INFINITE SERIES, POWER SERIES, AND TAYLOR SERIES

The principal concern of Math 527 is solving ordinary and partial differential equations as explicitly as possible. It turns out that simple combinations of the elementary functions learned in calculus—the algebraic, exponential, logarithmic and trigonometric functions—are usually not adequate to this task. However, infinite series of elementary functions, such as power series, which are infinite series of polynomial functions, or Fourier series, which are infinite series of trigonometric functions, do often work. The application of infinite series to differential equations is a major theme of Math 527, and so it is important to understand these series clearly.

Section 1. Infinite series

Let $\{a_n\} = \{a_1, a_2, \ldots\}$ be a sequence of real numbers. The expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_2 + \cdots,$$

is called an *infinite series*. If $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists and is finite, we say that the infinite series *converges*, and we identify the value of the infinite series with this limit:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \qquad \text{if the limit exists and is finite.}$$

If this limit does not exist or is infinite, the infinite series is said to *diverge*. The sums $\sum_{n=1}^{N} a_n$, where N is finite, are called *partial sums* of the infinite series.

The infinite series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If an infinite series converges absolutely, then it converges. A series which converges, but not absolutely, is said to be *conditionally convergent.*

Warning. An elementary mistake is to confuse convergence of an infinite series with convergence of the sequence of terms in the infinite sum. Keep clear the difference between $\lim_{n\to\infty} a_n$ and $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$; only the latter, if it exists, represents the value of the infinite series.

Example 1: The most important example of an infinite series is the *geometric series* $\sum_{n=0}^{\infty} r^n$. We have

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1. \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases} \tag{1}$$

For $r \neq 1$ this is easy to see from the formula for the partial sums of the geometric series,

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}, \qquad r \neq 1,$$

since we then have

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N} r^n &= \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r} = \frac{1}{1 - r}\left[1 - \lim_{N\to\infty} r^{N+1}\right] \\ &= \begin{cases} \dfrac{1}{1-r}, & \text{if } |r| < 1, \\ \text{diverges}, & \text{if } |r| > 1 \text{ or } r = -1. \end{cases} \end{aligned}$$

Note that if $r = -1$ then $\lim_{N\to\infty} r^{N+1}$ does not exist because the values r^{N+1} oscillate from 1 to -1 as N changes. If $r = 1$, then $\sum_{n=0}^{N} r^n = N + 1$, which grows to infinity as $N \to \infty$, implying divergence.

Section 2. Tests for Convergence and divergence

The elementary theory of infinite series is concerned mostly with techniques—called *tests*—for determining convergence or divergence. We give here some of the most important of these

The divergence test. A basic fact about infinite series is that if the series $\sum_1^\infty a_n$ converges then it must be true that $\lim_{n\to\infty} a_n = 0$. As a consequence we obtain the

Divergence test

If $\lim_{n\to\infty} a_n 0$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_1^\infty a_n$ diverges.

For example, the divergence test implies that $\sum_1^\infty n/(n+1)$ and $\sum_1^\infty (-1)^n$ both diverge; in the first case this is because $\lim_{n\to\infty} n/(n+1) = 1 \neq 0$ and in the second case because $\lim_{n\to\infty} (-1)^n$ does not exist. The divergence test also shows that the geometric series diverges whenever $|r| \geq 1$.

The divergence test is an easy first test. However, the converse of the divergence test does not hold: that is, $\lim_{n\to\infty} a_n = 0$ *does not imply* that $\sum_1^\infty a_n$ converges. The *harmonic series* $\sum_1^\infty n^{-1}$ is an example; although $\lim_{n\to\infty} n^{-1} = 0$, the harmonic series diverges. Therefore, when $\lim_{n\to\infty} a_n = 0$ one must resort to more refined tests to discriminate between convergence and divergence. These tests all involve comparison of the infinite series, either to an improper integral or to an infinite series with known convergence properties.

