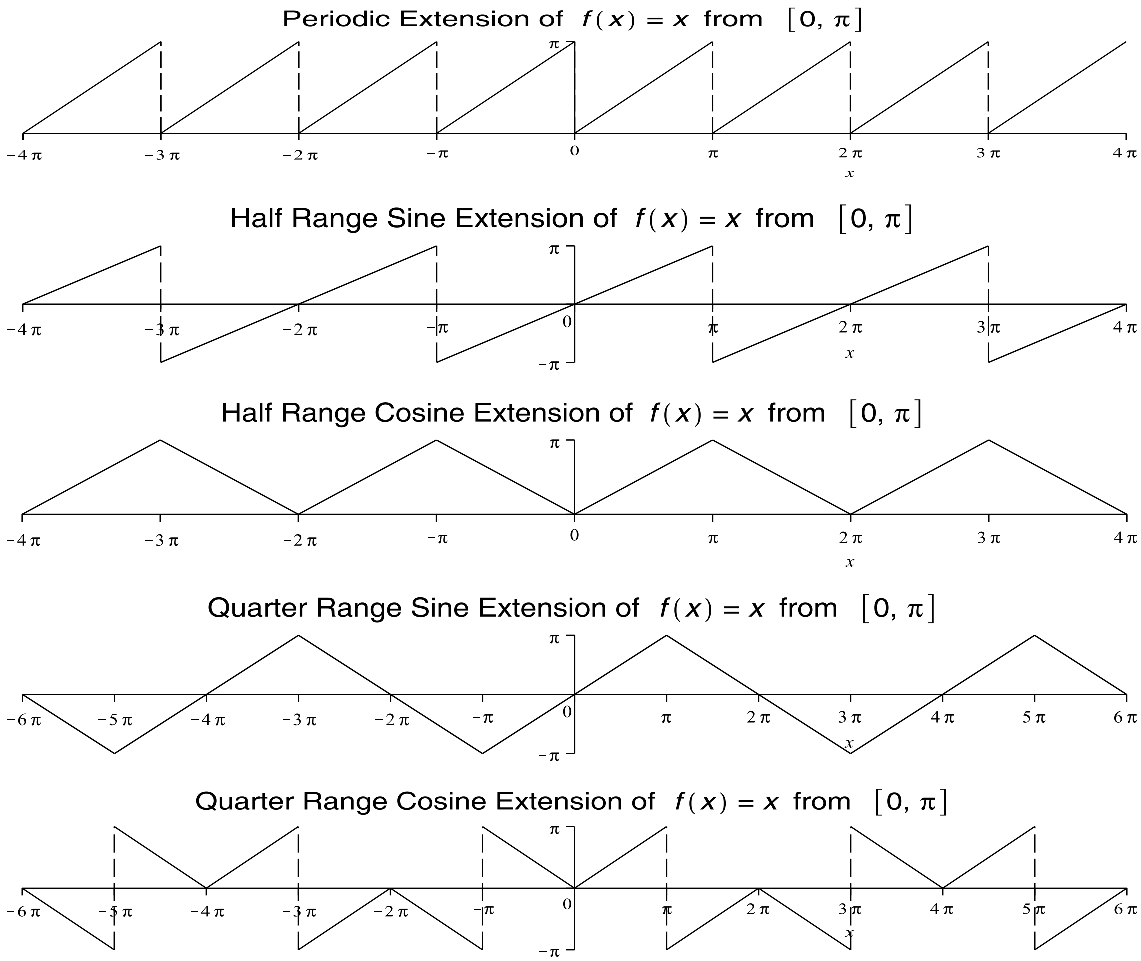

Figure 2.2: Various extensions of $f(x) = x$ from $[0, \pi]$ to $\mathbb{R}$.

with coefficients given by (2.14):

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx, \qquad n \geq 1,$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx, \qquad n \geq 1, \tag{2.35}$$

We know already that this series converges to $f(x)$ (in the sense of (1.32) or (1.32)).

B. The half range sine series of f. The series in (2.32) is called a *half range sine series*. Note that in comparison with (2.34), (2.32) involves only sine functions, and the n^{th} sine term is $\sin(n\pi x/L)$, not $\sin(2n\pi x/L)$; this is the origin of the name.