The ratio test. This is the most important test for Math 527. It can be used only if the limit of the ratio a_{n+1}/a_n, or more properly the limit of the absolute value of this ratio, exists.

Ratio test

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_1^\infty a_n$ converges absolutely;

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_1^\infty a_n$ diverges;

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive.

To say that the test is inconclusive means that the series may either converge or diverge, and some other method must be used to determine which.

The ratio test is best understood intuitively as a test that compares $\sum_1^\infty a_n$ to a geometric series. Let $r = \lim_{n\to\infty} a_{n+1}/a_n$. Heuristically, this says that for all large n, $a_{n+1} \approx a_n r$. Fixing a suitably large m, we get that $a_{m+1} \approx a_m r$, $a_{m+2} \approx a_{m+1} r \approx a_m r^2$, and, continuing in this manner, $a_{m+k} \approx a_m r^k$; thus, for large m, the terms a_m of the series look approximately like those of a geometric series, and accordingly the series converges if $|r| < 1$ and diverges if $|r| > 1$. This is not a rigorous argument, but it is the idea behind a rigorous proof.

Example 2: (a) The infinite series $\sum_1^\infty 1/n!$ converges, because

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

(b) Similarly, $\sum_1^\infty n/2^n$ converges, because

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

The root test. This is quite similar to the ratio test, but is based on the value of the limit of the quantities $\sqrt[n]{|a_n|}$. We state it briefly for completeness.

Ratio test

If $\lim_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum_1^\infty a_n$ converges absolutely;

If $\lim_{n\to\infty} |a_n|^{1/n} > 1$, then $\sum_1^\infty a_n$ diverges;

If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, then the test is inconclusive.

The root test is more powerful than the ratio test, but often more difficult to apply.

Other tests. There are other important tests for convergence of series—in particular, the *comparison test*, based on comparing one series with another, and the *integral test*, based on comparing a series to an improper integral, but these will not be so useful for us in this course, so we omit them here.

Section 3. Power series

A *power series* is an expression of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n, \tag{2}$$

or equivalently

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots. \tag{3}$$

In this series $c_0, c_1, \ldots$ are real numbers, called the *coefficients* of the power series, x_0 is a real number, called the *center* of the power series (one says that the power series is *centered at* x_0), and x is a variable. The first term $c_0(x - x_0)^0$ is always taken to be equal to c_0, as shown in the second form (3). In dealing with power series it is frequently a good idea to work with both the form (2), called *sigma notation* or *summation notation*, and the more explicit form (3).

In a sense, power series are generalizations of polynomials. In fact, polynomials are special cases of power series in which only a finite number of terms are nonzero.

Power series can be used to define functions. If we define a function f by the rule

$$f(x) = \sum_{0}^{\infty} c_n (x - x_0)^n, \tag{4}$$

we mean that f assigns to x the value of the sum of the infinite series on the right hand side of (4), *if the infinite series converges.* If the infinite series diverges, then x is not in the domain of f and $f(x)$ is undefined.

Example 3: Let us define a function $g(x)$ as the sum of a power series by the formula

$$g(x) = \sum_{n=0}^{\infty} (x - 1)^n.$$

With a little thought we can recognize that this series is in fact a geometric series, with ratio $r = (x - 1)$ so that from (1) we can conclude that

$$\sum_{n=0}^{\infty} (x-1)^n = \begin{cases} \dfrac{1}{1-(x-1)} = \dfrac{1}{2-x}, & \text{if } |x-1| < 1, \\ \text{diverges}, & \text{if } |x-1| \geq 1. \end{cases} \tag{5}$$

Hence the domain of g is the interval $0 < x < 2$, and $g(x)$ is equal to $1/(2 - x)$ on this interval. Notice that we are not asserting that $1/(2 - x)$ and $g(x) = \sum_{n=0}^{\infty}(x - 1)^n$ are identical functions, only that they coincide on $(0, 2)$, which happens to be the interval on which the power series converges. In working with power series, one must be attentive to intervals of convergence in this way.