There are two ways to think about (2.32). First, one can check easily that the functions $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$, form an orthonormal set in $C_p[0, L]$ with the usual inner product. (**Exercise:** verify this.) This immediately tells us that the correct formula for the coefficients b_n in (2.32) is

$$b_n = \frac{\langle f(x), \sin(n\pi x/L)\rangle}{\|\sin(n\pi x/L)\|^2} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{2.36}$$

The remaining question is whether or not this orthonormal set is complete, that is, whether or not equality holds in (2.32) for every function $f(x)$.

The second way to look at (2.32) also furnishes an answer to that question. Let us start with the function f defined on $[0, L]$, say $f \in C_p[0, L]$. We then define $f_1(x)$, the *odd extension of f with period $2L$*, which we will also call the *half range sine extension*, by first extending f to an odd function defined on $[-L, L]$, taking $f_1(x) = -f(-x)$ for $-L \le x < 0$, and then extending this odd function to a periodic function, with period $2L$, on all of $\mathbb{R}$ (see Figure 2.2). Now $f_1(x)$ has a Fourier series as in (2.5) (with $\ell = L$); by (2.7) the Fourier coefficients are given by

$$a_0 = a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n \ge 1, \tag{2.37}$$

where we have used the fact that $f_1(x) = f(x)$ for $0 \le x \le L$. The series is then just

$$f_1(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{2.38}$$

In fact, we know that here $\sim$ can be replaced by $=$ (in the usual sense) for all x, and in particular for all $x \in [0, L]$ where $f_1(x) = f(x)$; thus we obtain

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 \le x \le L,$$

again in the sense of (1.31) or (1.32). But this is just the half-range sine series (2.32), and (2.37) is just the formula for the coefficients that we obtained in (2.36) above. We summarize:

Half range sine series: The half range sine series (2.32) or (2.38) is the Fourier series of the odd periodic extension f_1 of f; it converges to $f(x)$ for x in $[0, L]$ and the coefficients $c_n = b_n$ are given by (2.36). The orthonormal set of all function $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$, is complete in $C_p[0, L]$.

In addition to the half range sine series there are three other commonly used Fourier-like series expansions of a function $f(x)$ defined on $[0, L]$. Each is associated with a particular boundary value problem like (2.33) (or equivalently an initial/boundary value problem like (2.22)) and is the true Fourier series of some extension of $f(x)$ from $[0, L]$ to the entire line. The development in each case is almost exactly parallel to the development of the half range sine series above. We will therefore discuss the first two of these—the half range cosine and quarter range sine series—very briefly, but to further illustrate the ideas we will give some of the computations for the third—the quarter range cosine series—in more detail.

C. The half range cosine series of f. Consider the initial/boundary value problem

$$\begin{aligned}
&\text{PDE:} && u_t(x,t) = \alpha^2 u_{xx}(x,t), && 0 < x < L, \quad t > 0, \\
&\text{BC:} && u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, && t > 0 \\
&\text{IC:} && u(x,0) = f(x), && 0 < x < L.
\end{aligned} \tag{2.39}$$

This is the same as the problem (2.22) considered above except that the boundary condition there has been replaced by a *homogeneous Neumann boundary condition*, that is, a condition involving only the partial derivative $u_x(x,t)$ at the boundaries. Applying separation of variables as in Section 2.2 leads to a boundary value problem similar to (2.33):

$$\text{ODE:} \qquad X''(x) = -\lambda X(x) \quad \text{for some } \lambda, \qquad 0 < x < L, \tag{2.40a}$$

$$\text{BC:} \qquad X'(0) = 0, \quad X'(L) = 0. \tag{2.40b}$$

This is analyzed just as in Section 2.2: this time we find that one possible value of λ is $\lambda_0 = 0$, with corresponding solution $X_0(x) = 1$. The other possible values of λ are $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$, with solutions $X_n(x) = \cos(n\pi x/L)$. In each case we solve (2.27a) to find $T_n(t) = \exp(-\alpha^2 \lambda_n t)$. Thus we can solve (2.39), with solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n X_n(x) T_n(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(\alpha n\pi/L)^2 t},$$

if every function $f(x)$ may we written as a *half range cosine series:*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad 0 \le x \le L. \tag{2.41}$$

To investigate (2.41) we first note that the functions $X_n(x)$, $n = 0, 1, 2, \ldots$, form an orthogonal set in $C_p[0,L]$ (**Exercise:** verify this), leading via (1.24) to the formulas

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots. \tag{2.42}$$

We then consider the *even periodic extension* $f_2(x)$ *of* $f(x)$ *with period* $2L$, which we also call the *half range cosine extension of* f*;* this is defined by first extending $f(x)$ to an even function on $[-L, L]$ and then to a periodic function on all of $\mathbb{R}$; see Figure 2.2. $f_2(x)$ has a Fourier series which, on the interval $[0,L]$, coincides with the series in (2.41), and has coefficients given by (2.42); the fact that the Fourier series converges to f_2 guarantees that (2.41) holds, i.e., that the orthogonal set of functions X_n, $n = 0, 1, 2, \ldots$, is complete. Briefly:

Half range cosine series: The half range cosine series (2.41) is the Fourier series of the even periodic extension f_2 of f; it converges to $f(x)$ for x in $[0, L]$ and the coefficients a_n are given by (2.42). The orthogonal set consisting of 1 and of all functions $\cos(n\pi x/L)$, $n = 1, 2, 3, \ldots$, is complete in $C_p[0, L]$.

D. The quarter range sine series of f. Consider the initial/boundary value problem

$$\begin{aligned} &\text{PDE:} && u_t(x,t) = \alpha^2 u_{xx}(x,t), && 0 < x < L, \quad t > 0, \\ &\text{BC:} && u(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, && t > 0 \\ &\text{IC:} && u(x,0) = f(x), && 0 < x < L. \end{aligned} \tag{2.43}$$

This is the same as the problems (2.22) and (2.39) considered above except that the we now have a homogeneous Dirichlet boundary condition at $x = 0$ and a homogeneous Neumann boundary condition at $x = L$. Separation of variables leads to the boundary value problem

$$\text{ODE:} \qquad X''(x) = -\lambda X(x) \quad \text{for some } \lambda, \qquad 0 < x < L, \tag{2.44a}$$

$$\text{BC:} \qquad X(0) = 0, \quad X'(L) = 0. \tag{2.44b}$$

Now the possible values of λ are of the form $\lambda_n = (n\pi/2L)^2$ with n odd; the solutions are $X_n(x) = \sin(n\pi x/2L)$, and we then solve (2.27a) to find $T_n(t) = \exp(-\alpha^2\lambda_n t)$. Thus we can solve (2.43), with solution of the form

$$u(x,t) = \sum_{n \text{ odd}} b_n X_n(x) T_n(t) = \sum_{n \text{ odd}} b_n \sin\frac{n\pi x}{2L} e^{-(\alpha n\pi/2L)^2 t},$$

if every function $f(x)$ may we written as a *quarter range sine series:*

$$f(x) = \sum_{n \text{ odd}} b_n \sin\frac{n\pi x}{2L}, \qquad 0 \le x \le L. \tag{2.45}$$

Again the functions $X_n(x)$, n odd, form an orthogonal set in $C_p[0, L]$ (**Exercise:** verify this), leading via (1.24) to the formulas

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{2L}\, dx, \qquad n \text{ odd}. \tag{2.46}$$

The *quarter range sine extension of* $f(x)$, $f_3(x)$, is an extension of period $4L$; to obtain it we first extend $f(x)$ to the interval $[0, 2L]$ in such a way that it is symmetric around $x = L$ (the formula is $f(x) = f(2L - x)$ for x in $[L, 2L]$), extend this function to an *odd* function on $[-2L, 2L]$, and then make a periodic extension to all of $\mathbb{R}$ (see Figure 2.2). $f_3(x)$ has a Fourier series which, on the interval $[0, L]$, coincides with the series in (2.51), and has coefficients given by (2.52); the fact that the Fourier series converges to f_3 guarantees that (2.51) holds, i.e., that the orthogonal set of functions X_n, $n = 1, 3, 5, \ldots$, is complete.

Quarter range sine series: The quarter range sine series (2.51) is the Fourier series of the quarter range sine periodic extension f_3 of f; it converges to $f(x)$ for x in $[0, L]$ and the coefficients b_n are given by (2.52). The orthogonal set consisting of all functions $\sin(n\pi x/2L)$, n odd, is complete in $C_p[0, L]$.