The important things to know about power series are: given a power series, how to find the set of values of x at which it converges; how to differentiate, integrate, and algebraically manipulate power series on their domains of convergence; and how to find power series representations (Taylor series) of a given function f.

Section 4. Domain of convergence of a power series

The basic fact is this: given a power series in the form (2) there exists a number R, called the *radius of convergence* of the power series, with $0 \le R \le \infty$, such that

Radius of convergence

- The power series converges if $|x - x_0| < R$, that is, on $(x_0 - R, x_0 + R)$;
- The power series diverges if $|x - x_0| > R$;
- If $|x - x_0| = R$ (and R is not 0 or ∞) then the power series may either converge or diverge.

In the special case $R = 0$, the series converges if and only if $x = x_0$, (convergence at $x = x_0$ is always true because then all terms in the series are zero, except possibly the first term c_0). In the case $R = \infty$, the power series converges for all x. When $0 < R < \infty$, the power series may diverge or converge at the endpoints $x_0 - R$ and $x_0 + R$.

The radius of convergence can often be determined in practice by applying the ratio test (or the root test); we now give several examples of this, and then a general discussion. **Another important method of finding R, which applies in a somewhat different context, is described in Section 8 below.**

Example 4: (a) Consider the power series $\sum_{n=0}^{\infty} \sqrt{n}\, x^n/2^n$. The n^{th} term of this series is $a_n = \sqrt{n}\, x^n/2^n$. To apply the ratio test we compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\, x^{n+1}/2^{n+1}}{\sqrt{n}\, x^n/2^n}\right| = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}\, \frac{|x|}{2} = \frac{|x|}{2}.$$

According to the ratio test, then, this series converges if $|x|/2 < 1$ or, equivalently, if $|x| < 2$, and diverges if $|x| > 2$. Thus the radius of convergence in this case is $R = 2$.

(b) The series $\sum_{n=0}^{\infty} n3^n x^{2n}$ converges if

$$\lim_{n\to\infty} \frac{\left|(n+1)3^{n+1}x^{2(n+1)}\right|}{\left|n3^n x^2\right|} = 3x^2 \lim_{n\to\infty} \frac{n+1}{n} = 3x^2 < 1,$$

and diverges if this limit is greater than 1. Thus the radius of convergence is determined by the inequality $3x^2 < 1$, or equivalently, $|x| < \sqrt{3}$, and hence $R = \sqrt{3}$.

Now we consider a general power series $\sum_0^\infty c_n(x-x_0)^n$ and try to use the ratio test to obtain a general formula for the radius of convergence. We have to study the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right| = \bigl|x-x_0\bigr|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|.$$

It then follows from the ratio test that *if the limit* $L=\lim_{n\to\infty}\bigl|c_{n+1}/c_n\bigr|$ *exists* then the series converges if $|x-x_0|L<1$ and diverges if $|x-x_0|L>1$, so that

$$R = \frac{1}{\lim_{n\to\infty}\left|\dfrac{c_{n+1}}{c_n}\right|} = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|. \tag{6}$$

Equation (6) furnishes a general formula for R when $\lim_{n\to\infty}\bigl|c_{n+1}/c_n\bigr|$ exists (if this limit is ∞ then $R=0$ and if it is 0 then $R=\infty$).

However, one must use formula (6) with great caution, because the limit may not exist. Thus if we write out the power series of Example 4(b) in full we have

$$\sum_{n=0}^{\infty} n3^n x^{2n} = 3x^2 + 18x^4 + 81x^6 + 324x^8 + \cdots.$$

This series contains only even powers of x, that is, $c_n=0$ if n is odd, so that the ratio $|c_{n+1}/c_n|$ alternates between 0 and ∞, and the limit in (6) does not exist. Our text gives a trick which furnishes one way to handle this problem (see Example 4 on page 178) but I think that **it is simpler, and safer, to use the ratio test directly**, as we did in Example 4(b).

Example 5: To find the radius of convergence of the series $\sum_{n=1}^\infty n^{-n}x^n$ it is convenient to use the root test:

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{|x|^n}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{|x|}{n} = 0.$$

The series converges for all x by the root test, so $R=\infty$.

Section 5. Algebra and calculus of power series

The rule of thumb is: *on their intervals of convergence, infinite series can be treated as if they were polynomials.* This means:

- Different power series which have the same center x_0 and which all converge for x satisfying $|x-x_0|<R$ may be added, multiplied, and divided as if they were polynomials, and the resulting power series converge for $|x-x_0|<R$, except in the case of division, in which they converge so long as the denominator is not zero. See Example 8(e) for an example of multiplication of series. (Note that all the series involved must have radius of convergence at least as big as R, but some of them may have a larger radius of convergence.)

- If a power series $\sum_0^\infty c_n(x-x_0)^n$ converges for x in the interval (x_0-R, x_0+R) then it is differentiable on (x_0-R, x_0+R) and its derivative is obtained by term-by-term differentiation:

$$\frac{d}{dx}\left[\sum_0^\infty c_n(x-x_0)^n\right] = \sum_{n=0}^\infty \frac{d}{dx}c_n(x-x_0)^n = \sum_{n=1}^\infty nc_n(x-x_0)^{n-1} \tag{7}$$

for $x_0-R<x<x_0+R$. Notice that we were able to change the lower limit of the summation to $n=1$ because the $n=0$ term of the derivative vanished. By repeating this procedure, one finds that power series have derivatives of all orders on their intervals of convergence and

$$\frac{d^k}{dx^k}\left[\sum_0^\infty c_n(x-x_0)^n\right] = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)\,c_n(x-x_0)^{n-k}, \tag{8}$$

again for $x_0-R<x<x_0+R$.

- Power series can be integrated term-by-term on their interval of convergence: if $\sum_0^\infty c_n(x-x_0)^n$ converges on (x_0-R, x_0+R), then on this interval

$$\int_{x_0}^x \sum_{n=0}^\infty c_n(u-x_0)^n\,du = \sum_{n=0}^\infty \int_{x_0}^x c_n(u-x_0)^n\,du = \sum_{n=0}^\infty \frac{c_n}{n+1}(x-x_0)^{n+1}. \tag{9}$$

Section 6. Representation of functions by power series; Taylor series

In equation (4) we used a power series to define a function. Now we approach the problem from another direction: we start with a given function f and try to represent it by a power series. More specifically, we suppose that x_0 is given point and are interested in representing f in an interval about x_0 by a convergent power series with center x_0:

$$f(x) = \sum_0^\infty c_n(x-x_0)^n, \quad x_0-R<x<x_0+R. \tag{10}$$

Suppose that this is in fact possible and that (10) holds. Then (8) tells us that f is infinitely differentiable on the interval $x_0-R<x<x_0+R$ and that on this interval

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)\,c_n(x-x_0)^{n-k} \\ &= k!\,c_k + [(k+1)k\cdots 2]\,c_{k+1}\,(x-x_0) + \cdots. \end{aligned} \tag{11}$$

where $f^{(k)}$ denotes the derivative of f of order k. If we set $x=x_0$ in (11) then all terms with $n>k$ vanish, and we find that $c_k = \dfrac{f^{(k)}(x_0)}{k!}$. It follows that the power series representation of f in (10) must have the form

$$f(x) = \sum_0^\infty \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{12}$$

This power series is called the *Taylor series* of f centered at x_0. The Taylor series of f at $x_0 = 0$ is sometimes called the *Maclaurin series* of f.