E. The quarter range cosine series of f. Now we consider the initial/boundary value problem

$$\begin{aligned}
&\text{PDE:} && u_t(x,t) = \alpha^2 u_{xx}(x,t), && 0 < x < L, \quad t > 0,\\
&\text{BC:} && u_x(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, && t > 0\\
&\text{IC:} && u(x,0) = f(x), && 0 < x < L.
\end{aligned} \tag{2.47}$$

that is, we impose a homogeneous Neumann boundary condition at $x = 0$ and a homogeneous Dirichlet condition at $x = L$. Separation of variables leads to the boundary value problem

$$\text{ODE:} \qquad X''(x) = -\lambda X(x) \quad \text{for some } \lambda, \qquad 0 < x < L, \tag{2.48a}$$

$$\text{BC:} \qquad X'(0) = 0, \quad X(L) = 0. \tag{2.48b}$$

Just as in Section 2.2 the form of the solutions of the differential equation (2.48a) depend on the sign of λ; the alternatives are as in (2.28), which we reproduce in part here:

$$\begin{array}{rcc}
 & \lambda & X(x)\\
\textbf{(I)} & \lambda = \kappa^2 > 0 & A\cos\kappa x + B\sin\kappa x\\
\textbf{(II)} & \lambda = 0 & A + Bx\\
\textbf{(III)} & \lambda = -k^2 & A\cosh kx + B\sinh kx
\end{array} \tag{2.49}$$

We now carry out, for each of the three cases in (2.49), the analysis of which solutions can satisfy the boundary condition.

(I) Since $X(x) = A\cos\kappa x + B\sin\kappa x$, $X'(0) = kB$ and since $k \neq 0$ (the case $\lambda = 0$ is case II), (2.48b) tells us that $B = 0$ and so $X(L) = A\cos\kappa L$. Now $X(L) = 0$ requires either that $A = 0$ or $\cos\kappa L = 0$. As usual we reject the first possibility (which leads to $X(x) = 0$ for all x); the second possibility requires that $\kappa = n\pi/2L$ for some odd value of n. Then $\lambda_n = (n\pi/2L)^2$, $n = 1, 3, 5, \ldots$, and we have solutions of (2.48):

$$X_n(x) = \cos\frac{n\pi x}{2L}, \qquad \lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 3, 5, \ldots. \tag{2.50}$$

(II) Since $X(x) = A + Bx$, $X'(0) = B$ and (2.29) requires that $B = 0$. Then $X(x) = A$ and in particular $X(L) = A$; thus (2.48b) requires that $B = 0$. Case II leads to no interesting solutions.

(III) Since $X(x) = A\cosh kx + B\sinh kx$, $X'(0) = b$ and (2.48b) requires that $B = 0$ and so $X(x) = A\cosh kx$ and $X(L) = A\cosh kL$; since $\cosh u > 0$ for all u, $X(L) = 0$ requires that $A = 0$ and we have only the (uninteresting) zero solution.

We conclude that (2.50) gives all solutions of the boundary value problem (2.48).

The function $T_n(t)$ associated with λ_n is still $T_n(t) = \exp(-\alpha^2\lambda_n t)$, and so we propose a solution of the initial/boundary value problem (2.47) in the form

$$u(x,t) = \sum_{n\ \mathrm{odd}} a_n X_n(x) T_n(t) = \sum_{n\ \mathrm{odd}} a_n \cos\frac{n\pi x}{2L} e^{-(\alpha n\pi/2L)^2 t}.$$

We can then solve the inital value problem if every function $f(x)$ defined on $[0, L]$ may we written as a *quarter range cosine series:*

$$f(x) = \sum_{n\ \mathrm{odd}} a_n \cos\frac{n\pi x}{2L}, \qquad 0 \le x \le L. \tag{2.51}$$

The investigation of (2.51) is parallel to similar discussions above. The functions $X_n(x)$, n odd, form an orthogonal set in $C_p[0, L]$ (**Exercise:** verify this). Thus we know that the right hand side of (2.51) is the best possible approximation to $f(x)$ if we take

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx, \qquad n = 1, 3, 5, \ldots. \tag{2.52}$$

In order to verify the equality in (2.51) we then consider the *quarter range cosine extension $f_4(x)$ of $f(x)$*, which has period $4L$. Let us define this very carefully. We first extend $f(x)$ to a function $g(x)$ on the interval $[0, 2L]$ in such a way that g is *antisymmetric* under reflection across $x = L$, that is, by defining $g(x) = f(x)$ for x in $[0, L]$ and $g(x) = -f(2L - x)$ for x in $[L, 2L]$. Next we extend $g(x)$ to an *even* function $h(x)$ on $[-2L, 2L]$, defining $h(x) = g(x)$ if x lies in $[0, 2L]$ and $h(x) = g(-x)$ if x lies in $[-2L, 0]$. Finally we take $f_4(x)$ to be the periodic extension h_{per} of h (which will have period $4L$): $f_4(x) = h_{\mathrm{per}}(x)$. See Figure 2.4.