It is important to realize that **a polynomial is its own Taylor series with center** $x_0 = 0$ (Maclaurin series). Consider for example $f(x) = 1 + 2x + 6x^2$; if we write

$$f(x) = 1 + 2x + 6x^2 = 1 + 2x + 6x^2 + 0\,x^3 + 0\,x^4 + 0\,x^5 + \cdots$$

we see that the polynomial is a Maclaurin series in which all but a finite number of coefficients are zero.

Example 6: If one wants the Taylor series with a different center, say $x_0 = 5$, there is a simple trick to find it:

$$\begin{aligned} f(x) &= 1 + 2[(x-5)+5] + 6[(x-5)+5]^2 \\ &= 1 + 2[(x-5)+5] + 6[(x-5)^2 + 10(x-5) + 25] \\ &= 161 + 62(x-5) + 6(x-5)^2. \end{aligned}$$

Here are some basic Taylor series, given with intervals of convergence. These are easily obtained from (12), but are so important that they should be known from memory. Notice that the first is just the geometric series.

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1; & \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty; \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty; & \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty. \end{aligned} \tag{13}$$

Now we can define a very important class of functions

Analytic functions

If the Taylor series of f at x_0 converges to f in some interval centered at x_0 of radius $R > 0$, then f is said to be *analytic at* x_0.

The Taylor series shown in (13) tell us immediately that the functions there are all analytic at $x = 0$. In fact, the elementary special functions that are studied in undergraduate calculus are analytic at most points. The exponential and trigonometric functions, as well as the polynomials, are analytic at all points. Rational functions, that is, ratios of polynomials, are analytic at all points at which they are defined. Fractional powers x^p, where p is not an integer, are analytic everywhere they are defined except at the origin. If f is analytic at x_0 and g is analytic at $f(x_0)$, then $g(f(x))$ will be analytic at x_0.

Example 7: (a) The function e^x is analytic at every x, and so are $\sin x$ and $e^{\sin(x^2)}$.
(b) The function $1/(x^2-1)$ is analytic everywhere except at $x = \pm 1$, where the denominator vanishes.

(c) $\sqrt{x}$ is analytic at every $x > 0$, but is not analytic at 0. $e^{\sqrt{x}}$ will be analytic at every $x > 0$, but not at 0.

Section 7. Finding Taylor series

Suppose that one is given a function f and a point x_0 and wants to find the Taylor series of f with center x_0. In the very simplest cases one can use (12). However, this is often very time consuming or completely impractical, and it may be simpler to exploit simple known power series, using substitution into a known Taylor series, differentiation or integration of such a series, and/or algebraic manipulations as illustrated in Example 8.

Example 8: (a) To find the Taylor series centered at $x = 0$ for $f(x) = e^{x^2}$ one simply substitutes x^2 for x in the Taylor series given in (13), to find

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

It is instructive to try to solve this one by direct use of (12); you will find that the complications are immense.

(b) Suppose that we want to find the Taylor series of $\dfrac{1}{x+2}$ with center $x_0 = 1$, and its radius of convergence. We are looking for a Taylor series in the form $\sum_{n=0}^{\infty} c_n(x-1)^n$, so we write

$$\frac{1}{x+2} = \frac{1}{3+(x-1)} = \frac{1}{3}\left[\frac{1}{1-(-(x-1)/3)}\right].$$

Substituting $-(x-1)/3$ for x in the geometric power series formula (a) above, we find

$$\frac{1}{x+2} = \frac{1}{3}\sum_{n=0}^{\infty}\left[-\frac{x-1}{3}\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(x-1)^n, \quad \left|\frac{x-1}{3}\right| < 1.$$

This is the Taylor series and its radius of convergence is $R = 3$. Note that in writing $x+2 = 3+(x-1)$ we are using essentially the same trick as in Example 6.

(c) Here is an example using differentiation:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{k=0}^{\infty}(k+1)x^k, \quad |x| < 1.$$

In the last step we have made a change of index, with $k = n-1$.

(d) An example using integration:

$$\ln(1-x) = -\int_0^x \left(\frac{1}{1-x}\right) dx = -\int_0^x \left(\sum_{n=0}^{\infty} x^n\right) dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}, \quad |x| < 1.$$

(e) Finally, an example using the multiplication of power series:

$$\begin{aligned}\frac{e^x}{1-x} &= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)(1+x+x^2+x^3+\cdots)\\ &= 1+(1+1)x+\left(1+1+\frac{1}{2}\right)x^2+\left(1+1+\frac{1}{2}+\frac{1}{6}\right)x^3+\cdots\\ &= 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots.\end{aligned}$$

Section 8. Finding the radius of convergence

In Section 5 above we said that power series could be divided, and the resulting series would converge as long as the denominator was not zero. Here we give a more precise statement of this idea.

Principle of the Radius of Convergence

Suppose that f and g are analytic at x_0 and that their power series with center x_0 have radii of convergence R_1 and R_2 respectively. Suppose also that $g(x_0) \neq 0$ and that R_3 is the distance from x_0 to the closest point in the complex plane at which g vanishes. Then f/g is analytic at x_0 and the radius of convergence of its Taylor series there is at least as big as the smallest of R_1, R_2, and R_3.

We can often use this principle to find the radius of convergence of a Taylor series, even if we do not work out the series itself.

Example 9: (a) We know that $f(x) = 1$ and $g(x) = 1 - x$ are analytic everywhere, in particular at $x = 0$, and that $g(0) \neq 0$, so we know that $f(x)/g(x) = 1/(1-x)$ is analytic at $x = 0$. Using the notation of the principle above we have $R_1 = R_2 = \infty$. But the only zero of the denominator $1 - x$ is at $x = 1$, a distance of $R_3 = 1$ from the origin, so the Taylor series of $1/1 - x$ must have $R = 1$. Of course, this is just the geometric series of (13), so we knew this already.

(b) The function $1/(x^2 + 4)$ is analytic at $x = 0$. The zeros of the denominator are at $\pm 2i$, each a distance 2 from the origin, so the radius of convergence of the Taylor series with center 0 will be 2. The Taylor series with center 3 will have $R = \sqrt{13}$, the distance from the center to either of these zeros.

Section 9. Exercises

1. Find explicitly the Taylor series with center $x_0 = 0$ of the function $1/(x^2 + 4)$ discussed in Example 9(b), and show using the ratio test that it has radius of convergence 2, as we concluded there. Hint: use the method of Example 8(b).

2. In each part below, find the Taylor series of the given function with the given center x_0. Try to use the methods of Section 7 to obtain these series from other ones which are already known, rather than the formula (12). Find also the radius of convergence, and explain how your answer is consistent with the Principle of the Radius of Convergence above.

(a) $1/(2+3x)$, $x_0 = 3$.

(b) $(1+x)/(1-x)$, $x_0 = 0$.

(c) e^x, $x_0 = -3$.

(d) $1/(1-x)^3$, $x_0 = 0$.

(e) $1/(1-x)^3$, $x_0 = 4$.

(f) $1/(1+x^3)$, $x_0 = 0$.

(g) $\sin(x^4)$, $x_0 = 0$.

(h) $\ln x$, $x_0 = 2$.

(i) $\sin 2x$, $x_0 = \pi/4$. Hint: $\sin(\theta - \pi/2) = -\cos\theta$.

3. Find the Taylor series with the given center x_0 for the given functions directly from (12), and show that the results agree with those obtained by other methods, as indicated.

(a) $1/(1-x)^2$, $x_0 = 0$. (Example 8(c))

(b) $\ln(1-x)$, $x_0 = 0$. (Example 8(d))

(c) $\sin 2x$, $x_0 = \pi/4$. (Exercise 2(i))

4. Show that the Taylor series for $\sin x$ with center $x_0 = 0$ may be obtained by differentiating the series for $\cos x$.

5. Find the Taylor series for $1/(1-x)^2$, given in Example 8(c), by squaring the geometric series for $1/(1-x)$.