Now $f_4(x)$ has a Fourier series of the form

$$f_4(x) \sim a_0 + \sum_{n=1}^{\infty}\Bigl(a_n\cos\frac{n\pi x}{2L} + b_n\sin\frac{n\pi x}{2L}\Bigr). \tag{2.53}$$

Because f_4 is an even function we know that the coefficients b_n all vanish, but we also want to show that the coefficients a_n are such that (2.53) reduces to (2.51) with coefficients given by (2.52). We evaluate the coefficients a_n in (2.53) using (2.6) and the fact that $f_4(x) = g(x)$ on the interval $[0, 2L]$:

$$\begin{aligned} a_0 = \frac{1}{2L}\int_0^{2L} f_4(x)\,dx &= \frac{1}{2L}\int_0^{2L} g(x)\,dx \\ &= \frac{1}{2L}\Bigl[\int_0^L g(x)\,dx + \int_L^{2L} g(x)\,dx\Bigr] \\ &= \frac{1}{2L}\Bigl[\int_0^L f(x)\,dx + \int_L^{2L} (-f(2L-x))\,dx\Bigr] \\ &= \frac{1}{2L}\Bigl[\int_0^L f(x)\,dx - \int_0^L f(u)\,dx\Bigr] = 0, \end{aligned} \tag{2.54}$$

where at the last step we have made the substitution $u = 2L - x$. We compute a_n for $n > 0$ similarly (we omit a few steps which can be easily reconstructed from (2.54)):

$$\begin{aligned}
a_n &= \frac{1}{L}\int_0^{2L} g(x)\cos\frac{n\pi x}{2L}\,dx \\
&= \frac{1}{L}\left[\int_0^{L} f(x)\cos\frac{n\pi x}{2L}\,dx - \int_L^{2L} f(2L-x)\cos\frac{n\pi x}{2L}\,dx\right] \\
&= \frac{1}{L}\left[\int_0^{L} f(x)\cos\frac{n\pi x}{2L}\,dx - \int_0^{L} f(u)\cos\frac{n\pi(2L-u)}{2L}\,dx\right] \\
&= \frac{1}{L}\left[\int_0^{L} f(x)\cos\frac{n\pi x}{2L}\,dx + (-1)^{n+1}\int_0^{L} f(u)\cos\frac{n\pi u}{2L}\,dx\right] \\
&= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{2}{L}\displaystyle\int_0^{L} f(x)\cos\frac{n\pi x}{2L}\,dx, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned} \tag{2.55}$$

Here we have first made the substitution $u = 2L - x$ and then used the identity $\cos(n\pi - \theta) = (-1)^n\cos(\theta)$. We see that inserting (2.54), (2.55), and $b_n = 0$ into (2.53) yields (2.51) with (2.52)—that is, the quarter range cosine series (2.51) is essentially just the Fourier series of f_4.

> **Quarter range cosine series:** The quarter range cosine series (2.51) is the Fourier series of the quarter range cosine extension f_4 of f; it converges to $f(x)$ for x in $[0, L]$ and the coefficients a_n are given by (2.52). The orthogonal set consisting of all functions $\cos(n\pi x/2L)$, $n = 1, 3, 5, \ldots$, is complete in $C_p[0, L]$.

Remark 2.6: With each of the half and quarter range series considered in B through E above we associated a certain boundary value problem. The Fourier series considered in A is similarly associated with such a problem—one with *periodic boundary conditions.* We will study this problem when we discuss Sturm-Liouville problems.

2.4 Sturm-Liouville problems

NO DETAILED NOTES ON THIS TOPIC HAVE BEEN PREPARED; SEE SECTIONS 17.7 AND 17.8 OF GREENBERG.